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Time-dependent singularities in the heat equation

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1 Introduction

In this article, we review recent works with Yanagida [16] and Kan [8] on removable and non-removable time-dependent singularities in parabolic equations. Throughout this article, we only consider the case $N\geq 3$. The case $N=2$ is considered in [16] and [8].

This article is organized as follows. In Section 1.1, we give a necessary and sufficient condition for the removability of a time-dependent singularity in the linear heat equation. Non-removable singularities are considered in Section 1.2. We devote Section 1.3 to state results on effects of a motion of the singular point. In Section 2, we prove Theorem 1.2.

1.1 Removable singularities

For a solution of the Laplace equation, the removability of a singularity is defined as follows. Let u be a solution of

$$\Delta u=0 \quad \text{in } \Omega\setminus\{\xi_{0}\},$$

where Ω is a domain in $\mathbb{R}^{N}$ and $\xi_{0}\in\Omega$. We say that the singularity of u at the point $x=\xi_{0}$ is removable if there exists a classical solution $\tilde{u}$ of the Laplace equation in Ω such that $\tilde{u}\equiv u$ in $\Omega\setminus\{\xi_{0}\}$. It is well known that the singularity of u at ξ_{0} is removable if and only if the singularity is weaker than that of the fundamental solution of the Laplace equation, that is, the condition for the removability is

$$|u(x)|=o(|x-\xi_{0}|^{2-N}) \quad \text{as} \quad x\rightarrow\xi_{0}.$$

For nonlinear elliptic equations, the removability of a singularity has been studied in many papers and various results have been obtained (see, e.g., Brezis and Véron [1], Véron [17], P.-L. Lions [10], Gidas and Spruck [3], the monograph Véron [18] and references cited therein).

Similarly, for the heat equation

$$u_t = \Delta u \quad \text{in } \Omega \setminus \{\xi_0\} \times (0,T)$$

with $T > 0$, Hsu [6] proved that the singularity of u at $x = \xi_0$ is removable if and only if

$$|u(x,t)| = o(|x - \xi_0|^{2-N}) \quad \text{as} \quad x \to \xi_0$$

for every $t \in (0,T)$. The proof is based on precise estimates of the heat kernel. Later, Hui [7] gave a simpler proof for this result by utilizing Schauder estimates and the maximum principle. For the semilinear parabolic equation

$$u_t = \Delta u + u^p, \quad x \in \mathbb{R}^N,\ t > 0, \tag{1.1}$$

Hirata [5] extended Hsu and Hui's result by an iteration technique.

For the case where a singular point may move in time, the problem on the removability is formulated as follows. Let $\xi : [0,T] \to \mathbb{R}^N$ be a continuous curve. We take a domain $\Omega \subset \mathbb{R}^N$ such that $\xi(t) \in \Omega$ for any $t \in [0,T]$ and define

$$D_\Omega := \{(x,t) \in \mathbb{R}^{N+1} : x \in \Omega \setminus \{\xi(t)\}, t \in (0,T)\}. \tag{1.2}$$

For a solution of

$$u_t - \Delta u = 0 \quad \text{in } D_\Omega, \tag{1.3}$$

the time-dependent singularity at $x = \xi(t)$ is said to be removable if there exists $\tilde{u}$ which satisfies the heat equation in $\Omega \times (0,T)$ in the classical sense and $\tilde{u} \equiv u$ on D. Our first theorem gives a necessary and sufficient condition for the removability of the time-dependent singularity. Roughly speaking, if ξ has some Hölder continuity, then the removability is analogous to Hsu and Hui's result. More precisely, the results is the following:

Theorem 1.1 ([16]). *Suppose that ξ is $1/2$-Hölder continuous on $[0,T]$ and that u satisfies (1.3) in the classical sense. Then the singularity of u at $x = \xi(t)$ is removable if and only if for any $t_1, t_2 \in (0,T)$ with $t_1 < t_2$ and $\varepsilon \in (0,1)$ there exists $r \in (0,1)$ depending on t_1, t_2 and ε such that*

$$|u(x,t)| \le \varepsilon |x - \xi(t)|^{2-N}$$

for any $x \in \mathbb{R}^N$ with $0 < |x - \xi(t)| < r$ and for any $t \in [t_1, t_2]$.

The proof is based on a construction of a suitable cut-off function. In order to construct the desired function, we suppose that ξ has 1/2-Hölder continuity. However, 1/2 is the critical Hölder exponent in some sense. Indeed, in Section 1.3, we see that the shape of time-dependent singular solutions are distorted when the motion of ξ is quicker than or equal to 1/2-Hölder continuous.

1.2 Non-removable singularities

In what follows, we consider singular solutions whose singularity moves in time and is not removable. For the semilinear heat equation (1.1), Sato and Yanagida [11] constructed the solution with a time-dependent singularity to the Cauchy problem for $N/(N-2) < p < (N+2\sqrt{N-1})/(N-4+2\sqrt{N-1})$. The solution is singular on given any smooth curve $\xi(t)$. Moreover, they also proved that the leading term of the expansion at $x = \xi(t)$ has the same form as that of the singular steady state of this equation, that is, the solution satisfies

$$u(x,t) = L|x-\xi(t)|^{-m} + o(|x-\xi(t)|^{-m}),$$

as $x = \xi(t)$, where $m := 2/(p-1)$ and $L := \{m(N-m-2)^{1/(p-1)}\}$. Afterward, they studied various properties of time-dependent singular solutions, for instance, the time-global existence [12], convergence to singular steady states [15] and appearance of anomalous singularities [13, 14].

In this article, we turn to the linear heat equation. To begin with, we recall that the linear heat equation has the singular steady state

$$\Psi(x) := A_N|x|^{2-N} \quad \left(A_N := 4^{-1}\pi^{-\frac{N}{2}}\Gamma\left(\frac{N}{2}-1\right)\right). \tag{1.4}$$

We remark that Ψ is the fundamental solution of the Laplace equation. Indeed, $A_N = 1/N(N-2)\omega_N$, where ω_N is the volume of the unit ball in $\mathbb{R}^N$. Analogous to the semilinear heat equation, it is expected that there exists a singular solution whose singular point moves in time, and the leading term of the expansion is $\Psi(x-\xi(t))$. Indeed, in [16], such solutions were constructed by utilizing the following equation:

$$u_t - \Delta u = \delta_{\xi(t)} \quad \text{in } \mathbb{R}^N \times (0,T), \tag{1.5}$$

where $\delta_{\xi(t)}$ is the Dirac distribution concentrated at the point $\xi(t) \in \mathbb{R}^N$ for each $t \in (0,T)$. We may represent a solution of (1.5) by using the representation formula for the inhomogeneous heat equation. More precisely, we denote the heat kernel by $\Phi(x,t) = (4\pi t)^{-N/2}\exp(-|x|^2/4t)$ and define F in $\mathbb{R}^N \times (0,T)$ by

$$F(x,t) := \int_0^t \Phi(x-\xi(s), t-s)\, ds.$$

Then, F satisfies the following:

Theorem 1.2 ([16]). *Suppose that $\xi : [0,T] \to \mathbb{R}^N$ is continuous. Then F satisfies* (1.5) *in $\mathbb{R}^N \times (0,T)$ in the distributional sense and* (1.3) *in $D_{\mathbb{R}^N}$ in the classical sense, where $D_{\mathbb{R}^N}$ is given by* (1.2).

Remark 1.1. *Theorem 1.2 also holds if $N=1$ and $N=2$.*

It was also shown that the leading term of the expansion of $F(x,t)$ at $x = \xi(t)$ is $\Psi(x-\xi(t))$ if ξ has some Hölder continuity.

Theorem 1.3 ([16]). *Suppose that ξ is α-Hölder continuous on $[0,T]$ with some $\alpha > 1/2$. Then for each $t \in (0,T)$,*

$$F(x,t) = \Psi(x-\xi(t)) + o(|x-\xi(t)|^{2-N})$$

as $x \to \xi(t)$, where A_N is given by (1.4).

Remark 1.2. *Another proof of Theorems* 1.2 *and* 1.3 *were given by Karch and Zheng* [9, Section 4]. *Their method is based on the Fourier transform.*

1.3 Effects of a motion of the singular point

Let us consider the effect of the motion of the singular point. To measure instantaneous quickness of the motion of the singular point $\xi(t)$, we make the following definition. In this article, we say that ξ has an α-velocity at t if

$$\lim_{s \uparrow t} \frac{\xi(t)-\xi(s)}{(t-s)^\alpha}$$

exists. When ξ has an α-velocity at t, we call the above limit α-velocity vector and denote it by $v_\alpha(t)$. Throughout this subsection, let us consider the case where ξ is continuous on $[0,T]$, and

$$\text{non-zero vector } v_\alpha(t_0) \text{ exists for some } \alpha \in (0,1] \text{ and } t_0 \in (0,T).$$

We introduce notation before stating our results. Put $\rho_0 := |v_\alpha(t_0)|$ and $\nu_0 := v_\alpha(t_0)/|v_\alpha(t_0)|$. For $z \in \mathbb{R}^N \setminus \{0\}$, we write $r = |z|$, $\omega = z/|z|$ and denote $\theta \in [0, \pi]$ by the angle between ω and $-\nu_0$, that is, $\cos\theta = -\omega \cdot \nu_0$. With this notation, we have the decomposition $\omega = -(\cos\theta)\nu_0 + (\sin\theta)n$ for some $n \in \mathbb{R}^N$ with $|n| = 1$, $n \cdot \nu_0 = 0$.

In what follows, the case $\alpha = 1$, $\alpha \in (1/2, 1)$, $\alpha = 1/2$ and $\alpha \in (0, 1/2)$ are considered, respectively. If $\alpha = 1$, then the expansion of F at $x = \xi(t_0)$ is as follows.

Theorem 1.4 ([8]). *Suppose $\alpha = 1$. Then the following* (i) *and* (ii) *hold as $z := x - \xi(t_0) \to 0$.*

(i) *If $N = 3$, then*

$$F(x, t_0) = \Psi(z) + (4\pi)^{-\frac{3}{2}} \Bigg[\Gamma\Big(\frac{1}{2}\Big)\rho_0 \cos\theta + \int_0^{t_0} \tau^{-\frac{3}{2}} \left(e^{-\frac{1}{4}\tau^{-1}|\xi(t_0)-\xi(t_0-\tau)|^2} - 1 \right) d\tau + \frac{2}{\sqrt{t_0}} \Bigg].$$

(ii) *If $N \geq 4$, then*

$$F(x, t_0) = \Psi(z) + \frac{\rho_0 \cos\theta}{8\pi^{\frac{N}{2}}} \Gamma\Big(\frac{N}{2} - 1\Big) r^{3-N} + o(r^{3-N}).$$

Remark 1.3. *The integral in Theorem* 1.4 (i) *is finite.*

If $\alpha \in (1/2, 1)$, then the effect of the motion also appears in the second term of the expansion of F.

Theorem 1.5 ([8]). *Suppose $\alpha \in (1/2, 1)$. Then*

$$F(x, t_0) = \Psi(z) + \frac{\rho_0 \cos\theta}{2^{2\alpha+1}\pi^{\frac{N}{2}}} \Gamma\Big(\frac{N}{2} - \alpha\Big) r^{2\alpha+1-N} + o(r^{2\alpha+1-N})$$

as $z := x - \xi(t_0) \to 0$.

When $\alpha = 1/2$, the effect appears in the leading term of the expansion. The expansion in the next result implies that the shape of the solution is distorted towards the back of the singular point.

Theorem 1.6 ([8]). *If $\alpha = 1/2$, then*

$$F(x,t_0) = (4\pi)^{-\frac{N}{2}} e^{-\frac{\rho_0^2}{4}} \left(\int_0^\infty \sigma^{\frac{N}{2}-2} e^{-\frac{1}{4}(\sigma - 2\sqrt{\sigma}\rho_0 \cos\theta)} \, d\sigma \right) r^{2-N} + o(r^{2-N})$$

as $z := x - \xi(t_0) \to 0$.

In what follows, let us consider the case $\alpha < 1/2$. We remark that under this assumption, the integral

$$\int_0^{t_0} (t_0 - s)^{-\frac{N}{2}} \exp\left\{ -\frac{|\xi(t_0) - \xi(s)|^2}{4(t_0 - s)} \right\} ds$$

is finite, because the integrand is bounded in $(0, t_0)$. Therefore the value of $F(x, t_0)$ at $x = \xi(t_0)$ can be defined as a finite value. This fact suggests that there is some region $\mathcal{N}$ containing the point $\xi(t_0)$ such that $F(\cdot, t_0)$ is bounded in $\mathcal{N}$. The problems in this case are to find such a region $\mathcal{N}$ and also to specify the behavior of $F(x, t_0)$ when $x \notin \mathcal{N}$, $x \to \xi(t_0)$.

In order to state our result, we define for $\varepsilon > 0$ and $M > 0$,

$$S_\varepsilon := \left\{ z \in \mathbb{R}^N \setminus \{0\} \, ; 1 - \cos\theta \geq 2\rho_0^{-\frac{1}{\alpha}} \left(\frac{N-3}{2\alpha} + 1 \right) (1+\varepsilon) r^{\frac{1}{\alpha}-2} \log \frac{1}{r} \right\},$$

$$T_M := \left\{ z \in \mathbb{R}^N \setminus \{0\} \, ; 1 - \cos\theta \leq M r^{\frac{1}{\alpha}-2} \right\}.$$

Our main result is the following.

Theorem 1.7 ([8]). *Suppose that $\alpha \in (0, 1/2)$ and that*

$$\xi(t_0) - \xi(s) = (t_0 - s)^\alpha v_\alpha(t_0) + (t_0 - s)^{\frac{1}{2}} w_0 + o((t_0 - s)^{\frac{1}{2}}) \tag{1.6}$$

for some $w_0 \in \mathbb{R}^N$ as $s \uparrow t_0$. Then, for any $\varepsilon > 0$ and $M > 0$,

$$\lim_{\substack{x - \xi(t_0) \in S_\varepsilon \\ x \to \xi(t_0)}} F(x, t_0) = F(\xi(t_0), t_0),$$

$$\lim_{\substack{x - \xi(t_0) \in T_M \\ x \to \xi(t_0)}} \left(r^{\frac{N-3}{2\alpha}+1} e^{\frac{1}{4} J(x - \xi(t_0))} F(x, t_0) \right) = (4\pi)^{-\frac{N-1}{2}} \alpha^{-1} \rho_0^{\frac{N-3}{2\alpha}} e^{-\frac{1}{4} c_0},$$

where $J(z) := 2\rho_0^{\frac{1}{\alpha}} r^{-(\frac{1}{\alpha}-2)} (1 - \cos\theta) + 2\rho_0^{\frac{1}{2\alpha}} (n \cdot w_0) r^{-(\frac{1}{2\alpha}-1)} \sin\theta$ and $c_0 := |w_0|^2 - (\nu_0 \cdot w_0)^2$. Furthermore,

$$\liminf_{x \to \xi(t_0)} F(x, t_0) = F(\xi(t_0), t_0),$$

$$\limsup_{x \to \xi(t_0)} \left(r^{\frac{N-3}{2\alpha}+1} F(x, t_0) \right) = (4\pi)^{-\frac{N-1}{2}} \alpha^{-1} \rho_0^{\frac{N-3}{2\alpha}}.$$

Remark 1.4 ([8]). *If $N = 2$ and $\alpha \in (0, 1/2)$, then we obtain*

$$\lim_{x \to \xi(t_0)} F(x, t_0) = F(\xi(t_0), t_0)$$

without using (1.6).

Theorem 1.7 implies that the shape of the solution is more distorted than that of the case $\alpha \in [1/2, 1]$. In particular, the solution is continuous along some directions and is not continuous towards the back of the singular point. To observe this phenomenon, we give a simpler version of Theorem 1.7. In this version, we only consider the limit of F when x approaches $\xi(t)$ along the direction ω.

Corollary 1.1. *Let $\omega \in S^{N-1}$. Suppose that $\alpha \in (0, 1/2)$ and that*

$$\xi(t_0) - \xi(s) = (t_0 - s)^\alpha v_\alpha(t_0) + o((t_0 - s)^{\frac{1}{2}})$$

as $s \uparrow t_0$. Then the following (i) *and* (ii) *hold.*

(i) *If $\omega = -v_\alpha(t_0)/|v_\alpha(t_0)|$, then*

$$F(x, t_0) = (4\pi)^{-\frac{N-1}{2}} \alpha^{-1} \rho_0^{\frac{N-3}{2\alpha}} |x - \xi(t_0)|^{-\frac{N-3}{2\alpha} - 1} + o(|x - \xi(t_0)|^{-\frac{N-3}{2\alpha} - 1})$$

as $x \to \xi(t_0)$ along the direction $-v_\alpha(t_0)/|v_\alpha(t_0)|$.

(ii) *If $\omega \neq -v_\alpha(t_0)/|v_\alpha(t_0)|$, then*

$$F(x, t_0) = F(\xi(t_0), t_0) + o(1)$$

as $x \to \xi(t_0)$ along the direction ω.

2 Proof of Theorem 1.2

We give a proof of Theorem 1.2 for $N \geq 1$. The proof is similar to [16, Section4]. In this article, we say that u satisfies (1.5) in the distributional sense if u belongs to $L^1_{\rm loc}(\mathbb{R}^N \times (0, T))$ and satisfies

$$\int_0^T \int_{\mathbb{R}^N} (-\phi_t - \Delta\phi) u \, dxdt = \int_0^T \phi(\xi(t), t) \, dt \tag{2.1}$$

for any $\phi \in C_0^\infty(\mathbb{R}^N \times (0, T))$.

proof of Theorem 1.2. Since

$$\int_0^T \int_{\mathbb{R}^N} F(x,t)\,dxdt = \frac{1}{2}T^2 < \infty,$$

the function F is integrable on $\mathbb{R}^N \times (0,T)$. In particular, F belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^N \times (0,T))$. In the following, we show that F satisfies (2.1) for all $\phi \in C_0^\infty(\mathbb{R}^N \times (0,T))$. For each $n \in \mathbb{N}$, we define

$$F_n(x,t) := \int_0^{\frac{n}{n+1}t} \Phi(x-\xi(s), t-s)\,ds.$$

Then, the integrating by parts yields

$$\int_0^T \int_{\mathbb{R}^N} (-\phi_t - \Delta\phi)F_n\,dxdt = \frac{n}{n+1}I_n,$$

where

$$I_n := \int_0^T \int_{\mathbb{R}^N} \phi(x,t)\Phi\Big(x - \xi(\frac{n}{n+1}t), \frac{1}{n+1}t\Big)\,dxdt.$$

First, we prove that

$$\lim_{n\to\infty} \frac{n}{n+1}I_n = \int_0^T \phi(\xi(t),t)\,dt. \tag{2.2}$$

To prove this, we rewrite

$$I_n = \int_0^T \phi(\xi(t),t)\,dt + I_n',$$

where

$$I_n' := \int_0^T J_n(t)\,dt$$

and

$$J_n(t) := \int_{\mathbb{R}^N} \{\phi(x,t) - \phi(\xi(t),t)\}\Phi\Big(x - \xi(\frac{n}{n+1}t), \frac{1}{n+1}t\Big)\,dx.$$

By a similar calculation to [2, Section 2.3.1], $\lim_{n\to\infty} J_n(t) = 0$ for each $t \in (0,T)$. Indeed, let $t \in (0,T)$ and $\varepsilon > 0$, since ξ is continuous on (0,T), there exists $\delta > 0$ such that $|\phi(x,t) - \phi(\xi(t),t)| < \varepsilon$ for any $x \in \mathbb{R}^N$ with $|x - \xi(t)| < \delta$. Then,

$$|J_n(t)| \leq \varepsilon + C_1 \int_{\{|x-\xi(t)|\geq\delta\}} \Phi\Big(x - \xi(\frac{n}{n+1}t), \frac{1}{n+1}t\Big)\,dx$$

for some constant $C_1 > 0$. Taking $n \in \mathbb{N}$ such that $|\xi(t) - \xi(\frac{n}{n+1}t)| \leq \frac{1}{2}\delta$, we have $\frac{1}{2}|x - \xi(t)| \leq |x - \xi(\frac{n}{n+1}t)|$ when $|x - \xi(t)| \geq \delta$. Thus, by the change of variables $r = |x - \xi(t)|$ and $s = \sqrt{n+1}r/4\sqrt{t}$, we calculate that

$$\begin{aligned}
&\int_{\{|x-\xi(t)|\geq\delta\}} \Phi\Big(x - \xi(\frac{n}{n+1}t), \frac{1}{n+1}t\Big)\,dx \\
&\quad\leq C_2\Big(\frac{t}{n+1}\Big)^{-\frac{N}{2}} \int_{\{|x-\xi(t)|\geq\delta\}} \exp\Big\{-\frac{|x-\xi(t)|^2}{16(n+1)^{-1}t}\Big\}\,dx \\
&\quad\leq C_3 \int_{\frac{\delta}{4}\sqrt{\frac{n+1}{t}}}^{\infty} s^{N-1}e^{-s^2}\,ds,
\end{aligned}$$

where $C_2, C_3 > 0$ are constants independent of n. By $N \geq 1$, we obtain $\lim_{n\to\infty} J_n(t) = 0$. Moreover, for any $n \in \mathbb{N}$ and $t \in (0,T)$, the integrand of I_n' is dominated by some constant $C_4 > 0$. Hence, $\lim_{n\to\infty} I_n' = 0$ by Lebesgue's dominated convergence thorem. Thus, (2.2) holds.

Next, direct calculation shows that

$$\left|\int_0^T \int_{\mathbb{R}^N} (-\phi_t - \Delta\phi)(F_n - F)\,dxdt\right| \leq C_5 \int_0^T \int_{\frac{n}{n+1}t}^{t} dsdt = \frac{C_5}{2(n+1)}T^2$$

for some constant $C_5 > 0$. Taking $n \to \infty$, we obtain

$$\lim_{n\to\infty} \int_0^T \int_{\mathbb{R}^N} (-\phi_t - \Delta\phi)F_n\,dxdt = \int_0^T \int_{\mathbb{R}^N} (-\phi_t - \Delta\phi)F\,dxdt. \tag{2.3}$$

Since (2.2) and (2.3) hold, F satisfies (2.1). Therefore, F satisfies (1.5) in $\mathbb{R}^N \times (0,T)$ in the distributional sense. Furthermore, (2.1) particularly shows that for any $\psi \in C_0^\infty(D_{\mathbb{R}^N})$,

$$\int_0^T \int_{\mathbb{R}^N} (-\psi_t - \Delta\psi)F\,dxdt = 0.$$

By the Weyl lemma for the heat equation (see, e.g., [4, Section 6]), we conclude that F satisfies (1.3) in $D_{\mathbb{R}^N}$ in the classical sense. ∎

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