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A reflection principle formulated in terms of games *

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Abstract

We introduce a principle formulated in terms of the existence of a winning strategy of a game and prove that this principle is placed between the reflection principle down to internally stationary sets ($\mathsf{RP}_{\mathsf{IS}}$) and the reflection principle down to internally club sets ($\mathsf{RP}_{\mathsf{IC}}$). In particular, under CH this principle gives a new characterization of Fleissner's $\mathsf{Axiom\ R}$.

1 Introduction

For a game $\mathcal{G}$ played by Players I and II, let $\mathsf{WS}_{II}(\mathcal{G})$ denote the assertion "Player II has a winning strategy in $\mathcal{G}$".

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An extended version of this paper with more details and proofs can be found at:
`http://kurt.scitec.kobe-u.ac.jp/~fuchino/preprints.html`

In [9], we introduced a game $G^{\downarrow}_{\omega}(\kappa)$ for uncountable cardinals κ (see Section 3 for the definition of this and other games mentioned here) and proved that the Rado Conjecture (RC, see Section 2 for the definition of this principle) implies the assertion

(G_0) $\mathsf{WS}_{II}(G^{\downarrow}_{\omega}(\kappa))$ holds for all uncountable κ.

Further, it is proved in [9] that (G_0) implies the Fodor-type Reflection Principle (FRP, see Section 2 for the definition of this principle and, [4] and [5] for basic facts of this principle).

In [1], Philipp Doebler introduced a similar game he called $G_{\omega}([\kappa]^{\omega_1}, \omega_1)$ and proved that the Rado Conjecture also implies the principle

(G_1) $\mathsf{WS}_{II}(G_{\omega}([\kappa]^{\aleph_1}, \omega_1))$ holds for all $\kappa \geq \aleph_2$.

He then proved that the principle (G_1) implies the Semistationary Reflection (SSR).

In this paper, we introduce a game $G^{\downarrow\downarrow}_{\omega}([\kappa]^{\aleph_1})$ which generalizes both $G^{\downarrow}(\kappa)$ and $G_{\omega}([\kappa]^{\omega_1}, \omega_1)$. Unfortunately the principle

($\mathsf{G}^{\downarrow\downarrow}$) $\mathsf{WS}_{II}(G^{\downarrow\downarrow}_{\omega}([\kappa]^{\aleph_1}))$ for all $\kappa \geq \aleph_2$

is not a consequence of the Rado Conjecture: In Section 4, we show that the principle ($\mathsf{G}^{\downarrow\downarrow}$) implies the reflection principle $\mathsf{RP}_{\mathsf{IS}}$. It is known that $\mathsf{RP}_{\mathsf{IS}}$ (or even RP) is not a consequence of RC (see Sakai [14]).

2 Reflection Principles

Let us first review the reflection principles we mentioned in the previous section.

We shall call here a partial ordering $T = \langle T, \leq_T \rangle$ a *tree* if the initial segment $\{u \in T : u \leq_T t\}$ in T below each $t \in T$ is well-ordered. In particular, we assume here that a tree may have multiple roots.

A tree T is *special* if there are $T_i \subseteq T$, $i \in \omega$ such that each of T_i's is pairwise incomparable and $T = \bigcup_{i \in \omega} T_i$.

Rado's Conjecture (RC) is the assertion:

(RC): Any tree T is special if and only if all subtrees of T of cardinality $\aleph_1$ are special.

RC is known to be consistent (modulo a large large cardinal). E.g., Todorčević showed that, if κ is strongly compact and $\mathbb{P} = Col(\omega_1, {<}\kappa)$, then we have

$$\Vdash_{\mathbb{P}} \text{“Rado's Conjecture”}.$$

For a cardinal κ and a regular cardinal $\delta < \kappa$, we denote

$$E^{\kappa}_{\delta} = \{\alpha < \kappa : cf(\alpha) = \delta\}.$$

a mapping $g : E \to \kappa$ for $E \subseteq E^\kappa_\delta$ is called a ladder system if $\sup g(\alpha) = \alpha$ and $otp(g(\alpha)) = \delta$ hold for all $\alpha \in E$.

For a regular uncountable cardinal κ, we define the Fodor-type Reflection Principle for κ by

$\mathsf{FRP}(\kappa)$: For all stationary $E \subseteq E^\kappa_\omega$ and for all ladder system $g : E \to [\kappa]^{\aleph_0}$, there exists $\alpha^* \in E^\kappa_{\omega_1}$ such that

$$\{x \in [\alpha^*]^{\aleph_0} : sup(x) \in E,\ g(sup(x)) \subseteq x\}$$

is stationary in $[\alpha^*]^{\aleph_0}$.

The Fodor-type Reflection Principle (FRP) is the assertion:

(FRP): $\mathsf{FRP}(\kappa)$ holds for all regular $\kappa > \aleph_1$.

FRP is known to be equivalent to many mathematical reflection principles over ZFC (see [3], [4], [5], [6], [7], see also [8]).

(2.1) Any locally countably compact topological space X is metrizable if and only if all subspaces of X of cardinality $\leq \aleph_1$ are metrizable

is one of such assertions equivalent to FRP over ZFC (see [4] and [5]).

FRP implies Shelah's Strong Hypothesis and hence, in particular, Singular Cardinal Hypothesis (see [7]). It also implies the total failure of square principles $\square_\kappa$ for all cardinals $\kappa \geq \aleph_1$.

Suppose that $M \prec \mathcal{H}(\lambda)$ for some regular $\lambda \geq \aleph_2$ and $|M| = \aleph_1$.

M is said to be *internally cofinal* (abbreviation: IU)[1] if $[M]^{\aleph_0} \cap M$ is cofinal in $[M]^{\aleph_0}$ with respect to $\subseteq$. M is *internally stationary* (abbreviation: IS) if $[M]^{\aleph_0} \cap M$ is stationary in $[M]^{\aleph_0}$. M is *internally club* (abbreviation: IC) if $[M]^{\aleph_0} \cap M$ contains a closed unbounded set in $[M]^{\aleph_0}$. Finally, M is *internally approachable* (abbreviation: IA) if M is the union of a continuously increasing sequence $\langle M_\alpha : \alpha < \omega_1\rangle$ countable sets such that $\langle M_\alpha : \alpha \leq \delta\rangle \in M_{\delta+1}$ for all $\delta < \omega_1$ [2].

It is clear from the definition that, for any $M \prec \mathcal{H}(\lambda)$, we have the implication: M is IA $\Rightarrow$ M is IC $\Rightarrow$ M is IS $\Rightarrow$ M is IU. It is easy to see that all of these notions can be characterized in terms of filtration (see footnote 2)):

Lemma 2.1 *Suppose that $M \prec \mathcal{H}(\lambda)$ for some regular $\lambda \geq \omega_2$ and $|M| = \aleph_1$.*

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[1] Internally cofinal M is also called *internally unbounded* in the literature (see e.g. Krueger [11]).

[2] For a structure M of cardinality $\aleph_1$, we shall call a continuously increasing sequence $\langle M_\alpha : \alpha < \omega_1\rangle$ of countable subsets of M with $\bigcup_{\alpha<\omega_1} M_\alpha = M$ a *filtration* of M. By thinning out the index set ω_1, we may assume in some cases that the filtration $\langle M_\alpha : \alpha < \omega_1\rangle$ consists of elementary structures.

(1) *M is internally cofinal if and only if there is a filtration $\langle a_\alpha : \alpha < \omega_1 \rangle$ of M such that $a_{\alpha+1} \in M$ for every $\alpha < \omega_1$.*

(2) *M is internally stationary if and only if $\{\alpha < \omega_1 : M_\alpha \in M\}$ is stationary for a/any filtration $\langle M_\alpha : \alpha < \omega_1 \rangle$ of M.*

(3) *M is internally club if and only if there is a filtration $\langle M_\alpha : \alpha < \omega_1 \rangle$ of M such that $M_\alpha \in M_{\alpha+1}$ for all $\alpha < \omega_1$.* $\square$

These notions can be different: e.g. John Krueger proved under PFA, there are stationarily many internally club but not internally approachable $M \prec \mathcal{H}(\lambda)$ for all regular $\lambda > \aleph_1$ (for this and other results of this line see Krueger [11] and [12]). However this is not the case under CH:

Lemma 2.2 *Under* CH, *any $M \prec \mathcal{H}(\lambda)$ is* IU *if and only if it is* IS *if and only if it is* IC *if and only if it is* IA. $\square$

In the following, we shall always denote one of the properties IU, IS, IC or IA with $\mathcal{P}$. "$\sqsubset$" in connection with a cardinal, say λ, denotes a(n arbitrary) well-ordering of the set $\mathcal{H}(\lambda)$ of all sets of hereditarily of cardinality $< \lambda$. If we have to emphasize that the well-ordering $\sqsubset$ refers to $\mathcal{H}(\lambda)$, we write $\sqsubset_{\mathcal{H}(\lambda)}$.

For a cardinal $\lambda > \aleph_1$ let

$\mathsf{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$: For any stationary $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ there is a $\mathcal{P}$ elementary substructure M of the structure $\langle \mathcal{H}(\lambda), \in, \sqsubset \rangle$ (of cardinality $\aleph_1$) such that

$$S \cap [M]^{\aleph_0}\ ^{3)} \text{ is stationary in } [M]^{\aleph_0}. \tag{2.2}$$

We define the global version of the reflection principle $\mathsf{RP}_{\mathcal{P}}$ down to a structure with the property $\mathcal{P}$ to be $\mathsf{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$ for all cardinal $\lambda > \aleph_1$.

$\mathsf{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$ is equivalent with seemingly stronger variants of the assertion:

Lemma 2.3 *the following are equivalent for a regular cardinal $\lambda > \aleph_1$:*

(a) $\mathsf{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$.

(b) *For any stationary $S \subseteq [\mathcal{H}(\kappa)]^{\aleph_0}$ and any expansion $\mathcal{M}$ of the structure $\langle \mathcal{H}(\kappa), \in, \sqsubset \rangle$ in an arbitrary countable language, there is a $\mathcal{P}$ elementary substructure M of $\mathcal{M}$ (of cardinality $\aleph_1$) with* (2.2).

(c) *For any stationary $S \subseteq [\mathcal{H}(\kappa)]^{\aleph_0}$ and any expansion $\mathcal{M}$ of the structure $\langle \mathcal{H}(\kappa), \in, \sqsubset \rangle$ in an arbitrary countable language, there are stationarily many $\mathcal{P}$ elementary substructures M of $\mathcal{M}$ (of cardinality $\aleph_1$) with* (2.2). $\square$

Using Lemma 2.3 we can prove the following downward transfer property of $\mathsf{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$:

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3) That is, S intersection with the set of all countable subsets of the underlying set of the structure M.

Lemma 2.4 *For regular cardinals* $\aleph_1 < \lambda' < \lambda$, *if* $\mathsf{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$ *holds then* $\mathsf{RP}_{\mathcal{P}}([\mathcal{H}(\lambda')]^{\aleph_0})$ *also holds.* □

Lemma 2.5 *The following are equivalent:* (a) $\mathsf{RP}_{\mathcal{P}}$.

(b) *For any uncountable* X, *stationary* $S \subseteq [X]^{\aleph_0}$, *regular* θ *with* $X \subseteq \mathcal{H}(\theta)$ *and any expansion* $\mathcal{M}$ *of* $\langle \mathcal{H}(\theta), \in, \sqsubset, X\rangle$ *in a countable language, there is a* $\mathcal{P}$ *elementary substructure* M *of* $\mathcal{M}$ *of cardinality* $\aleph_1$ *such that* $S \cap [X \cap M]^{\aleph_0}$ *is stationary in* $[X \cap M]^{\aleph_0}$.

(c) *For any uncountable cardinal* λ, *stationary* $S \subseteq [\lambda]^{\aleph_0}$, *regular* $\theta \geq \lambda$ *and any expansion* $\mathcal{M}$ *of* $\langle \mathcal{H}(\theta), \in, \sqsubset, \lambda\rangle$ *in a countable language, there is a* $\mathcal{P}$ *elementary substructure* M *of* $\mathcal{M}$ *of cardinality* $\aleph_1$ *such that* $S \cap [\lambda \cap M]^{\aleph_0}$ *is stationary in* $[\lambda \cap M]^{\aleph_0}$. □

Fleissner's Axiom R ([2]) is equivalent to $\mathsf{RP}_{\mathsf{IU}}$ in our notation. For a any set X of cardinality $> \aleph_1$, let

($\mathsf{AR}([X]^{\aleph_0})$): For any stationary $S \subseteq [X]^{\aleph_0}$ and ω_1-club[4] $T \subseteq [X]^{\aleph_1}$, there is $U \in T$ such that $S \cap [U]^{\aleph_0}$ is stationary in $[U]^{\aleph_0}$.

Then we define Axiom R to be the assertion that $\mathsf{AR}([\lambda]^{\aleph_0})$ holds for all cardinal $\alpha > \aleph_1$. Since $\mathsf{AR}([\lambda]^{\aleph_0})$, for cardinals $\lambda > \aleph_1$ also satisfy the downward transfer similar to Lemma 2.4, the following Lemma implies the equivalence of $\mathsf{RP}_{\mathsf{IU}}$ and Axiom R:

Lemma 2.6 *For any* $\lambda > \aleph_1$, *we have* $\mathsf{AR}([2^{<\lambda}]^{\aleph_0})$ *if and only if* $\mathsf{RP}_{\mathsf{IU}}([\mathcal{H}(\lambda)]^{\aleph_0})$. □

Proof. Note that $|\,\mathcal{H}(\lambda)\,| = 2^{<\lambda}$ and hence $\mathsf{AR}([2^{<\lambda}]^{\aleph_0})$ is equivalent to $\mathsf{AR}([\mathcal{H}(\lambda)]^{\aleph_0})$.

First, assume $\mathsf{RP}_{\mathsf{IU}}([\mathcal{H}(\lambda)]^{\aleph_0})$. Suppose that $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ is stationary and $T \subseteq [\mathcal{H}(\lambda)]^{\aleph_1}$ is ω_1-club.

Let $\mathcal{M} = \langle \mathcal{H}(\lambda), \in, \sqsubset, T\rangle$. By Lemma 2.5, there is $M \prec \mathcal{M}$ such that

(2.3) $|\,M\,| = \aleph_1$;

(2.4) $M \models \mathsf{IU}$ and

(2.5) $S \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$.

By (2.3), (2.4) and $M \prec \mathcal{M}$, it is easy to see that M is the union of an ω_1 chain of elements of T. By ω_1-clubness of T it follows that $M \in T$. This shows that $\mathsf{AR}([\mathcal{H}(\lambda)]^{\aleph_0})$ holds.

Assume now $\mathsf{AR}([2^{<\lambda}]^{\aleph_0})$ and suppose that $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ is stationary. Let

$$T = \{M \in [\mathcal{H}(\lambda)]^{\aleph_1} : M \prec \mathcal{H}(\lambda),\ M \models \mathsf{IU}\}.$$

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[4] $T \subseteq [X]^{\aleph_1}$ for an uncountable set X is said to be ω_1-*club* (or "tight and unbounded" in Fleissner's terminology in [2]) if T is cofinal in $[X]^{\aleph_1}$ with respect to $\subseteq$ and for any increasing chain $\langle U_\alpha : \alpha < \omega_1\rangle$ in T of length ω_1, we have $\bigcup_{\alpha<\omega_1} U_\alpha \in T$.

Then T is ω_1-club. By $\mathsf{AR}([2^{<\lambda}]^{\aleph_0})$ or by its equivalent $\mathsf{AR}([\mathcal{H}(\lambda)]^{\aleph_0})$, there is $M \in T$ such that $S \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$. This shows that $\mathsf{RP}_{\mathsf{IU}}([\mathcal{H}(\lambda)]^{\aleph_0})$ holds. $\square$ (Lemma 2.6)

3 Definition of the games

For a cardinal κ, let

(3.1) $\quad {}^{\kappa\downarrow}\kappa = \{f \in {}^{\kappa}\kappa \,:\, f \text{ is regressive}\}$.

The game $G^{\downarrow}_{\omega}(\kappa)$ for Players I and II is defined as follows: A match in $G^{\downarrow}_{\omega}(\kappa)$ is a sequence of the form:

$$\begin{array}{c|cccccccc} I & f_0 \in {}^{\kappa\downarrow}\kappa & & f_1 \in {}^{\kappa\downarrow}\kappa & & \cdots & f_n \in {}^{\kappa\downarrow}\kappa & & \cdots \\ \hline II & & \delta_0 \in \kappa & & \delta_1 \in \kappa & \cdots & & \delta_n \in \kappa & \cdots \end{array} \quad (n < \omega)$$

Player II wins in a match of $G^{\downarrow}_{\omega}(\kappa)$ as above if

(3.2) $\quad \{\alpha \in E^{\kappa}_{\omega_1} \,:\, f_n(\alpha) < \sup\{\delta_i \,:\, i \in \omega\} \text{ for all } n \in \omega\}$ is unbounded.

The game $G^{\downarrow}_{\omega}(\kappa)$ was introduced in [9]. It is used there to prove the implication of FRP from RC by showing that the assertion (G_0) as in Section 1 defined in terms of this game interpolates the implication.

The following game $G_{\omega}([\kappa]^{\aleph_1}, \omega_1)$ for Players I and II for a cardinal κ was introduced by Doebler in [1]: A match in $G_{\omega}([\kappa]^{\aleph_1}, \omega_1)$ is a sequence of the form:

$$\begin{array}{c|cccccccc} I & f_0 \in {}^{[\kappa]^{\aleph_1}}\omega_1 & & f_1 \in {}^{[\kappa]^{\aleph_1}}\omega_1 & & \cdots & f_n \in {}^{[\kappa]^{\aleph_1}}\omega_1 & & \cdots \\ \hline II & & \delta_0 \in \omega_1 & & \delta_1 \in \omega_1 & \cdots & & \delta_n \in \omega_1 & \cdots \end{array} \quad (n < \omega)$$

II wins in a match of $G_{\omega}([\kappa]^{\aleph_1}, \omega_1)$ as above if

$$\{a \in [\kappa]^{\aleph_1} \,:\, f_n(a) < \sup\{\delta_i \,:\, i \in \omega\} \text{ for all } n \in \omega\}$$

is cofinal in $[\kappa]^{\aleph_1}$.

Doebler proved that the principle (G_1) as defined in Section 1 in terms of this game follows also from RC and it implies SSR.

It is easy to see that both of (G_0) and (G_1) are consequences of $\mathsf{RP}_{\mathsf{IC}}$ (this also follows from Corollary 4.4). Hence we have the diagram on the right:

$$\begin{tikzcd} \mathsf{RC} \arrow[d] \arrow[dr] & \mathsf{RP}_{\mathsf{IC}} \arrow[dl] \arrow[d] \\ (\mathsf{G}_0) \arrow[d] & (\mathsf{G}_1) \arrow[d] \\ \mathsf{FRP} & \mathsf{SSR} \arrow[d] \\ & \mathsf{CC} \end{tikzcd}$$

Since FRP and SSR imply almost all known consequences of RC[5], it seems to be an interesting question what is the natural principle which is still a consequence of both RC and $\mathsf{RP}_{\mathsf{IC}}$ while which implies both FRP and SSR.

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[5] Perhaps with the exception of the negation of Martin's Axiom for $\aleph_1$ dense sets which is a consequence of RC while $\mathsf{RC}_{\mathcal{P}}$'s are consistent with Martin's Axiom since they all follow from $\mathsf{MA}^{+}(\sigma\text{-closed})$.

The assertion of the existence of the winning strategy for player *II* (the principle $(\mathsf{G}^{\downarrow\downarrow})$ introduced in Section 1) in the following game $G_\omega^{\downarrow\downarrow}([\kappa]^{\aleph_1})$ for all $\kappa > \aleph_1$ seemed to be a natural candidate for such an interpolant. Unfortunately, this principle turned out to be too strong to be a consequence of RC while it is still a consequence of $\mathsf{RP}_{\mathsf{IC}}$ as we shall see in Section 4. In [9] we introduce a weakening of $(\mathsf{G}^{\downarrow\downarrow})$ which is an interpolant of RC and $\mathsf{RP}_{\mathsf{IC}}$ on one side and FRP and SSR on the other.

$$\begin{tikzcd}
\mathsf{RC} \arrow[drr] & & & & \mathsf{RP}_{\mathsf{IC}} \arrow[dll] \\
& & ? \arrow[dl] \arrow[dr] & & \\
& (\mathsf{G}_0) \arrow[d] & & (\mathsf{G}_1) \arrow[d] & \\
& \mathsf{FRP} & & \mathsf{SSR} \arrow[d] & \\
& & & \mathsf{CC} &
\end{tikzcd}$$

Here is the definition of $G_\omega^{\downarrow\downarrow}([\kappa]^{\aleph_1})$ for a cardinal $\kappa > \aleph_1$. We call a function $f : [\kappa]^{\aleph_1} \to \kappa$ regressive if $f(a) \in a$ holds for all $a \in [\kappa]^{\aleph_1}$. Similarly to the definition (3.1), let

$$(3.3) \qquad {}^{[\kappa]^{\aleph_1}\downarrow}\kappa = \{f \in {}^{[\kappa]^{\aleph_1}}\kappa \, : \, f \text{ is regressive}\}.$$

A match in $G_\omega^{\downarrow\downarrow}([\kappa]^{\aleph_1})$ for Players *I* and *II* is a sequence of the form:

$$\begin{array}{c|cccccccc}
I & f_0 \in {}^{[\kappa]^{\aleph_1}\downarrow}\kappa & & f_1 \in {}^{[\kappa]^{\aleph_1}\downarrow}\kappa & & \cdots & f_n \in {}^{[\kappa]^{\aleph_1}\downarrow}\kappa & & \cdots \\
\hline
II & & d_0 \in [\kappa]^{\aleph_0} & & d_1 \in [\kappa]^{\aleph_0} & \cdots & & d_n \in [\kappa]^{\aleph_0} & \cdots
\end{array} \quad (n < \omega)$$

II wins in a match in $G_\omega^{\downarrow\downarrow}([\kappa]^{\aleph_1})$ as above if

$$\{a \in [\kappa]^{\aleph_1} \, : \, f_n(a) \in \bigcup\{d_i \, : \, i \in \omega\} \text{ for all } n \in \omega\}$$

is cofinal in $[\kappa]^{\aleph_1}$.

Note that by the definition of the games, it is clear that $(\mathsf{G}^{\downarrow\downarrow})$ implies both of (G_0) and (G_1).

$$\begin{tikzcd}
& & & \mathsf{RP}_{\mathsf{IC}} \arrow[d] \\
\mathsf{RC} \arrow[ddr] \arrow[ddrr] & & & (\mathsf{G}^{\downarrow\downarrow}) \arrow[ddll] \arrow[ddl] \arrow[d] \\
& & & \mathsf{RP}_{\mathsf{IS}} \\
& (\mathsf{G}_0) \arrow[d] & (\mathsf{G}_1) \arrow[d] & \\
& \mathsf{FRP} & \mathsf{SSR} \arrow[d] & \\
& & \mathsf{CC} &
\end{tikzcd}$$

4 Characterizations of $(\mathsf{G}^{\downarrow\downarrow})$

The following characterization of $(\mathsf{G}^{\downarrow\downarrow})$ can be obtained easily by regarding the moves of Player *I* in $G_\omega^{\downarrow\downarrow}([\kappa]^{\aleph_1})$ as an enumeration of Skolem functions with parameters in some model M and the moves of Player *II* as the gradual capturing of $\kappa \cap M$:

Lemma 4.1 *For any cardinal $\kappa > \aleph_1$ the following are equivalent:*

(a) $\mathsf{WS}_{II}(G_\omega^{\downarrow\downarrow}([\kappa]^{\aleph_1}))$.

(b) *For sufficiently large regular θ with $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \sqsubset \rangle$, for any $M \prec \mathcal{M}$ with $|M| = \aleph_0$ and $\kappa \in M$, we have: for any $a \in [\kappa]^{\aleph_1}$, there are $b \in [\kappa]^{\aleph_1}$ and countable $N \prec \mathcal{M}$ such that $a \subseteq b$, $b \in N$, $M \subseteq N$ and $b \cap N = b \cap M$.*

(c) *For sufficiently large regular θ with $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \sqsubset \rangle$, for club many*[6] *countable $M \prec \mathcal{M}$ with $\kappa \in M$, we have: for any $a \in [\kappa]^{\aleph_1}$, there are $b \in [\kappa]^{\aleph_1}$ and countable $N \prec \mathcal{M}$ such that $a \subseteq b$, $b \in N$, $M \subseteq N$ and $b \cap N = b \cap M$.* □

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[6] We can also express this "club many ..." in terms of expansion of the structure $\mathcal{M}$ similarly to Lemma 2.3 or Lemma 2.4.

By Lemma 4.1, (c), we see immediately that $\mathsf{WS}_{II}(G_\omega^{\downarrow\downarrow}([\kappa]^{\aleph_1}))$ for cardinals $\kappa > \aleph_1$ also enjoy the downward transfer property:

Corollary 4.2 *Suppose* $\aleph_1 < \kappa' < \kappa$ *and* $\mathsf{WS}_{II}(G_\omega^{\downarrow\downarrow}([\kappa]^{\aleph_1}))$ *holds. Then we also have* $\mathsf{WS}_{II}(G_\omega^{\downarrow\downarrow}([\kappa']^{\aleph_1}))$.

Theorem 4.3 *The following are equivalent:* (a) $(\mathsf{G}^{\downarrow\downarrow})$.

(b) *For all* $\kappa > \aleph_1$, *for all sufficiently large regular* θ *with* $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \sqsubset \rangle$, *there are club many countable* $M \prec \mathcal{M}$ *such that* $\kappa \in M$ *and for any* $X \in [\mathcal{H}(\kappa)]^{\aleph_1}$, *there are* $Y \in [\mathcal{H}(\kappa)]^{\aleph_1}$ *and countable* $N \prec \mathcal{M}$ *such that* $X \subseteq Y$, $Y \in N$, $M \subseteq N$ *and* $Y \cap N = Y \cap M$.

(c) *For all* $\kappa > \aleph_1$, *for all sufficiently large regular* θ *with* $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \sqsubset \rangle$, *there are club many countable* $M \prec \mathcal{M}$ *such that* $\kappa \in M$ *and for any* $X \in [\mathcal{H}(\kappa)]^{\aleph_1}$, *there are* $Z \prec \langle \mathcal{H}(\kappa), \in, \sqsubseteq_{\mathcal{H}(\kappa)} \rangle$ *of cardinality* $\aleph_1$ *and countable* $N \prec \mathcal{M}$ *such that* $X \subseteq Z$, $Z \in N$, $M \subseteq N$ *and* $Z \cap N = Z \cap M$.

(d) *For any* $\kappa > \aleph_1$ *and stationary* $S \subseteq [\mathcal{H}(\kappa)]^{\aleph_0}$, *for any* $X \in [\mathcal{H}(\kappa)]^{\aleph_1}$ *there is a* $Z \prec \mathcal{H}(\kappa)$ *such that* $X \subseteq Z$ $|Z| = \aleph_1$ *and* $S \cap Z$ *is stationary in* $[Z]^{\aleph_0}$.

(e) *For all* $\kappa > \aleph_1$, *for all sufficiently large regular* θ *with* $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \sqsubset \rangle$, *there are club many countable* $M \prec \mathcal{M}$ *such that* $\kappa \in M$ *and for any* $X \in [\mathcal{H}(\kappa)]^{\aleph_1}$, *there are* IS $Z \prec \langle \mathcal{H}(\kappa), \in, \sqsubseteq_{\mathcal{H}(\kappa)} \rangle$ *of cardinality* $\aleph_1$ *and countable* $N \prec \mathcal{H}(\theta)$ *such that* $X \subseteq Z$, $Z \in N$, $M \subseteq N$ *and* $Z \cap N = Z \cap M$.

Proof. (a) $\Rightarrow$ (b): Let $\lambda = 2^{<\kappa} = |\mathcal{H}(\kappa)|$ and let $\varphi : \lambda \to \mathcal{H}(\kappa)$ be a bijection. Then all countable $M \prec \mathcal{M}$ with $\varphi \in M$ satisfies the condition in (b): the situation of Lemma 4.1, (b) (for κ there $= \lambda$) is translated to the desired condition in the present (b) by φ.

(b) $\Rightarrow$ (a): The back-translation by the mapping φ as in the proof of (a) $\Rightarrow$ (b) implies $\mathsf{WS}_{II}(G_\omega^{\downarrow\downarrow}([2^{<\kappa}]^{\aleph_1}))$ for all $\kappa > \aleph_1$. By Corollary 4.2, it follows that $\mathsf{WS}_{II}(G_\omega^{\downarrow\downarrow}([\kappa]^{\aleph_1}))$ for all $\kappa > \aleph_1$.

(b) $\Rightarrow$ (c): Suppose that κ, θ, $\mathcal{M}$, M, X, Y, N are as in (b). Then $Z = sk_{\mathcal{M}}(Y)$ witnesses (c).

(c) $\Rightarrow$ (d): Assume that (c) holds and suppose that $S \subseteq [\mathcal{H}(\kappa)]^{\aleph_0}$ is stationary. Let θ $\mathcal{M}$, M be as in (c). Since there are club many M's as in (c), we may assume that

(4.1) $\quad S \in M$ and $\mathcal{H}(\kappa) \cap M \in S$.

Let $X \in [\mathcal{H}(\kappa)]^{\aleph_1}$ be defined by

(4.2) $\quad X = \omega_1 \cup (\mathcal{H}(\kappa) \cap M) \cup \{M \cap \mathcal{H}(\kappa)\}$.

Let $Z \prec \mathcal{H}(\kappa)$ and $N \prec \mathcal{H}(\theta)$ be as in (c) for this X. Thus we have N is countable, Z is of cardinality $\aleph_1$, $X \subseteq Z$, $Z \in N$, $M \subseteq N$ and

$$Z \cap N = Z \cap M. \tag{4.3}$$

We are done by showing that $S \cap Z$ is stationary in $[Z]^{\aleph_0}$. Since S, $Z \in N$, it is enough to show that any club $C \subseteq [Z]^{\aleph_0}$ with $C \in N$ intersects with S: Note that we have

$$Z \cap M = \mathcal{H}(\kappa) \cap M \tag{4.4}$$

by (4.2). For such a club C we have

$$\begin{aligned} S \ni M \cap \mathcal{H}(\kappa) & \qquad \text{by (4.1)} \\ = Z \cap M & \qquad \text{by (4.4)} \\ = Z \cap N & \qquad \text{by (4.3)} \\ = \bigcup (C \cap N) \in C. & \qquad \text{by elementarity, } C \in N \text{ and closedness of } C \end{aligned}$$

Thus $S \cap C \neq \emptyset$ as desired.

(c) $\Rightarrow$ (e): The proof of (c) $\Rightarrow$ (d) above for $S = [\mathcal{H}(\kappa)]^{\aleph_0}$ shows this.

(e) $\Rightarrow$ (c): trivial.

(c) $\Rightarrow$ (b): trivial.

(d) $\Rightarrow$ (e): Assume that (d) holds. For $\kappa > \aleph_1$, let θ a sufficiently large regular cardinal and $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \sqsubset \rangle$. Let

$$\begin{array}{lll} (4.5) & S = \{ M \in [\mathcal{M}]^{\aleph_0} : M \prec \mathcal{M},\ \kappa \in M, \text{ there is } X_M \in [\mathcal{H}(\kappa)]^{\aleph_1} \text{ such that} & \\ & \qquad\qquad (4.6) \ \text{there are no countable } N \prec \mathcal{M} & \\ & \qquad\qquad\qquad \text{and } Y \prec \langle \mathcal{H}(\kappa), \in, \sqsubset_{\mathcal{H}(\kappa)} \rangle \text{ such that} & \\ & \qquad\qquad\qquad M \prec N,\ X_M \subseteq Y,\ Y \text{ is } \mathsf{IS} \text{ and of size } \aleph_1, & \\ & \qquad\qquad\qquad Y \in N \text{ and } M \cap Y = N \cap Y & \}. \end{array}$$

It is enough to show that S is non-stationary. In the following we show this indirectly: We assume that S is stationary and drive a contradiction from this assumption.

For each $M \in S$ we choose $X_M \in [\mathcal{H}]^{\aleph_1}$ such that

$$X_M \supseteq M \cup \omega_1,\ X_M \prec \mathcal{M} \text{ and (4.6) holds for } M \text{ and } X_M. \tag{4.7}$$

Let $\chi > 2^{<\theta}$ be regular. Note that we have $\mathcal{H}(\theta) \in \mathcal{H}(\chi)$. Let

$$\begin{aligned} \tilde{S} = \{ M \in [\mathcal{H}(\chi)]^{\aleph_0} : {} & M \prec \langle \mathcal{H}(\chi), \in, \sqsubset_{\mathcal{H}(\chi)} \rangle, \\ & \kappa, \theta, \cdots \in M,\ M \cap \mathcal{H}(\theta) \in S \ \}. \end{aligned} \tag{4.8}$$

By the assumption of the stationarity of S, $\tilde{S}$ is also stationary. Thus, by (d), there is $Z \prec \langle \mathcal{H}(\chi), \in, \sqsubset \rangle$ such that

$$|Z| = \aleph_1,\ \omega_1 \subseteq Z, \tag{4.9}$$

$$\kappa,\ \theta,\ S,\ \langle X_M : M \in S \rangle,\ \sqsubset_{\mathcal{H}(\kappa)},\ \sqsubset_{\mathcal{H}(\theta)}, \cdots \in Z \text{ and} \tag{4.10}$$

(4.11) $\tilde{S} \cap Z$ is stationary in $[Z]^{\aleph_0}$.

Let

(4.12) $Y = Z \cap \mathcal{H}(\kappa)$.

Then we have $\omega_1 \subseteq Y$ and hence $|Y| = \aleph_1$ and $Y \prec \langle \mathcal{H}(\kappa), \in, \sqsubseteq_{\mathcal{H}(\kappa)} \rangle$.

Y is stationary in $[Y]^{\aleph_0}$: Suppose that $C \subseteq [Y]^{\aleph_0}$ is a club. Let $\tilde{C} = \{x \in [Z]^{\aleph_0} : x \cap Y \in C\}$. Then $\tilde{C}$ is a club subset of $[Z]^{\aleph_0}$. By (4.11), $Z \cap [Z]^{\aleph_0}$ is stationary in $[Z]^{\aleph_0}$. Hence there is an $x \in \tilde{C} \cap Z$. By definition of $\tilde{C}$, we have $x \cap \mathcal{H}(\kappa) \in C$.

Since $Z \prec \langle \mathcal{H}(\chi), \in, \sqsubset_{\mathcal{H}(\chi)} \rangle$ and $\mathcal{H}(\kappa) \in Z$ by (4.10), we have $x \cap \mathcal{H}(\kappa) \in Z$. By $x \cap \mathcal{H}(\kappa) \in C$ we have $x \cap \mathcal{H}(\kappa) \in \mathcal{H}(\kappa)$. It follows that $x \cap \mathcal{H}(\kappa) \in Y$. Thus $x \cap \mathcal{H}(\kappa) \in C \cap Y$.

For each $M \in \tilde{S} \cap Z$ we have $M \cap \mathcal{H}(\theta) \in S \cap Z$ as we just saw and, by (4.10), $X_{M \cap \mathcal{H}(\theta)} \in Z \cap \mathcal{H}(\kappa) = Y$. Since $\omega_1 \subseteq Y$, it follows that

(4.13) $X_{M \cap \mathcal{H}(\theta)} \subseteq Y$.

By (4.11), there is countable $N^* \prec \langle \mathcal{H}(\chi), \in, \sqsubset_{\mathcal{H}(\chi)} \rangle$ such that

(4.14) $N^* \cap Z \in \tilde{S} \cap Z$ and

(4.15) $X, Y, Z, \cdots \in N^*$.

Let $M^* = (N^* \cap Z) \cap \mathcal{H}(\theta)$. Then we have $M^* \in S$ by (4.14). $M^* \subseteq N^* \cap \mathcal{H}(\theta)$ by the definition of M^* and $X_{M^*} \subseteq Y$ by (4.13). $Y \in N^* \cap \mathcal{H}(\theta)$ by (4.12) and (4.15). We also have

(4.16) $M^* \cap Y = ((N^* \cap Z) \cap \mathcal{H}(\theta)) \cap (Z \cap \mathcal{H}(\kappa)) = M^* \cap \mathcal{H}(\kappa) = (N^* \cap \mathcal{H}(\theta)) \cap Y$.

Thus $N^* \cap \mathcal{H}(\theta)$ and Y contradict to the choice of X_{M^*}. □ (Theorem 4.3)

Corollary 4.4 *The following implications hold:*

$$\mathsf{RP}_{\mathsf{IC}} \Rightarrow (\mathsf{G}^{\downarrow\downarrow}) \Rightarrow \mathsf{RP}_{\mathsf{IS}}. \quad \square$$

Proof. By Theorem 4.3, (d). The implication "$\mathsf{RP}_{\mathsf{IC}} \Rightarrow (\mathsf{G}^{\downarrow\downarrow})$" follows from the following trivial observation. □ (Corollary 4.4)

Lemma 4.5 *If $M \prec \mathcal{H}(\theta)$ is* IC *and $S \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$, then $S \cap (M \cap [M]^{\aleph_0})$ is stationary in $[M]^{\aleph_0}$ as well.* □

Corollary 4.6 *Under the* CH, *we have:*

$$\text{Axiom R} \Leftrightarrow \mathsf{RP}_{\mathsf{IU}} \Leftrightarrow \mathsf{RP}_{\mathsf{IS}} \Leftrightarrow (\mathsf{G}^{\downarrow\downarrow}) \Leftrightarrow \mathsf{RP}_{\mathsf{IC}} \Leftrightarrow \mathsf{RP}_{\mathsf{IA}}.$$

Proof. By Lemma 2.2, Lemma 2.6 and Corollary 4.4. □ (Corollary 4.6)

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