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# The least energy positive solution for the nonlinear elliptic three-systems with attractive and repulsive interaction terms

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## 0. Introduction

In this report, for the bounded domain $\Omega \subset \mathbf{R}^n$ $(n \leq 3)$ with smooth boundary, we consider the following nonlinear elliptic 3-system:

$$\begin{gathered} -\Delta u_i + \lambda_i u_i = \mu_i u_i^3 + \sum_{j=1}^{3} \beta_{i,j} u_i u_j^2 \quad \text{in } \Omega, \quad (i = 1,2,3), \\ u_i \in H_0^1(\Omega) \quad (i = 1,2,3). \end{gathered} \tag{$*$}$$

where $\lambda_i, \mu_i > 0$ $(i = 1,2,3)$ and $\beta_{i,j} = \beta_{j,i}$ $(1 \leq i < j \leq 3)$. We consider about a least energy positive solution of $(*)$ for the case $\beta_{1,2} > 0$ and $\beta_{1,3}, \beta_{2,3} \leq 0$. Here we call a solution $\vec{u} = (u_1, u_2, u_3) \in H^1(\mathbf{R}^n)^3$ is a least energy positive solution of $(*)$ if and only if $\vec{u}$ achieves $\inf\{I(\vec{u}) \,|\, I'(\vec{u}) = 0,\ u_i > 0\ (i = 1,2,3)\}$ where $I(\vec{u})$ is a functional corresponding to $(*)$. There are many papers for the existence of non-trivial solutions of $k$-system $(k \geq 3)$. (cf. **[LWe1]**, **[LW1]**, **[LW2]**, **[SW2]**, **[S]**, **[TT]**, **[TTVW]**, **[TV]**...) To author's knowledge, almost existence results for $(*)$ were given under the conditions that interaction terms $\beta_{i,j}$ are negative or not large positive.

In this report, we introduce results of our two papers **[SW3]**– **[SW4]** but we omit those proofs. For the proofs, see **[SW3]**–**[SW4]**. Roughly speaking our results, when $\beta_{1,3}, \beta_{2,3} \leq 0$ and $\beta_{1,2} > 0$ is sufficiently large, we observe the existence of least energy positive solution of $(*)$ in Section 1 and the multiple existence of positive solution of $(*)$ in Section 2.

## 1. The Existence of least energy solutions

In this section, when $\beta_{1,3}, \beta_{2,3} \le 0$ and $\beta_{1,2} > 0$ is sufficiently large, we observe the existence of least energy positive solution of $(*)$. Moreover, we observe that, even if $\Omega$ is ball, that solution is not radial symmetric. This is a different property from the single homogeneous equations. It is well-known in **[GNN]** that if $\Omega$ is ball and $f(u)$ is of class $C^1$, then any positive solutions in $C^2$ of $-\Delta u = f(u)$ in $\Omega$ $u = 0$ on $\partial\Omega$ are radial symmetric. Also, in **[LWe1]**, for a $k$-system on $\Omega = \mathbf{R}^n$, Lin and Wei showed that, if all interaction terms are positive, then, the least energy positive solutions must be radially symmetric by the Schwartz symmetrization.

Since we treat $\beta_{1,2}$ as a parameter which plays an important role, for simplicities, we often write $\beta \equiv \beta_{1,2}$. We also use the following notations:

$$||u||^p_{L^p(\Omega)} = \int_\Omega |u|^p \quad \text{for } u \in L^p(\Omega) \quad (1 \le p \le \infty),$$
$$|||u|||^2_{\lambda,\Omega} = ||\nabla u||^2_{L^2(\Omega)} + \lambda ||u||^2_{L^2(\Omega)} \quad \text{for } u \in H^1_0(\Omega).$$

The first theorem is about the existence of a least energy positive solution of $(*)$.

**Theorem 1.1.** *We suppose that $\beta \equiv \beta_{1,2} > 0$ and $\beta_{1,3}, \beta_{2,3} \le 0$. Then, there exists a $\beta_* > 0$ such that, for any $\beta > \beta_*$, $(*)$ has a least energy positive solution $\vec{u}_\beta = (u_{1,\beta}, u_{2,\beta}, u_{3,\beta})$. Moreover, there exist a sequence $\beta_m \to \infty$ and $U_i \in H^1_0(\Omega)$ $(i = 1, 2, 3)$ such that*

$$(\sqrt{\beta_m} u_{1,\beta_m}, \sqrt{\beta_m} u_{2,\beta_m}, u_{3,\beta_m}) \to (U_1, U_2, U_3) \quad \textit{strongly in } H^1_0(\Omega)^3.$$

*Here $U_3$ is a positive least energy solution of*

$$\begin{aligned} -\Delta u_3 + \lambda_3 u_3 &= \mu_3 u_3^3 \quad \textit{in } \Omega, \\ u_3 &\in H^1_0(\Omega), \end{aligned} \tag{1.1}$$

*and $(U_1, U_2)$ is a positive least energy solution of*

$$\begin{aligned} -\Delta u_1 + (\lambda_1 - \beta_{1,3} U_3^2) u_1 &= u_1 u_2^2 \quad \textit{in } \Omega, \\ -\Delta u_2 + (\lambda_2 - \beta_{2,3} U_3^2) u_2 &= u_1^2 u_2 \quad \textit{in } \Omega, \\ u_1, u_2 &\in H^1_0(\Omega). \end{aligned} \tag{1.2}$$

*In particular $(U_1, U_2, U_3)$ is a minimizer of the following minimizing problem:*

$$e = \inf_{u_3 \in K_3} \inf_{(u_1,u_2) \in N_{u_3}} \left( |||u_1|||^2_{\lambda_1,\Omega} + |||u_2|||^2_{\lambda_2,\Omega} - \beta_{1,3} ||u_1 u_3||^2_{L^2(\Omega)} - \beta_{2,3} ||u_2 u_3||^2_{L^2(\Omega)} \right), \tag{1.3}$$

*where*

$$K_3 = \{u \in H_0^1(\Omega) \mid u \textit{ is a least energy solution of } (1.1)\},$$

$$N_{u_3} = \left\{ (u_1, u_2) \in H_0^1(\Omega)^2 \;\middle|\; \begin{array}{r} |||u_1|||^2_{\lambda_1,\Omega} + |||u_2|||^2_{\lambda_2,\Omega} - \beta_{1,3}||u_1u_3||^2_{L^2(\Omega)} \\ -\beta_{2,3}||u_2u_3||^2_{L^2(\Omega)} = 2||u_1u_2||^2_{L^2(\Omega)} \neq 0 \end{array} \right\}.$$

**Remark 1.2.** The infimum $e$ is also written as $e = \inf_{u_3 \in K_3} \bar{e}(u_3)$ where

$$\bar{e}(u_3) = \inf_{(u_1,u_2)\in N_{u_3}} \left( |||u_1|||^2_{\lambda_1,\Omega} + |||u_2|||^2_{\lambda_2,\Omega} - \beta_{1,3}||u_1u_3||^2_{L^2(\Omega)} - \beta_{2,3}||u_2u_3||^2_{L^2(\Omega)} \right).$$

For any given $u_3 \in H_0^1(\Omega)$, $\bar{e}(u_3)$ is achieved by a minimizer $(u_1, u_2) \in N_{u_3}$ which is a non-trivial least energy solution of

$$\begin{aligned} -\Delta u_1 + (\lambda_1 - \beta_{1,3}u_3^2)u_1 &= u_1u_2^2 \quad \text{in } \Omega, \\ -\Delta u_2 + (\lambda_2 - \beta_{2,3}u_3^2)u_2 &= u_1^2u_2 \quad \text{in } \Omega, \\ u_1, u_2 &\in H_0^1(\Omega). \end{aligned} \tag{1.3}$$

In fact, in **[SW1]**, we showed the existence of a minimizer for $\bar{e}(u_3)$ when $V_i(x) \equiv \lambda_i - \beta_{i,3}u_3^2$ $(i = 1, 2)$ are positive constants. When $V_i(x)$ are non-negative functions, we can show the existence of a minimizer by the same way.

**Remark 1.3.** The solution $\vec{u}_\beta$ of Theorem 1.1 was given as a minimizer of the following minimizing problem:

$$c_\beta = \inf_{\vec{u}\in M_\beta} \left( |||u_1|||^2_{\lambda_1,\Omega} + |||u_2|||^2_{\lambda_2,\Omega} + |||u_3|||^2_{\lambda_3,\Omega} \right),$$

$$M_\beta = \left\{ \vec{u} \in H_0^1(\Omega)^3 \;\middle|\; \begin{array}{rl} f_1(\vec{u}) + f_2(\vec{u}) = 0, & (u_1, u_2) \neq (0,0), \\ f_3(\vec{u}) = 0, & u_3 \neq 0. \end{array} \right\},$$

where $f_i(\vec{u}) = |||u_i|||^2_{\lambda_i,\Omega} - \mu_i||u_i||^4_{L^4(\Omega)} - \sum_{j\neq i} \beta_{i,j}||u_iu_j||^2_{L^2(\Omega)}$. For details, see our paper **[SW3]**.

Next, we will observe the non-radial symmetry for a least energy positive solution of $(*)$ even if $\Omega$ is ball. When $\Omega$ is ball, it is well-known that least energy solution of (1.1) is a unique positive radial symmetric solution satisfying $U_3'(r) < 0, r = |x|$. (The uniqueness was proved in **[K]**, The radial symmetry was proved in **[GNN]**.) Let $\vec{u}_\beta = (u_{1,\beta}, u_{2,\beta}, u_{3,\beta})$ be a least energy solution of $(*)$. From Theorem 1.1, $u_{3,\beta}$ converges to the unique positive radial symmetric solution $U_3$ and a subsequence of $(\sqrt{\beta}u_{1,\beta}, \sqrt{\beta}u_{2,\beta})$ converges to a least energy solution $(U_1, U_2)$ of (1.2) whose potential functions $\lambda_i - \beta_{i,3}U_3(x)$ $(i = 1, 2)$ have minimum on the boundary $\partial\Omega$. Then, we can show that $(U_1, U_2)$ has a concentrating point near $\partial\Omega$. That is, $(U_1, U_2)$ is not radial symmetric.

In fact, in this report, we will observe such concentrating phenomenon for the following system including (1.2):

$$
\begin{aligned}
-\epsilon^2\Delta u_1 + V_1(x)u_1 &= u_1u_2^2 \quad \text{in } \Omega, \\
-\epsilon^2\Delta u_2 + V_2(x)u_2 &= u_1^2u_2 \quad \text{in } \Omega, \\
u_1, u_2 &\in H_0^1(\Omega).
\end{aligned}
\tag{1.4$_\epsilon$}
$$

where $\epsilon > 0$ is a parameter, $V_i(x) \in C(\overline{\Omega}, \mathbf{R})$ $(i = 1, 2)$ are positive functions. Let $(\vec{u}_\epsilon)_{\epsilon>0}$ be a family of positive least energy solutions of $(1.4)_\epsilon$. To state the concentrating point of $(\vec{u}_\epsilon)_{\epsilon>0}$, we need the following $b(\lambda_1, \lambda_2)$:

$$
b(\lambda_1, \lambda_2) = \inf_{(u_1,u_2)\in N_{\lambda_1,\lambda_2}} \left( |||u_1|||^2_{\lambda_1,\mathbf{R}^n} + |||u_2|||^2_{\lambda_2,\mathbf{R}^n} \right), \tag{1.5}
$$

$$
N_{\lambda_1,\lambda_2} = \left\{ (u_1, u_2) \in H^1(\mathbf{R}^n)^2 \;\middle|\; |||u_1|||^2_{\lambda_1,\mathbf{R}^n} + |||u_2|||^2_{\lambda_2,\mathbf{R}^n} = 2||u_1u_2||^2_{L^2(\mathbf{R}^n)} \neq 0 \right\}.
$$

We easily see that $b(\lambda_1, \lambda_2)$ is achieved for some $\vec{u} = (u_1, u_2) \in H^1(\mathbf{R}^n)^2$ and $\vec{u}$ is a least energy solution of

$$
\begin{aligned}
-\Delta u_1 + \lambda_1 u_1 &= u_1u_2^2 \quad \text{in } \mathbf{R}^n, \\
-\Delta u_2 + \lambda_2 u_2 &= u_1^2u_2 \quad \text{in } \mathbf{R}^n, \\
u_1, u_2 &\in H^1(\mathbf{R}^n).
\end{aligned}
\tag{1.6$_{(\lambda_1,\lambda_2)}$}
$$

Also, for $b(\lambda_1, \lambda_2)$, we have the following:

**Lemma 1.4.**

(i) $b(\lambda_1, \lambda_2) : (0, \infty)^2 \to \mathbf{R}$ is a continuous function.

(ii) $b(\lambda_1, \lambda_2)$ is increasing with respect to $\lambda_i$ $(i = 1, 2)$.

(iii) $b(\eta\lambda_1, \eta\lambda_2) = \eta^2 b(\lambda_1, \lambda_2)$ for all $\eta, \lambda_1, \lambda_2 > 0$.

We regard $u \in H_0^1(\Omega)$ as $u \in H^1(\mathbf{R}^n)$ by setting $u = 0$ on $\mathbf{R}^n \setminus \Omega$. Now we have the following theorem.

**Theorem 1.5.** *There exist sequences $\epsilon_m \to 0$, $x_m \to x_0$ in $\overline{\Omega}$ and $\vec{u}_0 = (u_{1,0}, u_{2,0}) \in H^1(\mathbf{R}^n)^2$ which is a positive least energy solution of $(1.6)_{V_1(x_0),V_2(x_0)}$ such that*

$$
\begin{aligned}
&u_{i,\epsilon_m}(\epsilon_m x - x_m) \to u_{i,0}(x) \quad \text{strongly in } H^1(\mathbf{R}^n) \quad (i = 1, 2), \\
&b(V_1(x_m), V_2(x_m)) \to b(V_1(x_0), V_2(x_0)) = \underline{b}.
\end{aligned}
$$

*Here $\underline{b} = \min_{x\in\overline{\Omega}} b(V_1(x), V_2(x))$.*

**Remark 1.6.**

(i) For an unique positive radial symmetric solution $U_3$ of (1.1), setting $V_i(x) = \lambda_i - \beta_{i,3}U_3(x)$ $(i = 1, 2)$, from (ii) of Lemma 1.4, $b(V_1(x), V_2(x))$ has minimum on the boundary $\partial\Omega$. Thus, the least energy solution $(U_1, U_2)$ of (1.2) with suitable coefficients, has a concentrating point near $\partial\Omega$.

(ii) When a positive interaction term $\beta$ closes to 0, Lin–Wei **[LWe2]** and Ikoma–Tanaka **[IT]** studied a singular perturbation problem for

$$\begin{aligned} &-\epsilon^2\Delta u_1 + V_1(x)u_1 = \mu_1 u_1^3 + \beta u_1 u_2^2 \quad \text{in } \mathbf{R}^n, \\ &-\epsilon^2\Delta u_2 + V_2(x)u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2 \quad \text{in } \mathbf{R}^n. \end{aligned}$$

In this case, there are possibilities that least energy positive solution $u_{1,\epsilon}$ and $u_{2,\epsilon}$ have different concentrating points. But, in our case, $u_{1,\epsilon}$ and $u_{2,\epsilon}$ always must concentrate a same point.

## 2. The Existence of $G$-symmetric least energy solutions

In this section, when $\Omega$ is a ball $B = \{x \in \mathbf{R}^N \mid |x| = 1\}$, we observe the multiple existence of positive solutions of $(*)$. When $\Omega$ is a ball, a positive solution $U_3$ of (1.1) is unique and radially symmetrically. Thus, for group actions $G \subset O(n)$ ($O(n)$ is the orthogonal group for $n = 2, 3$), by solving minimizing problems on $G$-symmetric function's set $H_0^G(B) = \{u \in H_0^1(B) \mid u(gx) = u(x) \text{ for all } g \in G\}$, we can expect multiple existence of positive solutions of $(*)$. In fact, we can show the following theorem by a similar way of Theorem 1.1.

**Theorem 2.1.** *Assume $\Omega = B$, $\beta \equiv \beta_{1,2} > 0$ and $\beta_{1,3}, \beta_{2,3} \le 0$. Let $G \subset O(n)$ be a group action. Then, there exists $\beta^G > 0$ such that, for any $\beta > \beta^G$, $(*)$ has a $G$-symmetric positive solution $\vec{u}_\beta^G(x) = (u_{1,\beta}^G(x), u_{2,\beta}^G(x), u_{3,\beta}^G(x))$. Moreover, there exist a sequence $\beta_m \to \infty$ and $U_i \in H_0^1(B)$ $(i = 1, 2, 3)$ such that*

$$(\sqrt{\beta_m}u_{1,\beta_m}^G, \sqrt{\beta_m}u_{2,\beta_m}^G, u_{3,\beta_m}^G) \to (U_1^G, U_2^G, U_3) \quad \textit{strongly in } H_0^1(B)^3.$$

*Here $U_3$ is a unique positive radial least energy solution of (1.1) with $\Omega = B$ and $(U_1^G, U_2^G)$ is a positive least energy $G$-symmetric solution of (1.2) with $\Omega = B$.*

Theorem 2.1 suggests the multiplicity of positive solutions of $(*)$. However, in order to get a multiple existence for $(*)$, we need to show the $\vec{u}^G \neq \vec{u}^{G'}$ for group actions $G \neq G'$. To observe this, we will discuss an asymptotically behavior of limit equation (1.2) by regarding some coefficients as parameters. That is, for group actions $G \neq G'$, we show that $(U_1^G, U_2^G)$ and $(U_1^{G'}, U_2^{G'})$ has different asymptotically behaviors when some parameters go to limits. More precisely, we will show $(U_1^G, U_2^G)$ has $k^G$-peaks near the boundary $\partial B$ where $k^G$ is a number of the minimum orbit for $G$. This argument is similar to the arguments for the multiplicity of positive solutions of $-\Delta u + u = u^p$ on annulus domain.

We will observe such asymptotically results for more general equations which including (1.2). For radial positive functions $V_i(x) \in C(B, \mathbf{R})$ $(i = 1, 2)$, we consider the following system

$$\begin{aligned} -\Delta u_1 + \eta V_1(x) u_1 &= u_1 u_2^2 \text{ in } B, \quad u_1 \in H_0^1(B), \\ -\Delta u_2 + \eta V_2(x) u_2 &= u_1^2 u_2 \text{ in } B, \quad u_2 \in H_0^1(B). \end{aligned} \tag{2.1}$$

For a group action $G \subset O(n)$ $(n = 2, 3)$, to get G-symmetric solutions of (2.1), we solve the following minimizing problem on $H_0^G(B)$:

$$\hat{b}_\eta^G = \inf_{(u_1, u_2) \in \hat{N}_\eta^G} \left( |||u_1|||_{\eta V_1, B}^2 + |||u_2|||_{\eta V_2, B}^2 \right), \tag{2.2}$$

$$\hat{N}_\eta^G = \left\{ (u_1, u_2) \in H_0^G(B)^2 \,\middle|\, |||u_1|||_{\eta V_1, B}^2 + |||u_2|||_{\eta V_2, B}^2 = 2||u_1 u_2||_{L^2(B)}^2 \neq 0 \right\}.$$

By standard ways, we see that $\hat{b}_\eta^G$ is achieved for some $(u_{1,\eta}^G, u_{2,\eta}^G) \in \hat{N}_\eta^G$ which is a $G$-symmetric positive solution of (2.1). To discuss an asymptotically behavior of $(u_{1,\eta}^G, u_{2,\eta}^G)$ as $\eta \to \infty$, the function $b(\lambda_1, \lambda_2)$ which was defined in (1.5) also plays important roles. By using the Schwarz symmetrization, we see that least energy solutions of $(1.6)_{(\lambda_1, \lambda_2)}$ is radial symmetry. For $G \subset O(n)$, let $G[x] = \{gx \mid g \in G\}$ be an orbit of $x \in \mathbf{R}^n \setminus \{0\}$ and $k^G = \min\{\#G[x] \mid x \in \mathbf{R}^n \setminus \{0\}\}$ be a element number of the minimum orbit. Now, we obtain the following theorem which is essential in our arguments.

**Theorem 2.2.** *Assume that $V_i(x) \in C(B, \mathbf{R})$ $(i = 1, 2)$ are positive radial functions and a finite group $G \subset O(n)$ satisfies $k^G < \frac{b(V_1(0), V_2(0))}{\underline{b}}$ where $\underline{b} = \min_{x \in B} b(V_1(x), V_2(x))$. Then we have*

$$\eta^{\frac{n}{2} - 2} \hat{b}_\eta^G \to k^G \underline{b} \quad \text{as } \eta \to \infty.$$

*Moreover, for a family of G-symmetric positive solutions $(u_{1,\eta}^G, u_{2,\eta}^G)$ of (2.1) which achieves the minimizing problem (2.2), there exist a subsequence $\eta_m \to \infty$ and a sequence $x_m \to x_0$ in $B$ with $\#G[x_m] = \#G[x_0] = k^G$ and $b(V_1(x_0), V_2(x_0)) = \underline{b}$ such that*

$$\left|\left|\left| u_{i,\eta_m}^G - \sum_{z \in G[x_m]} \sqrt{\eta_m} w_{i,0}(\sqrt{\eta_m}(\cdot - z)) \right|\right|\right|_{\eta_m V_i, B}^2 = o(\eta_m^{1 - \frac{n}{2}}) \quad (i = 1, 2).$$

*Here $(w_{1,0}, w_{2,0})$ is a positive least energy solution of (1.5) with $(\lambda_1, \lambda_2) = (V_1(x_0), V_2(x_0))$ and $o(\eta_m^{1-\frac{n}{2}})\eta_m^{\frac{n}{2}-1} \to 0$ as $\eta_m \to \infty$. That is, for large $\eta_m$, $(u_{1,\eta_m}^G, u_{2,\eta_m}^G)$ is close to $k^G$-peak functions.*

**Remark 2.3.**

(i) For the case $k^G \geq \frac{b(V_1(0),V_2(0))}{\underline{b}}$, $G$-symmetric positive solutions $(u_{1,\eta}^G, u_{2,\eta}^G)$ achieving (2.2) may be radial symmetric. Thus we can't look for any more positive solutions by only Theorem 2.2.

(ii) Theorem 2.1 and 2.2 still hold for $G$-invariant domain $\Omega$. The other corollaries and theorems below follow from Theorem 2.1 and Theorem 2.2.

From Theorem 2.2, we have the following

**Corollary 2.4.** *Suppose that the same assumptions as Theorem 2.2 hold. Let $K_\eta^G$ be a set of least energy $G$-symmetric solution of (2.1), $K$ be a set of least energy solution of (1.6) with $(\lambda_1, \lambda_2) = (V_1(x_0), V_2(x_0))$ and $X = \{x \in B \mid \#G[x] = k^G,\ b(V_1(x), V_2(x)) = \underline{b}\}$ and $X_\rho = \{x \in B \mid \#G[x] = k^G,\ dist(x, X) < \rho\}$. Then, for any $\rho > 0$, we have*

$$\sup_{(u_1,u_2)\in K_\eta^G} \inf_{(w_1,w_2)\in K, x\in X_\rho} \sum_{i=1,2} \left|\left|\left| u_i - \sum_{z\in G[x]} \sqrt{\eta} w_i(\sqrt{\eta}(\cdot - z)) \right|\right|\right|_{\eta V_i, B}^2 = o(\eta^{1-\frac{n}{2}}) \quad (2.3)$$

*Here $o(\eta^{1-\frac{n}{2}})\eta^{\frac{n}{2}-1} \to 0$ as $\eta \to \infty$.*

**Proof.** Suppose that Corollary 2.4 does not hold. Then there exists $c_0 > 0$, $\eta_m \to \infty$ and $(u_{1,\eta_m}, u_{2,\eta_m}) \in K_{\eta_m}^G$ such that

$$\inf_{(w_1,w_2)\in K, x\in X_\rho} \sum_{i=1,2} \left|\left|\left| u_{i,\eta_m} - \sum_{z\in G[x]} \sqrt{\eta_m} w_i(\sqrt{\eta_m}(\cdot - z)) \right|\right|\right|_{\eta_m V_i, B}^2 \geq c_0 \eta^{1-\frac{n}{2}}$$

But, this contradicts to Theorem 2.2. ∎

When $n = 2$, for any $k \in \mathbf{N}$, the cyclic group $\mathbf{Z}_k \subset O(2)$ satisfies $k^{\mathbf{Z}_k} = k$. Thus, from Corollary 2.4, we easily find the following multiple existence of positive solutions.

**Corollary 2.5.** *Suppose that $n = 2$ and $V_i(x) \in C(B, \mathbf{R})$ $(i = 1, 2)$ are positive radial functions and $k < \frac{b(V_1(0),V_2(0))}{\underline{b}}$ where $\underline{b} = \min_{x\in B} b(V_1(x), V_2(x))$. Then, there exists $\eta_k > 0$ such that, for any $\eta > \eta_k$, (2.1) has a positive solution close to $\ell$-peak function in* *the sense of* (2.3) *with* $G = \mathbf{Z}_\ell$ *and* (2.1) *has a radial positive solution. That is,* (2.1) *has at least* $k+1$ *positive solutions.*

**Proof.** From Corollary 2.4, for any $\ell < \frac{b(V_1(0),V_2(0))}{\underline{b}}$, there exists $\eta^{\mathbf{Z}_\ell} > 0$ such that, for any $\eta > \eta^{\mathbf{Z}_\ell}$, (2.1) has a positive solution close to $\ell$-peak solutions in the sense of (2.3). On the other hand, (2.1) always has a radial positive solution. Thus, for $\eta \geq \eta_k \equiv \max\{\eta^{\mathbf{Z}_1}, \cdots, \eta^{\mathbf{Z}_k}\}$, (2.1) has at least $k+1$ positive solutions. ∎

When $n = 3$, the subgroups $G = \mathbf{Z}_2, P_4, P_8, P_{12} \subset O(3)$ satisfy $k^G = 2, 4, 8$, or 12, respectively. Here $P_q$ is the $q$-regular polyhedron group. Thus, from Corollary 2.4, we also find the following corollary.

**Corollary 2.6.** *Suppose that* $n = 3$ *and* $V_i(x) \in C(B, \mathbf{R})$ $(i = 1, 2)$ *are positive radial functions and* 2 *(or* 4, 8, 12*, respectively)* $< \frac{b(V_1(0),V_2(0))}{\underline{b}}$ *where* $\underline{b} = \min_{x\in B} b(V_1(x), V_2(x))$*. Then, there exists* $\eta_0 > 0$ *such that, for any* $\eta > \eta_0$*,* (2.1) *has a positive solution close to* 2 *(or* 4, 8, 12*, respectively)-peak functions in the sense of* (2.3) *with* $G = \mathbf{Z}_2$ *(or* $P_4, P_8, P_{12}$*, respectively) and* (2.1) *has a radial positive solution.*

Here, we return to the our original equation $(*)$ and limit equation (1.2). For $\eta > 0$, $\lambda_i' > 0$ and $\beta_{i,3}' < 0$ $(i = 1, 2)$, we set

$$\lambda_i = \eta\lambda_i', \quad \beta_{i,3} = \eta\beta_{i,3}' \quad (i = 1, 2).$$

For (1.2), since $U_3(x)$ is radial and decreasing with respect to $r = |x|$, from (b1)–(b2), $b(\lambda_1 - \beta_{1,3}U_3(x)^2, \lambda_2 - \beta_{2,3}U_3(x)^2) : (0,\infty)^2 \to \mathbf{R}$ has maximum at $x = 0$ and minimum on $\partial B$. We remark that, from (b3), $b(\lambda_1 - \beta_{1,3}U_3(x)^2, \lambda_2 - \beta_{2,3}U_3(x)^2) = \eta^2 b(\lambda_1' - \beta_{1,3}'U_3(x)^2, \lambda_2' - \beta_{2,3}'U_3(x)^2)$. From Corollary 2.5 and Corollary 2.6, we have the following multiple existence for $(*)$.

**Theorem 2.7.** *Assume that* $\Omega = B$*. For* $\eta > 0$*, we assume that* $\beta \equiv \beta_{1,2} > 0$*,* $\lambda_i = \eta\lambda_i' > 0$*,* $\beta_{i,3} = \eta\beta_{i,3}' < 0$ $(i = 1, 2)$*. Let* $k$ *be the maximum integer satisfying* $k < \frac{b(\lambda_1' - \beta_{1,3}'U_3(0)^2, \lambda_2' - \beta_{2,3}'U_3(0)^2)}{b(\lambda_1', \lambda_2')}$.

*(i) When* $n = 2$*, for any* $\epsilon, \rho > 0$*, there exists* $\eta_k > 0$ *such that, for any* $\eta > \eta_k$*, there exists* $\beta_k(\eta) > 0$ *such that, for any* $\beta > \beta_k(\eta)$ *and* $\ell = 1, \cdots, k$*, we have*

$$\sup_{\vec{u}\in L_\beta^{\mathbf{Z}_\ell}} \inf_{(w_1,w_2)\in K, x\in X_\rho} \sum_{i=1,2} \left|\left|\left| \sqrt{\beta}u_i - \sum_{z\in \mathbf{Z}_\ell[x]} \sqrt{\eta}w_i(\sqrt{\eta}(\cdot - z)) \right|\right|\right|_{V_i,B}^2 \leq \eta^{1-\frac{n}{2}}\epsilon.$$

*Here* $L_\beta^{\mathbf{Z}_\ell}$ *is a set of least energy* $\mathbf{Z}_\ell$*-symmetric solutions of* $(*)$ *and* $K$ *is a set of least energy solutions of* (1.6) *and* $X_\rho = \{1 - \rho < |x| \leq 1\}$*. In particular,* $(*)$ *has* *at least $k+1$ positive solutions $\vec{u}^\ell = (u_1^\ell, u_2^\ell, u_3^\ell)$ of $(*)$ $(1 \le \ell \le k+1)$. Here $\vec{u}^\ell$ is $\mathbf{Z}_\ell$-symmetry and $(u_1^\ell, u_2^\ell)$ is close to $\ell$-peak functions which peak's locations are near $\partial B$ $(1 \le \ell \le k)$ and $\vec{u}^{k+1}$ is radial symmetry.*

*(ii) When $n = 3$, if 2 (or 4, 8, 12, respectively) $\le k$ holds, for any $\epsilon, \rho > 0$, there exists $\eta_k > 0$ such that, for any $\eta > \eta_k$, there exists $\beta_k(\eta) > 0$ such that, for any $\beta > \beta_k(\eta)$ and $G = \mathbf{Z}_2$ (or $P_4, P_8, P_{12}$, respectively), we have*

$$\sup_{\vec{u}\in L_\beta^G} \inf_{(w_1,w_2)\in K, x\in X_\rho} \sum_{i=1,2} \left|\left|\left| \sqrt{\beta}u_i - \sum_{z\in G[x]} \sqrt{\eta}w_i(\sqrt{\eta}(\cdot - z)) \right|\right|\right|_{V_i,B}^2 \le \eta^{1-\frac{n}{2}}\epsilon.$$

*Here $L_\beta^G$ is a set of least energy $G$-symmetric solutions of $(*)$ and $K$ is a set of least energy solutions of (1.6) and $X_\rho = \{1-\rho < |x| \le 1\}$.*

**Proof.** From Corollary 2.5 and Corollary 2.6, there exists $\eta_k > 0$ such that, for all $\eta > \eta_k$,

$$\sup_{(U_1,U_2)\in K_\eta^G} \inf_{(w_1,w_2)\in K, x\in X_\rho} \sum_{i=1,2} \left|\left|\left| U_i - \sum_{z\in G[x]} \sqrt{\eta}w_i(\sqrt{\eta}(\cdot - z)) \right|\right|\right|_{\eta V_i,B}^2 \le \eta^{1-\frac{n}{2}}\epsilon. \tag{2.4}$$

Here $K_\eta^G$ is a set of least energy $G$-symmetric solution of (2.1). Next, from Theorem 2.2, there exists $\beta(\eta) > 0$ such that, for all $\beta > \beta(\eta)$, we have

$$\sup_{\vec{u}\in L_\eta^G} \inf_{(U_1,U_2)\in K_\eta^G} \sum_{i=1,2} \left|\left|\left| \sqrt{\beta}u_i - U_i \right|\right|\right|_{\eta V_i,B}^2 \le \eta^{1-\frac{n}{2}}\epsilon. \tag{2.5}$$

From (2.4)–(2.5), we obtain Theorem 2.7. ∎

Moreover, since $U_3(0) \to \infty$ as $\lambda_3 \to \infty$, from (b3), we observe that

$$\frac{b(\lambda_1' - \beta_{1,3}'U_3(0)^2, \lambda_2' - \beta_{2,3}'U_3(0)^2)}{b(\lambda_1', \lambda_2')} \to \infty \quad \text{as } \lambda_3 \to \infty. \tag{2.6}$$

Therefore, from Theorem 2.7 and (2.6), we easily lead the following theorem.

**Theorem 2.8.** *Assume that $\Omega = B$ and $n = 2$. Then, for any $k \in \mathbf{N}$, there exists $\lambda_k > 0$, such that, for any $\lambda_3 > \lambda_k$, there exists $\eta_k(\lambda_3) > 0$, such that, for any $\eta > \eta_k(\lambda_3)$, there exists $\beta_k(\eta, \lambda_3) > 0$, such that, for any $\beta \equiv \beta_{1,2} > \beta_k(\eta, \lambda_3)$, $(*)$ has at least $k+1$ positive solutions $\vec{u}^\ell = (u_1^\ell, u_2^\ell, u_3^\ell)$ $(1 \le \ell \le k+1)$. Here $\vec{u}^\ell$ is $\mathbf{Z}_\ell$-symmetry and $(u_1^\ell, u_2^\ell)$ is close to $\ell$-peak functions which peak's locations are near $\partial B$ $(1 \le \ell \le k)$ and $\vec{u}^{k+1}$ is radial symmetry.*

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