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Fano fibrations and its applications

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In this report we consider a proper surjective morphism $f : X \to Y$ between smooth projective varieties whose general fiber is a Fano variety (resp. weak Fano variety), called a Fano fibration (resp. weak Fano fibration). We study when such a fibration is locally trivial. I think that the point of view is related with the consideration of the difference between smooth morphism and local triviality, and that it is useful for the classification of higher dimensional Fano varieties. Here we recall a sufficient conditions for a proper surjective morphism $f : X \to Y$ to be local trivial in terms of the anti-canonical line bundle $-K_{X/Y}(=: \pi^*K_Y - K_X)$. Next under the smooth morphism $\pi : X \to Y$ we consider the two cases 1) every fiber is a hypersurface 2) X is Fano 4-fold where Picard number of X is 1 and $Y = \mathbf{P}^1$.

Let $\pi: X \to Y$ be a proper surjective morphism of non-singular projective varieties with connected fibers.

Def. $\pi: X \to Y$ is called a Fano fibration (or weak Fano fibration)

$\leftrightarrow$ a general fiber π is a smooth Fano (or weak Fano resp.) variety F.

Def. Smooth proj var Z is Fano (or, weak Fano)

$\leftrightarrow$ the anti-canonical line bundle $-K_Z$ is ample (or, nef big respectively)

Aim. When is a Fano fibration locally trivial ? Find a (sufficient) condition for the above ?

Motivation. Kollár, Miyaoka and Mori showed in "Rational connectedness and boundedness of Fano manifolds. J. Differential Geom. 36 (1992), no. 3, 765-779."

Corollary 2.8 (char ≥ 0). Let $\pi : X \to Y$ be a surjective smooth morphism between smooth projective varieties. If $dimY > 0$, then $-K_{X/Y}$ cannot be ample.

Corollary 2.9 (char ≥ 0). Let $\pi: X \to Y$ be a surjective smooth morphism between smooth projective varieties. If X is a Fano manifold, then so is Y.

We do not know an example of a smooth Fano variety X where $\pi : X \to Y$ has a smooth fiber structure to a **projective variety** Y which is not locally trivial. In this note it is assumed that the base space Y is a **projective variety**.

We state the following

Theorem A Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with dim $Y \geq 1$. Assume that

1) every fiber $X_y := f^{-1}(y)$ of f is isomorphic to a hypersurface of degree d in P^n.

2) $n \geq 3$ (when n=3, $d \neq 4$ is assumed.)

3) there is a line bundle L on X which is relatively ample over Y such that $L|_{X_y} = \mathcal{O}_{X_y}(1)(:= \mathcal{O}_{P^n}(1)|_{X_y})$.

If f is a smooth morphism, then it is locally trivial.

More precisely if $d \geq 3$, the automorphism group of a general fiber F of f is a finite group. Thus there is an etale covering $\phi : \bar{Y} \to Y$ where $\bar{Y} \times_Y X$ is isomorphic to $\bar{Y} \times F$. F denotes the fiber of f.

For the proof we use the **discriminant on hypersurface in P^n.**

Theorem B Let X be a Fano 4-fold of Picard number 2 which has a surjective morphism $p : X \to \mathbf{P}^1$ with connected fibers. Moreover let the index of a general fiber F of p be one in the meaning of Iskovskih. Suppose that p is smooth and $-K_F$ is very ample. Then X is isomorphic to $\mathbf{P}^1 \times F$, unless F has the property: $4|(-K_F)^3$.

In this report we recall a condition for a fano fibration to be locally trivial and the related examples in §1. Next in §.2 we consider Theorem A. Finally we study Theorem B from §.3 to §.5. We heavily depend on the results of Fano 3-folds due to [I77].

§.1 Preliminary

We begin with

Ex. 1.1 (char ≥ 0) $\pi: X \to Y$ a smooth surjective morphism. Assume any fiber F of π is a Del Pezzo surface. $\longrightarrow$ π is locally trivial.

If $K_F^2 < 6(\leftrightarrow$ Aut F finite group) $\longrightarrow \exists\, Y'$, $\exists$ a finite surjectice morphism $g : Y' \to Y$ s. t. $X \times_Y Y' \cong F \times Y'$ ($\leftrightarrow -K_{X/Y}$ is semi-ample)

Ex.1.2 (char ≥ 0) smooth morphism, not locally trivial, weak Fano fibration

$A_1, ..., A_r$ $(2 \leq r \leq 7)$ r points in a general position in $\mathbf{P^2}$,
$l \subset P^2$ a line on $\mathbf{P^2}$ s.t . none of $\{A_i\}$ is in l,
$Z := \cup_{i=1}^{r}(\mathbf{P^1} \times \{A_i\}) \cup \Delta_l$,
Δ_l: = the diagonal of l in $P^1 \times P^1 (\subset \mathbf{P^1} \times \mathbf{P^2})$,
$X :=$ blowing up of $\mathbf{P^1} \times \mathbf{P^2}$ along Z.
$\longrightarrow$ $X \to \mathbf{P^1}$ is a smooth morphism but not locally trivial.

The reporter states the following theorem which is necessary for the proof of Theorem B

Theorem 1.3. $\pi: X \to Y$ a Fano fibration between smooth projective varieties. Assume $-K_{X/Y} := -K_X + \pi^* K_Y$ is semi-ample.

$\longrightarrow \exists$ a finite Galois étale covering $g : Y_1 \to Y$ s.t.
$X \times_Y Y_1 \cong F \times Y_1$ over Y_1 (F= a fiber of π)
If Y is simply connected, g is an identity map.

Corollary 1.4 Assume that $-K_{X/Y}$ is nef.
$\longrightarrow X \cong F \times Y$ for a fiber F of π
if one of the following conditions are satisfied:
1) Y is weak Fano.
2) X is weak Fano.
3) Y is rationally connected and $-K_{X/Y}$ is semi-ample.

Remark 1.5 Y is rationally connected and $-K_{X/Y}$ is nef. Assume π is smooth.
$\longrightarrow \pi$ is locally trivial.

Theorem 1.6. X , Y smooth projective varieties. $\pi: X \to Y$ a weak Fano fibration Assume $-K_{X/Y}$ is nef and Y is weak Fano.

$\longrightarrow X \cong F \times Y$ (F= a fiber of π)

For the proof of Theorem 1.3 the following two Propositions are necessary.

Proposition I $\pi: X \to Y$ a surj. mor. between sm. proj. varieties with $\dim X = n+1$, $\dim Y = 1$ with connected fibers.
$\longrightarrow -K_{X/Y}$ is not nef big.

It follows from Leray spectral sequence: $E^{p,q} = H^p(X, R^q p_*(p^* K_Y)) \to H^m(X, p^* K_Y)$ and Kawamata-Viehweg Vanishing Theorem.

Corollary II. Assume $-K_F$ is nef and big for a general fiber F and that $-K_{X/Y}$ is semi-ample. $\longrightarrow$ $\dim \phi(X) = \dim F$ where ϕ is a morphism by $-mK_{X/Y}$ for $m >> 0$.

Proposition III $\pi: X \to Y$ be a surj. mor. of non-singular proj. var.
$S \subset X$ a closed subvariety s.t.
1. $-K_{X/Y}|_S$ is nef $K_{X/Y} = K_X - \pi^*(K_Y)$,
2. π is smooth at a general point on S.

Let $\varphi: X \to Z$ be a mor. into another proj. var. Z s.t.

3. $\varphi(S)$ is a pt,
4. $\sigma = (\varphi, \pi): X \to Z \times Y$ is unramified at a general pt on S.

$\longrightarrow$ (i)-(iv) hold
(i) $\pi: X \to Y$ is smooth along S.
(ii) $\sigma: X \to Z \times Y$ is unramified along S.
(iii) $\pi|_S: S \to Y$ is unramified.
(iv) $\Omega^1_{X/Y}|_S$ is a free $\mathcal{O}_S$-module.
In particular, if $\pi(S) = Y$,

$\longrightarrow S$ is étale over Y,

Normal bundle $N_{S/X}$ = free $\mathcal{O}_S$-module = the dual of $\Omega^1_{X/Y}|_S$.

§.2 The proof of Theorem A.

We give the outline of the proof of Theorem A.

Let $f : X \to Y$ be a surjective morphism between smooth projective varieties with dim Y = 1.

Assume that

1) every fiber $X_y := f^{-1}(y)$ of f is isomorphic to a hypersurface of degree d in P^n.

2) $n \geq 3$ (if n = 3, $d \neq 4$)

3) there is a line bundle L on X which is relatively ample over Y such that $L|_{X_y} = \mathcal{O}_{X_y}(1)(:= \mathcal{O}_{P^n}(1)|_{X_y})$.

Then f is a smooth morphism, then it is locally trivial.

More precisely if $d \geq 3$, the automorphism group of a general fiber F of f is a finite group. Thus there is an etale covering $\phi : \bar{Y} \to Y$ where $\bar{Y} \times_Y X$ is isomorphic to $\bar{Y} \times F$. F denotes the fiber of f.

Proof. It is trivial in case of $d = 1, 2$.

Assume $d > 2$. We make use of the theory of the discriminant of hypersurface in P^n.

Step.1. There is a canonical surjection on X: $f^*f_*L \to L \to 0$, which yields a closed embedding $j : X \to P(f_*L)$ over Y.

From the exact sequence: $0 \to \mathcal{O}_{P^n}(1-d) \to \mathcal{O}_{P^n}(1) \to \mathcal{O}_{X_y}(1) \to 0$, we have an isomorphism $H^0(P^n, \mathcal{O}_{P^n}(1)) \cong H^0(X_y, \mathcal{O}_{X_y}(1))$. f_*L is a locally free of rank $n+1$ on Y where $f_*L \otimes k(y) \cong H^0(X_y, \mathcal{O}_{X_y}(1))$. Thus we get the desired fact.

Next to continue the argument we follow the notations on forms in §.5 by [Mu].

A form of degree d in n+1 coordinates $x_0, x_1, ..., x_n$ can be written

$f(x_0, x_1, ..., x_n) = \Sigma_{|I|=d} a_I x^I$..

Let us denote by $V_{n,d}$ the vector space of homogeneous polynomials $f(x)$ of degree d and consider the associated affine space written by the same notation $V_{n,d}$. $GL(n+1)$ acts on $V_{n,d}$ on the right by $f(x) \to f(gx)$.

Let $H_{n,d}$ be $P(V_{n,d})$ which is isomorphic to $Proj\ k[..., \xi_I, ...]$. Here $k[..., \xi_I, ...]$ is a polynomial ring with ${}_{n+d}C_d$ independent variable ξ_I.

We recall the discriminant.

$H^{sing}_{n,d}$ denotes the set of all singular hypersurfaces in $H_{n,d}$. Then it is shown that $H^{sing}_{n,d}$ is defined by the form $D(\xi)$ of $k[\xi]$ as an ample divisor in $H_{n,d}$ and that $D(\xi)$ is $SL(n+1)-$ invariant, equivalently the subvariety $D(\xi)=0$ in $H_{n,d}$ is $GL(n+1)-$ invariant.

Let $U_{n,d} := V_{n,d} - \{D(\xi)=0\}$.

Note that for $d \geq 3$, every point of $U_{n,d} \subset V_{n,d}$ is stable for the action of $GL(n+1,k)$ by virtue of the following:

Theorem Any smooth homogeneous polynomial $f \in k[x_0, ..., x_n]$, with degree ≥ 3 and $n \geq 3$ is invariant under at most finitely many $g \in GL(n+1)$.

Thus we have a good quotient

$$\Phi : U_{n,d} \to U_{n,d}/GL(n+1,k)$$

whose points parameterise precisely the $GL(n+1,k)$ orbit.

To complete the proof we state the following

Lemma. Let H_1, H_2 be hypersurfaces of degree d in P^n with $n \geq 3$ and $d \geq 2(\neq 4)$ except for $(n,d)=(3,4)$. Assume H_1, H_2 are smooth.

If H_1 is isomorphic to H_2, then H_1 is projective equivalent to H_2, namely there is an element g in $PGL(n,k)$ with $g(H_1)=H_2$.

Theorem implies that smooth hypersurfaces which is isomorphic to each other is in the same $GL(n+1,k)-$ orbit namely in one fiber of Φ.

Now we return Step 1and get Theorem A.

Remark. The 3) of the assumption in Example is not needed.

§.3 The proof of Theorem B

We study the following

Theorem B Let X be a Fano 4-fold of Picard number 2 which has a surjective morphism $p : X \to \mathbf{P}^1$ with connected fibers. Moreover let the index of a general fiber F of p be one in the meaning of Iskovskih. Suppose that p is smooth and $-K_F$ is very ample. Then X is isomorphic to $\mathbf{P}^1 \times F$, unless F has the property: $4|(-K_F)^3$.

Hereafter the condition of Thereom B is maintained. The outline of the proof of Th B are stated in §.3-§.5.

First we have

3.1) X has another contraction: $\pi : X \to Z$ by virtue of [An85] as follows:

3.1.1) a conic bundle on a smooth Fano 3-fold Z

3.1.1.i) $\mathbf{P}^1$-bundle on Z. (see Corollary 3.3.1.)

3.1.1.ii) a standard conic bundle on Z.(treated in §.4)

3.1.2) a divisorial contraction $g : X \to Z$ which is the blowing-up of a smooth fano 4-fold along a smooth subvariety B of codimension 2 in Z treated in §.5.

Thus both Z are of Picard number 1 and $PicZ \cong \mathbf{Z}L$ with the ample generater L.

Hereafter in this section we state a condition on triviality of $\mathbf{P}^1$-bundle on Z and basic properties on dual varieties.

(3.2) First we give a sufficient condition for $\mathbf{P}^1$-bundle on Z to be trivial.

(3.2.1) Let $E \to Z$ be a rank 2-vector bundle on Z with a canonical morphism $\pi : X(:= P(E)) \to Z$ enjoying

1) there is a surjective morphism $p : X \to \mathbf{P}^1$ with connected fibers.

2) each fiber of p is a finite covering on Z via π.

Then we get

Proposition 3.3. Let us maintain the above conditions (3.2.1). Then we have

1) There is a finite surjective morphism $h : Z' \to Z$ where the induced $\mathbf{P}^1$-bundle $\pi' : X'(:= P(h^*E)) \to Z'$ is trivial on Z'.

2) The relative anti-canonical line bundle $-K'_\pi$ of π' is $p'^*(-K_{\mathbf{P}^1})$ which is semi-ample where $p' : X'(:= \mathbf{P}^1 \times Z') \to \mathbf{P}^1$ is the first projection.

3) The relative anti-canonical line bundle $-K_\pi$ of π is semi-ample.

4) If Z is algebraically simply-connected, then $\pi : X \to Z$ is trivial $\mathbf{P}^1$-bundle on Z.

We treat the case 3.1.1.i) .

Corollary 3.3.1 Let X be a Fano 4-fold of Picard number 2 which has a surjective morphism $p : X \to \mathbf{P}^1$ with connected fibers. Assume X has another contraction $\pi : X \to Z$ which is $\mathbf{P}^1$-bundle on a smooth Fano 3-fold Z.

Then letting F a general fiber of p, the restricted map $\pi|_F : F \to Z$ is an isomorphism and X is isomorphic to $\mathbf{P}^1 \times F$.

3.4. Next we consider "Dual variety" which is useful for the proof of Theorem B.

Given a smooth projective variety X and a line bundle L on X , we recall a condition for the complte linear system $|L|$ to contain a subfamily consisting smooth divisors parameterized by a projective curve.

We suppose that X is a smooth and non-degenerate projective subvariety in $\mathbf{P}^n$. For such X the dual variety $X^\vee$ of X denotes the following:

$X^\vee = \{H \in \mathbf{P}^{n\vee} |$ there is a point x in X $s.t$ $T_{X,x} \subset H\}$

where $\mathbf{P}^{n\vee}$ is the dual projective space of $\mathbf{P}^n$ and $T_{X,x}$ a tangent space of X at x in $\mathbf{P}^n$.

Let us set $\mathrm{def}(X) = n - 1 - \dim X^\vee$.

The two results below are shown in [E85]

Proposition 3.4 (Proposition 3.1 in [E85]) Let X be an m-dimensional smooth and non-degenerate projective subvariety in $\mathbf{P}^n$. Then $\dim(X) = 0$, if X is one of the folllowing:

(a) X is a complete intersection.

(b) X is a curve.

(c) X is a surface.

Proposition 3.5. Let X be a non-degenerate 4-fold. If $\mathrm{def}(X) > 0$ then X is a scroll.

We have

Remark 3.6. def $(X) > 0$, namely, $\dim X^\vee < n - 1$ if and only if there is a family of smooth hyperplane section parameterized by a projective curve.

Corollary 3.7. If X is a smooth and non-degenerate 4-fold of the Picard number one, then there is no algebraic family of smooth hyperplane sections parameterized by projective curve.

We state one of the relations between the discriminant and dual variety.

Let us consider the d-uple embedding $j : \mathbf{P}^n \hookrightarrow \mathbf{P}^N$ of $\mathbf{P}^n$ with $N = \binom{n+d}{n}$.

Note that $j(\mathbf{P}^n)$ is non-degenerate in $\mathbf{P}^N$. Let $H_{n,d}$ be the set of hypersurfaces of the degree d in $\mathbf{P}^n$ and $H_{n,d}^{sing}$ the one of the singular hypersurfaces of $H_{n,d}$. It is well-known that $H_{n,d}^{sing}$ is a closed subscheme in $\mathbf{P}^N$.

Then

Fact 3.8 For $d > 1$ there is a natural isomorphism between the scheme $H_{n,d}^{sing}$ and the dual variety $j(\mathbf{P}^n)^\vee$ of $j(\mathbf{P}^n)$ in $\mathbf{P}^N$.

Finally we state a fact related with 3.1.2)

(3.9) Let H_1, H_2 be two smooth hypersurfaces of degree d in $\mathbf{P}^n$ with $n > 2$ and $B = H_1 \cap H_2$. Assume B is a smooth subvariety of codimension 2. Let $g : X \to \mathbf{P}^n$ be the blowing-up of $\mathbf{P}^n$ along B. Then there is a surjective morphism $p : X \to \mathbf{P}^1$ and each fiber of p corresponds to a member of linear system generating H_1, H_2.

Then we get

Proposition 3.9.1 Under the condition 3.9, assume $n \geq 4$. If $d > 1$, then $p : X \to \mathbf{P}^1$ has a singular fiber. If $d = 1$, then p is $\mathbf{P}^{n-1}$-bundle over $\mathbf{P}^1$ and $X \cong P(\mathcal{O}_{\mathbf{P}^1}^{\oplus(n-1)} \oplus \mathcal{O}_{\mathbf{P}^1}(1))$.

Proof. The former follows from 3.4 a) and 3.6.

§.4 The proof of Theorem B (a standard conic bundle 3.1.1. ii.)

We start with the definition of conic bundle.

(4.1) A surjecitive morphism $\pi : X \to Z$ between projective varieties X and Z is said to be conic bundle over Z if every fiber of π is conic. Namely, there are a rank 3-vector bundle E on Z and a line bundle J on Z so that X is a member of $|2\xi + \pi^* J|$. Here $\pi : V(=: P(E)) \to Z$ is a canonical projection and ξ the tautological line bundle of E. Note $\rho(X) = \rho(Z) + 1$.

(4.1.1) Let X be a Fano $n(\geq 4)$-fold of Picard number 2 which has a morphism $p : X \to \mathbf{P}^1$ with connected fibers. Moreover let us assume X has a standard conic bundle $\pi : X \to Z$ as another contraction.

Until 4.9 we consider an n-fold X.

We state an example of a Fano conic bundle $\pi : X \to Z$ with another surjective morphism $p : X \to \mathbf{P}^1$.

(4.2) Example. Let Z be a Fano variety of index $a(> 2)$ where $-K_Z = aM$. Let us assume M is an ample line bundle which is base point free e.g. Z is a hypersurface of degree d in the projective space $\mathbf{P}^n$ with $d \leq n-2$.

Let $E = \mathcal{O} \oplus M^{\oplus 2}$ be a rank-3 vector bundle on Z and X a general member of a line bundle 2ξ where ξ is the tautological line bundle of E. Then X is the desired thing.

(4.3) First we have

$PicV \cong \mathbf{Z}\xi + \mathbf{Z}\pi^* L$. Let $J = bL$, $K_Z = -zL$ with integers z, b $(b > 0)$ since Z is Fano. L is the ample generator of $PicZ$.

A natural homomorphism $i^* : PicV \to PicX$ via an inclusion $i : X \hookrightarrow V$ is an isomorphism since X is a standard conic bundle.

Moreover since $K_V = -3\xi + \pi^*(detE + K_Z) = -3\xi + \pi^*(c_1(E) - zL)$, we have

(4.4) $-K_X = \xi_X - (c_1 - z + b)\pi^* L$

where det $E = c_1 L$ and with $\xi|_X = \xi_X$.

$PicX \cong \mathbf{Z}\xi_X + \mathbf{Z}\pi^* L \cong \mathbf{Z}K_X + \mathbf{Z}\pi^* L$

(4.5) Assume that $p : X \to \mathbf{P}^1$ is smooth with connected fibers and that the index of a general fiber F of p is one

Then we easily see that

$PicX \cong \mathbf{Z}K_p + \mathbf{Z}p^*\mathcal{O}_{\mathbf{P}^1}(1)$

where K_p is the relative anti-canonical line bundle of p from the differentially local triviality and trivial monodromy-action of p. Thus we get

(4.5.1) $PicX \cong \mathbf{Z}K_X + \mathbf{Z}p^*\mathcal{O}_{\mathbf{P}^1}(1)$.

Let us set $M = \pi^*L,\ H = p^*\mathcal{O}_{\mathbf{P}^1}(1)$.

(4.6) Thus we have

$M + H = -aK_X$ with a positive integer a.

The morphism $p : X \to \mathbf{P}^1$ yields

(4.7) The self-intersection $(H \cdot H)_X$ is zero in $CH^2(X)$ where $CH(X) = \oplus CH^i(X)$ is the Chow ring of X and $CH^i(X)$ a module generated by cycles of codimension i modulo rational equivalence. The self-intersection $(H \cdot H)_X$ in $CH^2(X)$ is equal to the intersection $(\bar{H} \cdot \bar{H} \cdot X)_V$ in $CH^3(V)$.

In V we get

(4.8) $(\bar{H} \cdot \bar{H} \cdot X)_V = 0$

Thus we can assume

(4.9) $-K_X = \xi_E|_X$, (4.6.2) $M + H = -aK_X$

and X is linearly equivalent to $2\xi + b\pi^*L$ with an integer b without confusion. (4.8) turns to be

(4.9.1) $(a\xi - \pi^*L \cdot a\xi - \pi^*L \cdot 2\xi + bL) = 0$ in $CH^3(V)$.

the coefficients of $\xi^3,\ \xi^2\pi^*L,\ \xi\pi^*L^2$ are

$2a^2,\ -4a + a^2b,\ 2 - 2ab$.

Thus we have the Chern class of E:

$c_1(E) = (-\frac{b}{2} + \frac{2}{a})L$

$c_2(E) = (\frac{1}{a^2} - \frac{b}{a})L^2$

$c_3(E) = -\frac{b}{2a^3}L^3$

Here we consider the case:

X is 4-fold. Thus Z is 3-fold which is Fano of first species..

(4.10) Z has a line l with $(l \cdot L) = 1$.

Remark. Our vector bundle E never be an ample vector bundle. If othewise, each Chern class must be positive.

Recall that

$H^2(X, \mathbf{Z}) \cong \mathbf{Z}L$ since Fano is simply connceted,

$H^4(X, \mathbf{Z}) \cong \mathbf{Z}l$ modulo the torsion part,

$H^6(X, \mathbf{Z}) \cong \mathbf{Z}$,

2-cycle $L \cdot L = L^2$ in $H^4(X, \mathbf{Z})$ is homologically equivalent to dl from (1.10) with $d = L^3$.

(4.11) i) $(\frac{b}{2} - \frac{2}{a}) \in \mathbf{Z}$
ii) $(\frac{1}{a^2} - \frac{b}{a})d \in \mathbf{Z}$
iii) $\frac{b}{2a^3}d \in \mathbf{Z}$

(4.12) Let D be a general fiber of p. To continue this argument, we study the morphism $\pi_D (:= \pi|_D) : D \to Z$ between Fano varieties.

we have the following conditions:

1. $\rho(Z) = \rho(D) = 1$, namely $PicD \cong \mathbf{Z}W$, $PicZ \cong \mathbf{Z}L$ with ample line bundles L, W,
2. $\pi_D^* L = \alpha W$,
3. $-K_D = W$ and $-K_Z = zL$.

Thus we get

(4.12.1) $\pi_D^* L = aW$ and $d = a^3 W^3 / L^3$.

Recall that a,b are integers.

(4.13) We divide two cases

I. $b = 2b'$, $b' \in \mathbf{Z}$

II. $b = 2b' + 1$

Consequently we get

Proposition 4.15 Let X be a Fano 4-fold of Picard number 2 which has a surjective morphism $p : X \to \mathbf{P}^1$ with connected fibers. Assume that the index of a general fiber F of p is one and that X has a standard conic bundle structure $\pi : X \to Z$. Then p has a singular fiber, unless $4|(-K_F)^3$.

For the proof use the adjunction formula of $\pi_D : D \to Z$ for a general fiber D.

5. The proof of Theorem B (divisorial contraction 3.1.2.)

We discuss about the existence of divisorial contraction 3.1.2.

As a result, under the conditions and assumptions in Theorem B we show that the case 3.1.2 does not happen by Corollary 3.7 and Corollary 5.5.

(5.1) Let X be a Fano 4-fold of Picard number 2 which has a surjective morphism $p : X \to \mathbf{P}^1$ with connected fibers. Moreover let $g : X \to Z$ be a blowing-up of an $n(\geq 4)$-dimensional smooth projective variety Z along a smooth subvariety B of codimension 2. Let E be the exceptional locus of g.

(5.1.1) Assume that

1) a general smooth fiber of $p : X \to P^1$ is of Picard number 1. (if p is a smooth morphism, 1) holds.)

Note that each fiber of the induced morphism $p|_E : E \to \mathbf{P}^1$ is irreducible,

Let $\phi(= (p,g)) : X \to \mathbf{P}^1 \times Z$ be an induced morphism and $X_y = p^{-1}(y)$.
If $\phi|_E : E \to \mathbf{P}^1 \times B$ is an isomorphism, we get
(5.1.2) $(g^{-1}(b), X_y) = 1$ for a point b in B.

Thus $g(X_y)$ is normal and therefore is smooth by Zariski Main Theorem.
$g(X_y)$ is smooth aound a neighborhood of the closed subvariety F. Moreover we see the pull-back $g^*g(X_y)$ of a divisor $g(X_y)$ in X' is linearly equivalent to $X_y + E$ in X

We show the case 3.1.2 satisfies the condition 5.1.2.
Z is a Fano variety of Picard number 1 and $\{g(X_y)|y \in \mathbf{P}^1\}$ is an algebraic subfamily of $|\mathcal{O}_{X'}(c)|$ with a positive integer c. For each two point y, y' in $\mathbf{P}^1$ $g(X_y) \cap g(X_{y'})$ is purely 2 codimensional irreducible subscheme which is, as a set, equal to B.
We have

Proposition 5.2. Let us maintain the condition (5.1). Then we have isomorphisms: $\pi_1^{alg}(Z) \cong \pi_1^{alg}(g(X_y) \cap g(X_{y'})) \cong \pi_1^{alg}((g(X_y) \cap g(X_{y'}))_{red}) = \pi_1^{alg}(B)$. Thus B is algebraically simply connected.

From Theorem 1.3 we have
Corollary 5.3 Under 5.1.1 we have $\phi|_E : E \to \mathbf{P}^1 \times B$ is an isomorphism. Furthermore if a fiber $p^{-1}(y)$ is smooth, so is the image $g(p^{-1}(y))$

So far we dont assume p is smooth.
Remark 5.3.1 If the surjective morphism p is smooth, then the algebraic family $\{g(X_y)|y \in \mathbf{P}^1\}$ is s subfamily of complete linear system $|\mathcal{O}_{X'}(g(X_y))|$ which consists of smooth divisors in X' parameterized by $\mathbf{P}^1$.

Now we study to what extent the phenomena of Remark occurs.

Lemma 5.4 Let M be a smooth projective variety, L an ample line bundle on M and $\mathcal{D} = \{D_t|t \in C\}$ be an algebraic family which is a subset of complete linear system $|L|$ parameterized by a projective curve C.
Assume that
1. For each t, D_t is smooth and $D_t|_{D_t}$ is very ample in D_t.
2. $H^1(X, \mathcal{O}_X) = 0$
Then L is very ample.

Corollary 5.5 Let X be a smooth Fano 4-fold as in 5.1. Assume the index of a general smooth fiber F of p is 1. If $-K_F$ is very ample, then p has a singular fiber.

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