A reflection principle formulated in terms of games

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Abstract

We introduce a principle formulated in terms of the existence of a winning strategy of a game and prove that this principle is placed between the reflection principle down to internally stationary sets (RP$_S$) and the reflection principle down to internally club sets (RP$_C$). In particular, under CH this principle gives a new characterization of Fleissner’s Axiom R.

1 Introduction

For a game $\mathcal{G}$ played by Players $I$ and $II$, let $WS_{II}(\mathcal{G})$ denote the assertion “Player $II$ has a winning strategy in $\mathcal{G}$.”
In [9], we introduced a game $G_{\omega}^{{\downarrow}}(\kappa)$ for uncountable cardinals $\kappa$ (see Section 3 for the definition of this and other games mentioned here) and proved that the Rado Conjecture (RC, see Section 2 for the definition of this principle) implies the assertion

\[(G_0)\] $\text{WS}_\Pi(G_{\omega}^{{\downarrow}}(\kappa))$ holds for all uncountable $\kappa$.

Further, it is proved in [9] that $(G_0)$ implies the Fodor-type Reflection Principle (FRP, see Section 2 for the definition of this principle and, [4] and [5] for basic facts of this principle).

In [1], Philipp Doebler introduced a similar game he called $G_\omega([\kappa]^\omega_1,\omega_1)$ and proved that the Rado Conjecture also implies the principle

\[(G_1)\] $\text{WS}_\Pi(G_\omega([\kappa]^\aleph_1,\omega_1))$ holds for all $\kappa \geq \aleph_2$.

He then proved that the principle $(G_1)$ implies the Semistationary Reflection (SSR).

In this paper, we introduce a game $G_{\omega}^{{\downarrow\downarrow}}([\kappa]^\aleph_1)$ which generalizes both $G_{\omega}^{{\downarrow}}(\kappa)$ and $G_\omega([\kappa]^\omega_1,\omega_1)$. Unfortunately the principle

\[(G_{\downarrow\downarrow}) \text{WS}_\Pi(G_{\omega}^{{\downarrow\downarrow}}([\kappa]^\aleph_1))\] for all $\kappa \geq \aleph_2$

is not a consequence of the Rado Conjecture: In Section 4, we show that the principle $(G_{\downarrow\downarrow})$ implies the reflection principle $\text{RP}_{\text{IS}}$. It is known that $\text{RP}_{\text{IS}}$ (or even $\text{RP}$) is not a consequence of $\text{RC}$ (see Sakai [14]).

## 2 Reflection Principles

Let us first review the reflection principles we mentioned in the previous section.

We shall call here a partial ordering $T = \langle T, \leq_T \rangle$ a tree if the initial segment $\{u \in T : u \leq_T t\}$ in $T$ below each $t \in T$ is well-ordered. In particular, we assume here that a tree may have multiple roots.

A tree $T$ is special if there are $T_i \subseteq T, i \in \omega$ such that each of $T_i$'s is pairwise incomparable and $T = \bigcup_{i\in\omega} T_i$.

Rado’s Conjecture (RC) is the assertion:

\[(\text{RC})\] Any tree $T$ is special if and only if all subtrees of $T$ of cardinality $\aleph_1$ are special.

RC is known to be consistent (modulo a large large cardinal). E.g., Todorcevic showed that, if $\kappa$ is strongly compact and $P = \text{Col}(\omega_1, <\kappa)$, then we have

\[\models_P \text{“Rado’s Conjecture”}\].

For a cardinal $\kappa$ and a regular cardinal $\delta < \kappa$, we denote

$$E_\delta^\kappa = \{\alpha < \kappa : cf(\alpha) = \delta\}.$$
a mapping $g : E \rightarrow \kappa$ for $E \subseteq E_\delta^\kappa$ is called a ladder system if $\sup g(\alpha) = \alpha$ and $otp(g(\alpha)) = \delta$ hold for all $\alpha \in E$.

For a regular uncountable cardinal $\kappa$, we define the Fodor-type Reflection Principle for $\kappa$ by

\textbf{FRP}($\kappa$): For all stationary $E \subseteq E_\kappa^\kappa$ and for all ladder system $g : E \rightarrow [\kappa]^\aleph_0$, there exists $\alpha^* \in E_\omega^\kappa$ such that

$$\{ x \in [\alpha^*]^\aleph_0 : sup(x) \in E, g(sup(x)) \subseteq x \}$$

is stationary in $[\alpha^*]^\aleph_0$.

The Fodor-type Reflection Principle (FRP) is the assertion:

\textbf{(FRP): FRP}($\kappa$) holds for all regular $\kappa > \aleph_1$.

FRP is known to be equivalent to many mathematical reflection principles over ZFC (see [3], [4], [5], [6], [7], see also [8]).

(2.1) Any locally countably compact topological space $X$ is metrizable if and only if all subspaces of $X$ of cardinality $\leq \aleph_1$ are metrizable

is one of such assertions equivalent to FRP over ZFC (see [4] and [5]).

FRP implies Shelah's Strong Hypothesis and hence, in particular, Singular Cardinal Hypothesis (see [7]). It also implies the total failure of square principles $\square_\kappa$ for all cardinals $\kappa \geq \aleph_1$.

Suppose that $M \prec \mathcal{H} (\lambda)$ for some regular $\lambda \geq \aleph_2$ and $| M | = \aleph_1$.

$M$ is said to be \emph{internally cofinal} (abbreviation: $\text{IC}$)\footnote{Internally cofinal $M$ is also called \emph{internally unbounded} in the literature (see e.g. Krueger [11]).} if $[M]^\aleph_0 \cap M$ is cofinal in $[M]^\aleph_0$ with respect to $\subseteq$. $M$ is \emph{internally stationary} (abbreviation: $\text{IS}$) if $[M]^\aleph_0 \cap M$ is stationary in $[M]^\aleph_0$. $M$ is \emph{internally club} (abbreviation: $\text{IC}$) if $[M]^\aleph_0 \cap M$ contains a closed unbounded set in $[M]^\aleph_0$. Finally, $M$ is \emph{internally approachable} (abbreviation: $\text{IA}$) if $M$ is the union of a continuously increasing sequence $\langle M_\alpha : \alpha < \omega_1 \rangle$ countable sets such that $\langle M_\alpha : \alpha \leq \delta \rangle \in M_{\delta+1}$ for all $\delta < \omega_1$\footnote{For a structure $M$ of cardinality $\aleph_1$, we shall call a continuously increasing sequence $\langle M_\alpha : \alpha < \omega_1 \rangle$ of countable subsets of $M$ with $\bigcup_{\alpha < \omega_1} M_\alpha = M$ a \emph{filtration} of $M$. By thinning out the index set $\omega_1$, we may assume in some cases that the filtration $\langle M_\alpha : \alpha < \omega_1 \rangle$ consists of elementary structures.}.

It is clear from the definition that, for any $M \prec \mathcal{H} (\lambda)$, we have the implication: $M \text{ is } \text{IA} \Rightarrow M \text{ is } \text{IC} \Rightarrow M \text{ is } \text{IS} \Rightarrow M \text{ is } \text{IU}$. It is easy to see that all of these notions can be characterized in terms of filtration (see footnote 2)).

\textbf{Lemma 2.1} Suppose that $M \prec \mathcal{H} (\lambda)$ for some regular $\lambda \geq \omega_2$ and $| M | = \aleph_1$.
(1) $M$ is internally cofinal if and only if there is a filtration $\langle a_\alpha : \alpha < \omega_1 \rangle$ of $M$ such that $a_{\alpha+1} \in M$ for every $\alpha < \omega_1$.

(2) $M$ is internally stationary if and only if $\{ \alpha < \omega_1 : M_\alpha \in M \}$ is stationary for any filtration $\langle M_\alpha : \alpha < \omega_1 \rangle$ of $M$.

(3) $M$ is internally club if and only if there is a filtration $\langle M_\alpha : \alpha < \omega_1 \rangle$ of $M$ such that $M_\alpha \in M_{\alpha+1}$ for all $\alpha < \omega_1$.

These notions can be different: e.g. John Krueger proved under PFA, there are stationarily many internally club but not internally approachable $M < \mathcal{H}(\lambda)$ for all regular $\lambda > \aleph_1$ (for this and other results of this line see Krueger [11] and [12]). However this is not the case under CH:

**Lemma 2.2** Under CH, any $M < \mathcal{H}(\lambda)$ is IU if and only if it is IS if and only if it is IC if and only if it is IA. □

In the following, we shall always denote one of the properties IU, IS, IC or IA with $\mathcal{P}$. "\subseteq" in connection with a cardinal, say $\lambda$, denotes a(n arbitrary) well-ordering of the set $\mathcal{H}(\lambda)$ of all sets of hereditarily of cardinality $< \lambda$. If we have to emphasize that the well-ordering $\subseteq$ refers to $\mathcal{H}(\lambda)$, we write $\subseteq_{\mathcal{H}(\lambda)}$.

For a cardinal $\lambda > \aleph_1$ let $\text{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$: For any stationary $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ there is a $\mathcal{P}$ elementary substructure $M$ of the structure $\langle \mathcal{H}(\lambda), \in, \subseteq \rangle$ (of cardinality $\aleph_1$) such that

$$\text{(2.2)} \quad S \cap [M]^{\aleph_0} \text{ is stationary in } [M]^{\aleph_0}.\]$$

We define the global version of the reflection principle $\text{RP}_{\mathcal{P}}$ down to a structure with the property $\mathcal{P}$ to be $\text{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$ for all cardinal $\lambda > \aleph_1$.

$\text{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$ is equivalent with seemingly stronger variants of the assertion:

**Lemma 2.3** the following are equivalent for a regular cardinal $\lambda > \aleph_1$:

(a) $\text{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$.

(b) For any stationary $S \subseteq [\mathcal{H}(\kappa)]^{\aleph_0}$ and any expansion $\mathcal{M}$ of the structure $\langle \mathcal{H}(\kappa), \in, \subseteq \rangle$ in an arbitrary countable language, there is a $\mathcal{P}$ elementary substructure $M$ of $\mathcal{M}$ (of cardinality $\aleph_1$) with (2.2).

(c) For any stationary $S \subseteq [\mathcal{H}(\kappa)]^{\aleph_0}$ and any expansion $\mathcal{M}$ of the structure $\langle \mathcal{H}(\kappa), \in, \subseteq \rangle$ in an arbitrary countable language, there are stationarily many $\mathcal{P}$ elementary substructures $M$ of $\mathcal{M}$ (of cardinality $\aleph_1$) with (2.2). □

Using Lemma 2.3 we can prove the following downward transfer property of $\text{RP}_{\mathcal{P}}([\mathcal{H}(\lambda)]^{\aleph_0})$:

3) That is, $S$ intersection with the set of all countable subsets of the underlying set of the structure $M$. \[\]
Lemma 2.4 For regular cardinals $\aleph_1 < \lambda' < \lambda$, if $\text{RP}_P([\mathcal{H}(\lambda)]^{\aleph_0})$ holds then $\text{RP}_P([\mathcal{H}(\lambda')]^{\aleph_0})$ also holds. \hfill \Box

Lemma 2.5 The following are equivalent: (a) $\text{RP}_P$.

(b) For any uncountable $X$, stationary $S \subseteq [X]^{\aleph_0}$, regular $\theta$ with $X \subseteq \mathcal{H}(\theta)$ and any expansion $\mathcal{M}$ of $\langle \mathcal{H}(\theta), \in, \subseteq, X \rangle$ in a countable language, there is a $\mathcal{P}$ elementary substructure $M$ of $\mathcal{M}$ of cardinality $\aleph_1$ such that $S \cap [X \cap M]^{\aleph_0}$ is stationary in $[X \cap M]^{\aleph_0}$.

(c) For any uncountable cardinal $\lambda$, stationary $S \subseteq [\lambda]^{\aleph_0}$, regular $\theta \geq \lambda$ and any expansion $\mathcal{M}$ of $\langle \mathcal{H}(\theta), \in, \subseteq, \lambda \rangle$ in a countable language, there is a $\mathcal{P}$ elementary substructure $M$ of $\mathcal{M}$ of cardinality $\aleph_1$ such that $S \cap [\lambda \cap M]^{\aleph_0}$ is stationary in $[\lambda \cap M]^{\aleph_0}$.

\hfill \Box

Fleissner’s Axiom R ([2]) is equivalent to $\text{RP}_\text{IU}$ in our notation. For a any set $X$ of cardinality $\aleph_1$, let

$(\text{AR}([X]^{\aleph_0}))$: For any stationary $S \subseteq [X]^{\aleph_0}$ and $\omega_1$-club\footnote{For an uncountable set $X$ is said to be $\omega_1$-club (or “tight and unbounded” in Fleissner’s terminology in [2]) if $T$ is cofinal in $[X]^{\aleph_1}$ with respect to $\subseteq$ and for any increasing chain $(U_\alpha : \alpha < \omega_1)$ in $T$ of length $\omega_1$, we have $\bigcup_{\alpha < \omega_1} U_\alpha \in T$.} $T \subseteq [X]^{\aleph_1}$, there is $U \in T$ such that $S \cap [U]^{\aleph_0}$ is stationary in $[U]^{\aleph_0}$.

Then we define Axiom R to be the assertion that $\text{AR}([\lambda]^{\aleph_0})$ holds for all cardinal $\alpha > \aleph_1$. Since $\text{AR}([\lambda]^{\aleph_0})$, for cardinals $\lambda > \aleph_1$ also satisfy the downward transfer similar to Lemma 2.4, the following Lemma implies the equivalence of $\text{RP}_\text{IU}$ and Axiom R:

Lemma 2.6 For any $\lambda > \aleph_1$, we have $\text{AR}([2^{<\lambda}]^{\aleph_0})$ if and only if $\text{RP}_\text{IU}([\mathcal{H}(\lambda)]^{\aleph_0})$. \hfill \Box

Proof. Note that $|\mathcal{H}(\lambda)| = 2^{<\lambda}$ and hence $\text{AR}([2^{<\lambda}]^{\aleph_0})$ is equivalent to $\text{AR}([\mathcal{H}(\lambda)]^{\aleph_0})$.

First, assume $\text{RP}_\text{IU}([\mathcal{H}(\lambda)]^{\aleph_0})$. Suppose that $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ is stationary and $T \subseteq [\mathcal{H}(\lambda)]^{\aleph_1}$ is $\omega_1$-club.

Let $\mathcal{M} = \langle \mathcal{H}(\lambda), \in, \subseteq, T \rangle$. By Lemma 2.5, there is $M < \mathcal{M}$ such that

(2.3) $|M| = \aleph_1$;
(2.4) $M \models \text{IU}$ and
(2.5) $S \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$.

By (2.3), (2.4) and $M < \mathcal{M}$, it is easy to see that $M$ is the union of an $\omega_1$ chain of elements of $T$. By $\omega_1$-clubness of $T$ it follows that $M \in T$. This shows that $\text{AR}([\mathcal{H}(\lambda)]^{\aleph_0})$ holds.

Assume now $\text{AR}([2^{<\lambda}]^{\aleph_0})$ and suppose that $S \subseteq [\mathcal{H}(\lambda)]^{\aleph_0}$ is stationary. Let

$$T = \{M \in [\mathcal{H}(\lambda)]^{\aleph_1} : M < \mathcal{H}(\lambda), M \models \text{IU}\}.$$
Then $T$ is $\omega_1$-club. By $\text{AR}(\mathcal{H}(\lambda))$ or by its equivalent $\text{AR}(\mathcal{H}(\lambda))^{\aleph_0}$, there is $M \in T$ such that $S \cap [M]^{\aleph_0}$ is stationary in $[M]^{\aleph_0}$. This shows that $\text{RP}_{\text{IC}}(\mathcal{H}(\lambda))^{\aleph_0}$ holds. \hfill \blacksquare (Lemma 2.6)

### 3 Definition of the games

For a cardinal $\kappa$, let

$$\kappa^+_\kappa = \{ f \in {}^\kappa \kappa : f \text{ is regressive}. \}$$

The game $G^1_\omega(\kappa)$ for Players $I$ and $II$ is defined as follows: A match in $G^1_\omega(\kappa)$ is a sequence of the form:

$$
\begin{array}{c|cccc}
I & f_0 \in \kappa^+_\kappa & f_1 \in \kappa^+_\kappa & \cdots & f_n \in \kappa^+_\kappa & \cdots \\
II & \delta_0 \in \kappa & \delta_1 \in \kappa & \cdots & \delta_n \in \kappa & \cdots \\
\end{array} \quad (n < \omega)
$$

Player $II$ wins in a match of $G^1_\omega(\kappa)$ as above if

$$\{ \alpha \in E^\kappa_{\omega_1} : f_n(\alpha) < \sup \{ \delta_i : i \in \omega \} \text{ for all } n \in \omega \}$$

is unbounded.

The game $G^1_\omega(\kappa)$ was introduced in [9]. It is used there to prove the implication of FRP from RC by showing that the assertion $(G_0)$ as in Section 1 defined in terms of this game interpolates the implication.

The following game $G_\omega([\kappa]^1_{\omega_1}, \omega_1)$ for Players $I$ and $II$ for a cardinal $\kappa$ was introduced by Doebler in [1]: A match in $G_\omega([\kappa]^1_{\omega_1}, \omega_1)$ is a sequence of the form:

$$
\begin{array}{c|cccc}
I & f_0 \in [\kappa]^1_{\omega_1} & f_1 \in [\kappa]^1_{\omega_1} & \cdots & f_n \in [\kappa]^1_{\omega_1} & \cdots \\
II & \delta_0 \in \omega_1 & \delta_1 \in \omega_1 & \cdots & \delta_n \in \omega_1 & \cdots \\
\end{array} \quad (n < \omega)
$$

$II$ wins in a match of $G_\omega([\kappa]^1_{\omega_1}, \omega_1)$ as above if

$$\{ a \in [\kappa]^1_{\omega_1} : f_n(a) < \sup \{ \delta_i : i \in \omega \} \text{ for all } n \in \omega \}$$

is cofinal in $[\kappa]^1_{\omega_1}$.

Doebler proved that the principle $(G_1)$ as defined in Section 1 in terms of this game follows also from RC and it implies SSR.

It is easy to see that both of $(G_0)$ and $(G_1)$ are consequences of $\text{RP}_{\text{IC}}$ (this also follows from Corollary 4.4). Hence we have the diagram on the right:

Since FRP and SSR imply almost all known consequences of RC$^5$, it seems to be an interesting question what is the natural principle which is still a consequence of both RC and $\text{RP}_{\text{IC}}$ while which implies both FRP and SSR.

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$^5$ Perhaps with the exception of the negation of Martin's Axiom for $\mathcal{P}$ dense sets which is a consequence of RC while $\text{RC}_{\mathcal{P}}$'s are consistent with Martin's Axiom since they all follow from $\text{MA}^+(\sigma\text{-closed})$. 
The assertion of the existence of the winning strategy for player II (the principle (G↓) introduced in Section 1) in the following game $G_ω^{\downarrow}(\lceil\kappa\rceil_{N_1})$ for all $\kappa > N_1$ seemed to be a natural candidate for such an interpolant. Unfortunately, this principle turned out to be too strong to be a consequence of RC while it is still a consequence of RPIC as we shall see in Section 4. In [9] we introduce a weakening of (G↓) which is an interpolant of RC and RPIC on one side and FRP and SSR on the other.

Here is the definition of $G_ω^{\downarrow}(\lceil\kappa\rceil_{N_1})$ for a cardinal $\kappa > N_1$. We call a function $f : \lceil\kappa\rceil_{N_1} → \kappa$ regressive if $f(a) ∈ a$ holds for all $a ∈ \lceil\kappa\rceil_{N_1}$. Similarly to the definition (3.1), let

$$(3.3) \quad \lceil\kappa\rceil_{N_1} → \kappa = \{f ∈ \lceil\kappa\rceil_{N_1} : f \text{ is regressive}\}.$$ 

A match in $G_ω^{\downarrow}(\lceil\kappa\rceil_{N_1})$ for Players I and II is a sequence of the form:

$$\begin{array}{c|cccc} I & f_0 ∈ \lceil\kappa\rceil_{N_1} & f_1 ∈ \lceil\kappa\rceil_{N_1} & \cdots & f_n ∈ \lceil\kappa\rceil_{N_1} & \cdots \quad (n < \omega) \\ \hline II & d_0 ∈ [\kappa]^{\omega_0} & d_1 ∈ [\kappa]^{\omega_0} & \cdots & d_n ∈ [\kappa]^{\omega_0} & \cdots \end{array}$$

II wins in a match in $G_ω^{\downarrow}(\lceil\kappa\rceil_{N_1})$ as above if

$$\{a ∈ \lceil\kappa\rceil_{N_1} : f_n(a) ∈ \bigcup\{d_i : i ∈ \omega\} \text{ for all } n ∈ \omega\}$$

is cofinal in $\lceil\kappa\rceil_{N_1}$.

Note that by the definition of the games, it is clear that (G↓) implies both of (G₀) and (G₁).

### 4 Characterizations of (G↓)

The following characterization of (G↓) can be obtained easily by regarding the moves of Player I in $G_ω^{\downarrow}(\lceil\kappa\rceil_{N_1})$ as an enumeration of Skolem functions with parameters in some model $M$ and the moves of Player II as the gradual capturing of $\kappa \cap M$:

**Lemma 4.1** For any cardinal $\kappa > N_1$ the following are equivalent:

(a) $WS_II(G_ω^{\downarrow}(\lceil\kappa\rceil_{N_1}))$.

(b) For sufficiently large regular $\theta$ with $\mathcal{M} = \langle \mathcal{H}(\theta), ∈, ∪ \rangle$, for any $M < \mathcal{M}$ with $|M| = N_0$ and $\kappa ∈ M$, we have: for any $a ∈ \lceil\kappa\rceil_{N_1}$, there are $b ∈ \lceil\kappa\rceil_{N_1}$ and countable $N < \mathcal{M}$ such that $a ∩ b$, $b ∈ N$, $M ⊆ N$ and $b ∩ N = b ∩ M$.

(c) For sufficiently large regular $\theta$ with $\mathcal{M} = \langle \mathcal{H}(\theta), ∈, ∪ \rangle$, for club many$^6$ countable $M < \mathcal{M}$ with $\kappa ∈ M$, we have: for any $a ∈ \lceil\kappa\rceil_{N_1}$, there are $b ∈ \lceil\kappa\rceil_{N_1}$ and countable $N < \mathcal{M}$ such that $a ∩ b$, $b ∈ N$, $M ⊆ N$ and $b ∩ N = b ∩ M$.

$^6$We can also express this “club many ...” in terms of expansion of the structure $\mathcal{M}$ similarly to Lemma 2.3 or Lemma 2.4.
By Lemma 4.1, (c), we see immediately that $\text{WS}_\Pi(G^\Pi_{\kappa}(\kappa)^{\aleph_1}))$ for cardinals $\kappa > \aleph_1$ also enjoy the downward transfer property:

Corollary 4.2 Suppose $\aleph_1 < \kappa' < \kappa$ and $\text{WS}_\Pi(G^\Pi_{\kappa}(\kappa)^{\aleph_1}))$ holds. Then we also have $\text{WS}_\Pi(G^\Pi_{\kappa'}([\kappa']^{\aleph_1}))$.

Theorem 4.3 The following are equivalent: (a) $(G^{11})$.

(b) For all $\kappa > \aleph_1$, for all sufficiently large regular $\theta$ with $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \square \rangle$, there are club many countable $M \prec \mathcal{M}$ such that $\kappa \in M$ and for any $X \in [\mathcal{H}(\kappa)]^{\aleph_1}$, there are $Y \in [\mathcal{H}(\kappa)]^{\aleph_1}$ and countable $N \prec \mathcal{M}$ such that $X \subseteq Y$, $Y \in N$, $M \subseteq N$ and $Y \cap N = Y \cap M$.

(c) For all $\kappa > \aleph_1$, for all sufficiently large regular $\theta$ with $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \square \rangle$, there are club many countable $M \prec \mathcal{M}$ such that $\kappa \in M$ and for any $X \in [\mathcal{H}(\kappa)]^{\aleph_1}$, there are $Z \prec \langle \mathcal{H}(\kappa), \in, \subseteq \mathcal{H}(\kappa) \rangle$ of cardinality $\aleph_1$ and countable $N \prec \mathcal{M}$ such that $X \subseteq Z$, $Z \in N$, $M \subseteq N$ and $Z \cap N = Z \cap M$.

(d) For any $\kappa > \aleph_1$ and stationary $S \subseteq [\mathcal{H}(\kappa)]^{\aleph_0}$, for any $X \in [\mathcal{H}(\kappa)]^{\aleph_1}$ there is a $Z \prec \mathcal{H}(\kappa)$ such that $X \subseteq Z \upharpoonright |Z| = \aleph_1$ and $S \cap Z$ is stationary in $[Z]^{\aleph_0}$.

(e) For all $\kappa > \aleph_1$, for all sufficiently large regular $\theta$ with $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \square \rangle$, there are club many countable $M \prec \mathcal{M}$ such that $\kappa \in M$ and for any $X \in [\mathcal{H}(\kappa)]^{\aleph_1}$, there are $S \subseteq Z \prec \langle \mathcal{H}(\kappa), \in, \subseteq \mathcal{H}(\kappa) \rangle$ of cardinality $\aleph_1$ and countable $N \prec \mathcal{H}(\theta)$ such that $X \subseteq Z$, $Z \in N$, $M \subseteq N$ and $Z \cap N = Z \cap M$.

Proof. (a) $\Rightarrow$ (b): Let $\lambda = 2^{\kappa} = |\mathcal{H}(\kappa)|$ and let $\varphi: \lambda \rightarrow \mathcal{H}(\kappa)$ be a bijection. Then all countable $M \prec \mathcal{M}$ with $\varphi \in M$ satisfies the condition in (b): the situation of Lemma 4.1, (b) (for $\kappa$ there $= \lambda$) is translated to the desired condition in the present (b) by $\varphi$.

(b) $\Rightarrow$ (a): The back-translation by the mapping $\varphi$ as in the proof of (a) $\Rightarrow$ (b) implies $\text{WS}_\Pi(G_{\kappa}^{\Pi}(2^{\kappa})^{\aleph_1}))$ for all $\kappa > \aleph_1$. By Corollary 4.2, it follows that $\text{WS}_\Pi(G_{\kappa}^{\Pi}([\kappa]^{\aleph_1}))$ for all $\kappa > \aleph_1$.

(b) $\Rightarrow$ (c): Suppose that $\kappa$, $\theta$, $\mathcal{M}$, $M$, $X$, $Y$, $N$ are as in (b). Then $Z = s_{\mathcal{M}}(Y)$ witnesses (c).

(c) $\Rightarrow$ (d): Assume that (c) holds and suppose that $S \subseteq [\mathcal{H}(\kappa)]^{\aleph_0}$ is stationary. Let $\theta \mathcal{M}$, $M$ be as in (c). Since there are club many $M$’s as in (c), we may assume that

\begin{equation}
S \in M \text{ and } \mathcal{H}(\kappa) \cap M \in S.
\end{equation}

Let $X \in [\mathcal{H}(\kappa)]^{\aleph_1}$ be defined by

\begin{equation}
X = \omega_1 \cup (\mathcal{H}(\kappa) \cap M) \cup \{M \cap \mathcal{H}(\kappa)\}.
\end{equation}

Let $Z \prec \mathcal{H}(\kappa)$ and $N \prec \mathcal{H}(\theta)$ be as in (c) for this $X$. Thus we have $N$ is countable, $Z$ is of cardinality $\aleph_1$, $X \subseteq Z$, $Z \in N$, $M \subseteq N$ and
We are done by showing that $S \cap Z$ is stationary in $[Z]^{\aleph_0}$. Since $S, Z \in N$, it is enough to show that any club $C \subseteq [Z]^{\aleph_0}$ with $C \in N$ intersects with $S$: Note that we have

\begin{align*}
Z \cap N &= Z \cap M. \\

\end{align*}

Thus $S \cap C \neq \emptyset$ as desired.

(c) $\Rightarrow$ (e): The proof of (c) $\Rightarrow$ (d) above for $S = [\mathcal{H}(\kappa)]^{\aleph_0}$ shows this.

(e) $\Rightarrow$ (c): trivial.

(c) $\Rightarrow$ (b): trivial.

(d) $\Rightarrow$ (e): Assume that (d) holds. For $\kappa > \aleph_1$, let $\theta$ a sufficiently large regular cardinal and $\mathcal{M} = \langle \mathcal{H}(\theta), \in, \subset \rangle$. Let

\begin{align*}
S &= \{ M \in [\mathcal{M}]^{\aleph_0} : M \prec \mathcal{M}, \kappa \in M, \text{ there is } X_M \in [\mathcal{H}(\kappa)]^{\aleph_1} \text{ such that } \\
(4.6) &\text{ there are no countable } N \prec \mathcal{M} \\
&\text{ and } Y \prec \langle \mathcal{H}(\kappa), \in, \subset \mathcal{H}(\kappa) \rangle \text{ such that } \\
& M \prec N, X_M \subseteq Y, Y \text{ is IS and of size } \aleph_1, \\
& Y \in N \text{ and } M \cap Y = N \cap Y \}. \\
\end{align*}

It is enough to show that $S$ is non-stationary. In the following we show this indirectly: We assume that $S$ is stationary and drive a contradiction from this assumption.

For each $M \in S$ we choose $X_M \in [\mathcal{H}]^{\aleph_1}$ such that

\begin{align*}
X_M &\supseteq M \cup \omega_1, X_M \prec \mathcal{M} \text{ and (4.6) holds for } M \text{ and } X_M. \\
\end{align*}

Let $\chi > 2^{<\theta}$ be regular. Note that we have $\mathcal{H}(\theta) \in \mathcal{H}(\chi)$. Let

\begin{align*}
\tilde{S} &= \{ M \in [\mathcal{H}(\chi)]^{\aleph_0} : M \prec \langle \mathcal{H}(\chi), \in, \subset \mathcal{H}(\chi) \rangle, \\
&\kappa, \theta, \cdots \in M, M \cap \mathcal{H}(\theta) \in S \}. \\
\end{align*}

By the assumption of the stationarity of $S$, $\tilde{S}$ is also stationary. Thus, by (d), there is $Z \prec \langle \mathcal{H}(\chi), \in, \subset \rangle$ such that

\begin{align*}
(4.9) &\quad |Z| = \aleph_1, \omega_1 \subseteq Z, \\
(4.10) &\quad \kappa, \theta, S, \langle X_M : M \in S \rangle, \subset \mathcal{H}(\kappa), \subset \mathcal{H}(\theta), \cdots \in Z \text{ and } \\
\end{align*}
(4.11) \( \tilde{S} \cap Z \) is stationary in \([Z]^{\aleph_0} \).

Let

(4.12) \( Y = Z \cap \mathcal{H}(\kappa) \).

Then we have \( \omega_1 \subseteq Y \) and hence \( |Y| = \aleph_1 \) and \( Y \prec \langle \mathcal{H}(\kappa), \in, \sqsubset \rangle \).

By (4.11), \( Z \cap [Z]^{\aleph_0} \) is stationary in \([Z]^{\aleph_0} \). Hence there is an \( x \in \tilde{C} \cap Z \). By definition of \( \tilde{C} \), we have \( \omega_1 \subseteq Y \).

By (4.11), there is countable \( N^* \prec \langle \mathcal{H}(\chi), \in, \sqsubset \rangle \) such that

(4.13) \( X_{M \cap \mathcal{H}(\theta)} \subseteq Y \).

By (4.11), there is countable \( N^* \prec \langle \mathcal{H}(\chi), \in, \sqsubset \rangle \) such that

(4.14) \( N^* \cap Z \in \tilde{S} \cap Z \) and

(4.15) \( X, Y, Z, \cdots \in N^* \).

Let \( M^* = (N^* \cap Z) \cap \mathcal{H}(\theta) \). Then we have \( M^* \in S \) by (4.14). \( M^* \subseteq N^* \cap \mathcal{H}(\theta) \) by the definition of \( M^* \) and \( X_{M^*} \subseteq Y \) by (4.13). \( Y \in N^* \cap \mathcal{H}(\theta) \) by (4.12) and (4.15). We also have

(4.16) \( M^* \cap Y = ((N^* \cap Z) \cap \mathcal{H}(\theta)) \cap (Z \cap \mathcal{H}(\kappa)) = M^* \cap \mathcal{H}(\kappa) = (N^* \cap \mathcal{H}(\theta)) \cap Y \).

Thus \( N^* \cap \mathcal{H}(\theta) \) and \( Y \) contradict the choice of \( X_{M^*} \).

**Corollary 4.4** The following implications hold:

\[
\text{RP}_{IC} \Rightarrow (G^{\downarrow \downarrow}) \Rightarrow \text{RP}_{IS}.
\]

**Proof.** By Theorem 4.3, (d). The implication “\( \text{RP}_{IC} \Rightarrow (G^{\downarrow \downarrow}) \)” follows from the following trivial observation.

**Lemma 4.5** If \( M \prec \mathcal{H}(\theta) \) is IC and \( S \cap [M]^{\aleph_0} \) is stationary in \([M]^{\aleph_0} \), then \( S \cap (M \cap [M]^{\aleph_0}) \) is stationary in \([M]^{\aleph_0} \) as well.

**Corollary 4.6** Under the CH, we have:

\[
\text{Axiom R} \iff \text{RP}_{IU} \iff \text{RP}_{IS} \iff (G^{\downarrow \downarrow}) \iff \text{RP}_{IC} \iff \text{RP}_{IA}.
\]

**Proof.** By Lemma 2.2, Lemma 2.6 and Corollary 4.4.
References


