THE APPROXIMATION PROPERTY AND THE CHAIN CONDITION

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1. THE APPROXIMATION PROPERTY

Definition 1.1. Let $\mathbb{P}$ be a poset and $\kappa$ a cardinal. We say that the poset $\mathbb{P}$ has the $\kappa$-approximation property if for every ordinal $\tau$ and every $f \in (\tau^2)^V$, if $f|\tau \in V$ for every $x \in (|\tau|^\kappa)^V$, then $f \in V$.

It is known that for an uncountable $\kappa$, if $\mathbb{P}$ is an atomless poset of size $< \kappa$ and $\dot{Q}$ is a $\mathbb{P}$-name for a $\kappa$-closed poset, then $\mathbb{P} * \dot{Q}$ has the $\kappa$-approximation property (e.g., see Mitchell [1]). In this note, we show that the size assumption for a poset $\mathbb{P}$ can be relaxed to the chain condition assumption.

Definition 1.2. Let $\kappa$ be a regular uncountable cardinal. A poset $\mathbb{P}$ satisfies the strong $\kappa$-chain condition (strong $\kappa$-c.c., for short) if $\mathbb{P}$ satisfies the $\kappa$-c.c. and for every $\kappa$-Suslin tree $T$, $\mathbb{P}$ does not add a cofinal branch of $T$.

Note 1.3. (1) If there is no $\kappa$-Suslin tree, then the $\kappa$-c.c. is equivalent to the strong $\kappa$-c.c.

(2) For a poset $\mathbb{P}$, if $\mathbb{P} \times \mathbb{P}$ satisfies the $\kappa$-c.c., then $\mathbb{P}$ satisfies the strong $\kappa$-c.c.

Lemma 1.4. If a poset $\mathbb{P}$ satisfies the $\mu$-c.c. for some $\mu < \kappa$, then $\mathbb{P}$ satisfies the strong $\kappa$-c.c. In particular, every poset of size $< \kappa$ satisfies the strong $\kappa$-c.c.

Proof. Suppose to the contrary that there is a $\kappa$-Suslin tree $T$ such that $\models_\mathbb{P} "T$ has a cofinal branch $\dot{B}"$. Let $T' = \{ t \in T : p \models_\mathbb{P} "t \in \dot{B}" \text{ for some } p \in \mathbb{P} \}$. It is easy to check that $T'$ is a downward closed subtree of $T$ of height $\kappa$. Since $\mathbb{P}$ satisfies the $\mu$-c.c. and $\mu < \kappa$, each level of $T'$ has size $< \mu$. Now, by Kurepa's theorem, $T'$ has a cofinal branch. Then this branch is a cofinal branch of $T$, this is a contradiction. \hfill \Box

The following is a main result of this note:

Lemma 1.5. Let $\kappa$ be a regular uncountable cardinal. Let $\mathbb{P}$ be an atomless poset which satisfies the strong $\kappa$-c.c. Let $\dot{Q}$ be a $\mathbb{P}$-name for a $\kappa$-closed poset (trivial poset is possible). Then $\mathbb{P} * \dot{Q}$ has the $\kappa$-approximation property.
Proof. Let $\tilde{Q}$ be a term poset of $\hat{Q}$, that is, $\hat{Q}$ is the set of all $P$-names $\dot{q}$ with $\Vdash_{P} "\dot{q} \in \hat{Q}"$. For $\dot{q}_0, \dot{q}_1 \in \hat{Q}$, define $\dot{q}_0 \leq \dot{q}_1$ if $\Vdash_{P} "\dot{q}_0 \leq \dot{q}_1 \in \hat{Q}"$. Since $\hat{Q}$ is a name for a $\kappa$-closed poset, $\hat{Q}$ is $\kappa$-closed.

Let $\dot{x}$ be a $P \star \hat{Q}$-name such that $\Vdash "\dot{x} \in V"$. We say that a condition $\langle p, \dot{q} \rangle \in P \star \hat{Q}$ decides $\dot{x}$ if there is $y$ with $\langle p, \dot{q} \rangle \Vdash "\dot{x} = y"$.

Claim 1.6. Let $\tau$ be an ordinal and $\dot{f}$ be a $P \star \hat{Q}$-name such that $\Vdash "\dot{f} : \tau \rightarrow 2$ and $\dot{f}|x \in V$ for every $x \in ([\tau]^{<\kappa})^{V}"$. Let $\langle p, \dot{q} \rangle \in P \star \hat{Q}$ and $x \in [\tau]^{<\kappa}$. Then there are $\dot{q}^* \leq \dot{q}$ and $F \subseteq 2$ such that:

1. $|F| < \kappa$.
2. For every $g \in F$, there is $p' \leq p$ such that $\langle p', \dot{q}^* \rangle \Vdash "\dot{f}|x = g"$.
3. For every $p' \leq p$, there is $p'' \leq p'$ and $g \in F$ such that $\langle p'', \dot{q}^* \rangle \Vdash "\dot{f}|x = g"$.

Proof. It is easy to check that the set $\{p' \leq p : \exists \dot{q}' \langle p', \dot{q}' \rangle \leq \langle p, \dot{q} \rangle \text{ and } \langle p', \dot{q}' \rangle \text{ decides } \dot{f}|x \rangle \}$ is predense below $p$. Take a maximal antichain $A$ which is contained in this set. Since $P$ satisfies the $\kappa$-c.c., we know that $|A| < \kappa$. Then for each $r \in A$, there are $\dot{q}_r$ and $g_r$ such that $\langle r, \dot{q}_r \rangle \leq \langle p, \dot{q} \rangle$ and $\langle r, \dot{q}_r \rangle \Vdash "\dot{f}|x = g_r"$. Let $F = \{g_r : r \in A\}$ and one can take $\dot{q}^*$ such that $\dot{q}^* \leq \dot{q}$ and $r \Vdash "\dot{q}_r = \dot{q}^*"$ for every $r \in A$. Then $\dot{q}^*$ and $F$ work. $\Box$[Claim]

In order to show that $P \star \hat{Q}$ has the $\kappa$-approximation property, take $\langle p, \dot{q} \rangle \in P \star \hat{Q}$, an ordinal $\tau$, and a name $\dot{f}$ such that $\langle p, \dot{q} \rangle \Vdash "\dot{f} : \tau \rightarrow 2$ and $\dot{f}|x \in V$ for every $x \in ([\tau]^{<\kappa})^{V}"$. Suppose to the contrary that $\langle p, \dot{q} \rangle \Vdash "\dot{f} \notin V"$.

By induction on $\alpha < \kappa$, we would find $x_\alpha, \dot{q}_\alpha, F_\alpha (\alpha < \kappa)$ such that:

1. $x_\alpha \in [\tau]^{<\kappa}$ and $\langle x_\alpha : \alpha < \kappa \rangle$ is $\subseteq$-increasing.
2. $\langle \dot{q}_\alpha : \alpha < \kappa \rangle$ is decreasing in $\hat{Q}$ and $\dot{q}_0 \leq \dot{q}$.
3. $F_\alpha \subseteq 2$ and $|F_\alpha| < \kappa$.
4. For every $g \in F_\alpha$, there is $p' \leq p$ such that $\langle p', \dot{q}_\alpha \rangle \Vdash "\dot{f}|x_\alpha = g"$.
5. For every $p' \leq p$ there are $p'' \leq p'$ and $g \in F_\alpha$ such that $\langle p'', \dot{q}_\alpha \rangle \Vdash "\dot{f}|x_\alpha = g"$, i.e., the set $\{p' \leq p : \langle p', \dot{q}_\alpha \rangle \Vdash "\dot{f}|x_\alpha = g" \text{ for some } g \in F_\alpha\}$ is predense below $p$.
6. For every $g \in F_\alpha$, there are $g_0, g_1 \in F_{\alpha+1}$ such that $g \subseteq g_0, g_1$ and $g_0 \neq g_1$.

When $\alpha = 0$, pick an arbitrary $x_0 \in [\tau]^{<\kappa}$. Then we can find required $\dot{q}_0 \leq \dot{q}$ and $F_0$ by Claim 1.6.

Let $\alpha > 0$ and suppose $x_\beta, \dot{q}_\beta, F_\beta$ are defined for all $\beta < \alpha$.

Case 1: $\alpha$ is limit. We can find $x_\alpha \in [\tau]^{<\kappa}$ such that $x_\beta \subseteq x_\alpha$ for $\beta < \alpha$. Since $\hat{Q}$ is $\kappa$-closed, we can find $\dot{q}^* \leq \dot{q}_\beta$ for every $\beta < \alpha$. Then take $\dot{q}_\alpha \leq \dot{q}^*$ and $F_\alpha$ by Claim 1.6.

Case 2: $\alpha$ is successor, say $\alpha = \beta + 1$. Pick a maximal antichain $A \subseteq P$ below $p$ such that for every $p' \in A$ there is $g \in F_\beta$ such that $\langle p', \dot{q}_\beta \rangle \Vdash "\dot{f}|x_\beta = g"$. Note
that $|A| < \kappa$, and, for every $g \in F_\beta$, there is $p' \in A$ with $\langle p', \dot{q}_\beta \rangle \models \langle g \rangle^*_{x_\beta} = g$. Since $|A| < \kappa$ and $\langle p, \dot{q}_\beta \rangle \models \langle f \notin V \rangle$, we can find $x_\alpha \in [\tau]^{<\kappa}$ such that $x_\beta \subseteq x_\alpha$ for $\beta < \alpha$, but $\langle p', \dot{q}_\beta \rangle$ does not decide $\langle f \rangle^*_{x_\alpha}$ for every $p' \in A$.

Claim 1.7. For each $p' \in A$, there are $p'_0, p'_1 \leq p'$, $g'_0, g'_1 : x_\alpha \rightarrow 2$, and \( \dot{r} \leq \dot{q}_\beta \) such that $g'_0 \neq g'_1$ and $\langle p'_i, \dot{r} \rangle \models \langle f \rangle^*_{x_\alpha} = g_i$.

Proof. Since $\langle p', \dot{q}_\beta \rangle$ does not decide $\langle f \rangle^*_{x_\alpha}$, we can take $(p'_0, \dot{q}_0), (p'_1, \dot{q}_1) \leq (p', \dot{q}_\beta)$, and $g'_0, g'_1 : x_\alpha \rightarrow 2$ such that $g'_0 \neq g'_1$ and $\langle p'_i, \dot{q}_i \rangle \models \langle f \rangle^*_{x_\alpha} = g_i$'. We may assume that $p'_0$ is incompatible with $p'_1$; if $p'_0$ and $p'_1$ have a common extension $p_2$, take $p'_0, p'_1 \leq p_2$ such that $p'_0 \perp p'_1$ and replace $p'_1$ by $p'_0$.

Now take $\dot{r} \leq \dot{q}_\beta$ such that $p'_i \models \langle \dot{r} = \dot{q}_i \rangle$. Clearly $p'_i, g'_i$ and $\dot{r}$ work. $\square$[Claim]

For each $p' \in A$, pick $\dot{q}_\alpha \leq \dot{q}_\beta$ such that there are $p'_0, p'_1 \leq p'$, $g'_0, g'_1 : x_\alpha \rightarrow 2$ with $g'_0 \neq g'_1$ and $\langle p'_i, \dot{r}_p' \rangle \models \langle f \rangle^*_{x_\alpha} = g_i$'.

Then pick $q^* \leq q_\beta$ such that $p' \models \langle q^* = \dot{r}_p \rangle$ for every $p' \in A$. Finally, take $\dot{q}_\alpha \leq \dot{q}^*$ and $F_\alpha \subseteq F_{\alpha 2}$ as in Claim 1.6. The following claim shows that $x_\alpha, \dot{q}_\alpha$, and $F_{\alpha 2}$ work well:

Claim 1.8. For each $g \in F_\beta$, there are $g_0, g_1 \in F_{\alpha 2}$ such that $g_0 \neq g_1$ and $g \subseteq g_0, g_1$.

Proof. Take $p' \in A$ so that $\langle p', \dot{q}_\beta \rangle \models \langle f \rangle^*_{x_\beta} = g$. Then we can take $p'_0, p'_1 \leq p'$ and $g'_0, g'_1 : x_\alpha \rightarrow 2$ such that $g'_0 \neq g'_1$ and $\langle p'_i, \dot{q}^*_i \rangle \models \langle f \rangle^*_{x_\alpha} = g_i$. Clearly $g \subseteq g_0, g_1$. By the choice of $F_{\alpha 2}$ and $\dot{q}_\alpha$, for each $i < 2$, one can take $p_i \leq p'_i$ and $g_i \in F_{\alpha 2}$ such that $\langle p_i, \dot{q}_\alpha \rangle \models \langle f \rangle^*_{x_\alpha} = g_i$. Since $\dot{q}_\alpha \leq \dot{q}^*$, each $\langle p_i, \dot{q}_\alpha \rangle$ is compatible with $\langle p'_i, \dot{q}^*_i \rangle$. This means that $g'_i = g_i$, so $g_0 \neq g_1$ and $g \subseteq g_0, g_1$. $\square$[Claim]

Suppose $\dot{q}_\alpha, x_\alpha, F_{\alpha 2}$ are defined for $\alpha < \kappa$. Note that, for every $\alpha < \beta < \kappa$ and $g \in F_\beta$, we have $g|x_\alpha \in F_{\alpha 2}$; take $p' \leq p$ such that $\langle p', \dot{q}_\beta \rangle \models \langle f \rangle^*_{x_\beta} = g$. Then one can pick $p'' \leq p'$ and $h \in F_{\alpha 2}$ such that $\langle p'', \dot{q}_\alpha \rangle \models \langle f \rangle^*_{x_\alpha} = h$. $\langle p', \dot{q}_\beta \rangle$ is compatible with $\langle p'', \dot{q}_\alpha \rangle$. So $h = g|x_\alpha$.

Let $T = \bigcup_{\alpha < \kappa} F_{\alpha 2}$. $T$ with the inclusion forms a $\kappa$-tree, and each node of $T$ has at least two immediate successors.

Claim 1.9. $T$ has no antichain of size $\kappa$.

Proof. For each $g \in T$, there are $p_g$ and $\alpha_g < \kappa$ such that $\langle p_g, \dot{q}_{\alpha_g} \rangle \models \langle f \rangle^*_{x_{\alpha_g}} = g$. For $g, g'$ in $T$, if $g$ and $g'$ are incompatible in $T$, then $p_g$ is incompatible with $p_g'$ in $P$. This means that if $T$ has an antichain of size $\kappa$, then $P$ also has an antichain of size $\kappa$. This is impossible, hence $T$ does not have an antichain of size $\kappa$. $\square$[Claim]

Hence $T$ is a $\kappa$-Suslin tree. We finish the proof by showing the following claim, which contradicts the strong $\kappa$-c.c. of $P$:
Claim 1.10. \( p \vDash_{\mathbb{P}} \text{"T has a cofinal branch".} \)

Proof. Take a \((V, \mathbb{P})\)-generic \( G \) with \( p \in G \) and work in \( V[G] \). Let \( \alpha < \kappa \). Since \( \{p' \leq p : \langle p', \dot{q}_\alpha \rangle \vDash \text{"} f|_x = g \text{"} \text{ for some } g \in F_\alpha \} \) is predense below \( p \), we can find \( p_\alpha \in G \) and \( g_\alpha \in F_\alpha \subseteq T \) such that \( \langle p_\alpha, \dot{q}_\alpha \rangle \vDash \text{"} f|_x = g_\alpha \text{"} \). Now, for \( \alpha < \beta < \kappa \), \( p_\alpha \) is compatible with \( p_\beta \) and \( \dot{q}_\beta \leq \dot{q}_\alpha \). So \( \langle p_\alpha, \dot{q}_\alpha \rangle \) is compatible with \( \langle p_\beta, \dot{q}_\beta \rangle \). This means that \( g_\alpha \subseteq g_\beta \), so \( \{g_\alpha : \alpha < \kappa \} \) is a cofinal branch of \( T \). \( \square \)[Claim]

Note 1.11. If \( \mathbb{P} \) satisfies the \( \kappa \)-c.c. but does not the strong \( \kappa \)-c.c., then \( \mathbb{P} \) cannot have the \( \kappa \)-approximation property.

2. Applications

We consider some applications of Lemma 1.5.

Definition 2.1. Let \( \kappa \) be a regular uncountable cardinal and \( \lambda \geq \kappa \) a cardinal. A set \( X \subseteq \mathcal{P}_{\kappa} \lambda \) has the strong tree property if for every \( \langle d_x : x \in X \rangle \) with \( d_x \subseteq x \), if \( |\{d_x \cap a : x \in X\}| < \kappa \) for every \( a \in \mathcal{P}_{\kappa} \lambda \), then there is \( D \subseteq \lambda \) such that for every \( a \in \mathcal{P}_{\kappa} \lambda \) the set \( \{x \in X : d_x \cap a = D \cap a\} \) is unbounded in \( \mathcal{P}_{\kappa} \lambda \).

Fact 2.2 (Viale-Weiss [3]). (1) The following are equivalent:

- (a) \( \mathcal{P}_{\kappa} \lambda \) has the strong tree property.
- (b) There is some unbounded set \( X \subseteq \mathcal{P}_{\kappa} \lambda \) such that \( X \) has the strong tree property.
- (c) Every unbounded subset of \( \mathcal{P}_{\kappa} \lambda \) has the strong tree property.

(2) \( \kappa \) has the tree property if and only if \( \mathcal{P}_{\kappa} \kappa \) has the strong tree property.

(3) \( \kappa \) is strongly compact if and only if \( \kappa \) is inaccessible and \( \mathcal{P}_{\kappa} \lambda \) has the strong tree property for every \( \lambda \geq \kappa \).

(4) Suppose Proper Forcing Axiom. Then \( \mathcal{P}_{\omega_2} \lambda \) has the strong tree property for every \( \lambda \geq \omega_2 \).

Viale-Weiss [3] showed that for an inaccessible \( \kappa \), if a standard \( \kappa \)-stage iteration satisfying the \( \kappa \)-c.c. forces that \( \text{"} \kappa = \omega_2 \text{ and Proper forcing axiom"} \), then \( \kappa \) must be strongly compact in the ground model. The following is a slight improvement of their result.

Proposition 2.3. Let \( \kappa \) be a regular uncountable cardinal. Suppose that there is a poset \( \mathbb{P} \) which has the strong \( \kappa \)-c.c. and forces that \( \text{"} \mathcal{P}_{\kappa} \lambda \) has the strong tree property for every \( \lambda \geq \kappa \text{"} \). Then \( \mathcal{P}_{\kappa} \lambda \) has the strong tree property for every \( \lambda \geq \kappa \) in the ground model.
Proof. We check that $\mathcal{P}_{\kappa}\lambda$ has the strong tree property for every $\lambda \geq \kappa$. Fix $\lambda \geq \kappa$ and take $\langle d_{x} : x \in \mathcal{P}_{\kappa}\lambda \rangle$ such that $d_{x} \subseteq x$ and $|\{d_{x} \cap a : x \in \mathcal{P}_{\kappa}\lambda\}| < \kappa$ for every $a \in \mathcal{P}_{\kappa}\lambda$. Take a $(V, \mathbb{P})$-generic $G$ and work in $V[G]$. In $V[G]$, $\mathcal{P}_{\kappa}^{V}\lambda$ is unbounded in $\mathcal{P}_{\kappa}\lambda$ since $\mathbb{P}$ satisfies the $\kappa$-c.c. By the strong tree property of $\mathcal{P}_{\kappa}^{V}\lambda$ in $V[G]$, we can find $D \subseteq \lambda$ such that $\{x \in \mathcal{P}_{\kappa}^{V}\lambda : d_{x} \cap a = D \cap a\}$ is unbounded in $\mathcal{P}_{\kappa}\lambda$ for every $a \in \mathcal{P}_{\kappa}\lambda$. We see $D \in V$, this completes the proof. For each $a \in \mathcal{P}_{\kappa}^{V}\lambda$, there is $x \in \mathcal{P}_{\kappa}^{V}\lambda$ with $D \cap a = d_{x} \cap a \in V$. Thus, by the $\kappa$-approximation property of $\mathbb{P}$, we have $D \in V$. \hfill $\Box$

Next we look at the indestructibility of weak compactness.

Definition 2.4. Let $\kappa$ be weakly compact. If every $\kappa$-directed closed forcing preserves the weak compactness of $\kappa$, then $\kappa$ is said to be indestructibly weakly compact.

The existence of an indestructibly weakly compact cardinal is consistent (Laver [2]). The following theorem suggests that the consistency of the existence of an indestructibly weakly compact cardinal might be at least strongly compact cardinal.

Proposition 2.5. Let $\kappa$ be a regular uncountable cardinal. If there is a poset which satisfies the strong $\kappa$-c.c. and forces that "$\kappa$ is indestructibly weakly compact", then $\kappa$ is strongly compact.

Proof. Take $\lambda \geq \kappa$. We see that $\mathcal{P}_{\kappa}\lambda$ has the strong tree property. Take $\langle d_{x} : x \in \mathcal{P}_{\kappa}\lambda \rangle$ with $d_{x} \subseteq x$ and $|\{d_{x} \cap a : x \in \mathcal{P}_{\kappa}\lambda\}| < \kappa$ for every $a \in \mathcal{P}_{\kappa}\lambda$.

Take a $(V, \mathbb{P})$-generic $G$, and a $(V[G], \text{Col}(\kappa, \lambda))$-generic $H$. We work in $V[G][H]$. Fix a bijection $\pi : \lambda \rightarrow \kappa$. We know that $\{\pi^{-1}x : x \in \mathcal{P}_{\kappa}^{V}\lambda\}$ is unbounded in $\mathcal{P}_{\kappa}\kappa$. Since $\kappa$ is weakly compact in $V[G][H]$, by the tree property of $\kappa$, there is $C \subseteq \kappa$ such that $\{\pi^{-1}x \in \mathcal{P}_{\kappa}\kappa : \pi^{-1}(d_{x}) \cap a = C \cap a\}$ is unbounded for all $a \in \mathcal{P}_{\kappa}\kappa$. Put $D = \pi^{-1}C$. Then for every $a \in \mathcal{P}_{\kappa}\lambda$, the set $\{x \in \mathcal{P}_{\kappa}^{V}\lambda : d_{x} \cap a = D \cap a\}$ is unbounded in $\mathcal{P}_{\kappa}\lambda$. We know $D \in V$ since $\mathbb{P} \ast \text{Col}(\kappa, \lambda)$ has the $\kappa$-approximation property by Lemma 1.5. \hfill $\Box$

References


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