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Algebraic independence of the power series related to the beta expansions of real numbers

Hajime Kaneko *
JSPS, College of Science and Technology, Nihon University

Abstract

In this paper we review known results on the β-expansions of algebraic numbers. We also review applications to the transcendence of real numbers. Moreover, we give a new flexible criterion for the algebraic independence of two real numbers.

1 Introduction

In this paper we study the β-expansions of real numbers. There is little known on the digits of the β-expansions of given real numbers. For instance, let b be an integer greater than 1. Borel [3] conjectured that any algebraic irrational numbers are normal in base-b. However, there is no known examples of algebraic irrational number whose normality has been proved.

The study of base-b expansions and generally β-expansions of algebraic numbers is applicable to criteria for transcendence of real numbers. In this paper we introduce known results on the transcendence of real numbers related to the β-expansions. Moreover, we also study applications to algebraic independence of real numbers. In particular, in Section 2 we introduce criteria for algebraic independence. The criteria is flexible because it does not depend on functional equations. We prove main results in Section 3.

For a real number x, we denote the integral and fractional parts of x by $\lfloor x\rfloor$ and $\{x\}$, respectively. We use the Landau symbol o and the Vinogradov symbol $\ll$ with their regular meanings.

Let $\beta>1$ be a real number. We recall the definition of the β-expansions of real numbers introduced by Rényi [8]. Let $T_\beta:[0,1)\to[0,1)$ be the β-transformation defined by $T_\beta(x):=\{\beta x\}$ for $x\in[0,1)$. For a real number ξ with $\xi\in[0,1)$, the β-expansion of ξ is defined by

$$\xi=\sum_{n=1}^{\infty}t_n(\beta;\xi)\beta^{-n},$$

where $t_n(\beta;\xi)=\lfloor\beta T_\beta^{n-1}(\xi)\rfloor$ for $n=1,2,\ldots$.

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We introduce known results on the nonzero digits of the β-expansions of algebraic numbers. Put

$$S_\beta(\xi) := \{n \geq 1 \mid t_n(\beta;\xi) \neq 0\}$$

and, for a real number x,

$$\lambda_\beta(\xi;x) := \mathrm{Card}(S_\beta(x) \cap [1,x]),$$

where Card denotes the cardinality. If $\beta = b > 1$ is an integer, then put, for any real number $\xi > 0$,

$$S_b(\xi) := S_b(\{\xi\}), \lambda_b(\xi;x) := \lambda_b(\{\xi\};x)$$

for convenience. Bailey, Borwein, Crandall and Pomerance [2] showed that if $\beta = 2$, then, for any algebraic irrational number ξ of degree D, there exist positive constants C_1 and C_2, depending only on ξ, satisfying

$$\lambda_2(\xi;N) \geq C_1 N^{1/D}$$

for any integer $N \geq C_2$. Note that C_1 is effectively computable but C_2 is not. Adamczewski, Faverjon [1], and Bugeaud [4] independently proved effective versions of lower bounds for $\lambda_b(\xi;N)$ for an arbitrary integral base $b \geq 2$. Namely, if $\xi > 0$ is an algebraic number of degree D, then there exist effectively computable positive constants $C_3(b,\xi)$ and $C_4(b,\xi)$ such that

$$\lambda_b(\xi;N) \geq C_3(b,\xi) N^{1/D}$$

for any integer $N \geq C_4(b,\xi)$.

Next, we consider the case where β is a Pisot or Salem number. Recall that Pisot numbers are algebraic integers greater than 1 whose conjugates except themselves have absolute values less than 1. Salem numbers are algebraic integers greater than 1 such that the conjugates except themselves have absolute values not greater than 1 and that at least one conjugate has absolute value 1. Let β be a Pisot or Salem number and $\xi \in [0,1)$ an algebraic number such that there exists infinitely many nonzero digits in the β-expansion, namely,

$$\lim_{N\to\infty} \lambda_\beta(\xi;N) = \infty.$$

Put $D := [\mathbb{Q}(\beta,\xi) : \mathbb{Q}(\beta)]$ which denotes the degree of a field extension. Then the author [7] showed that there exist effectively computable positive constants $C_5(\beta,\xi)$ and $C_6(\beta,\xi)$ such that

$$\lambda_\beta(\xi;N) \geq C_5(\beta,\xi) \frac{N^{1/(2D-1)}}{(\log N)^{1/(2D-1)}} \tag{1.1}$$

for any integer $N \geq C_6(\beta,\xi)$. The inequality (1.1) gives criteria for transcendence of real numbers.

THEOREM 1.1 ([7]). *Let β be a Pisot or Salem number and $\xi \in [0,1)$ a real number such that*

$$\lim_{N\to\infty} \lambda_\beta(\xi;N) = \infty.$$

Assume for an arbitrary $\varepsilon > 0$ that

$$\liminf_{N\to\infty} \frac{\lambda_\beta(\xi;N)}{N^\varepsilon} = \infty.$$

Then ξ is transcendental.

If $\beta = b > 1$ is an integer, then the transcendence of ξ in Theorem 1.1 was essentially proved by Bailey, Borwein, Crandall and Pomerance [2].

In what follows, we consider the transcendence of the values of the form

$$\sum_{n=0}^{\infty} \alpha^{\lfloor f(n) \rfloor},$$

where α is an algebraic number with $0 < |\alpha| < 1$ and f is a nonnegative valued function such that

$$\lfloor f(n) \rfloor < \lfloor f(n+1) \rfloor$$

for any sufficiently large integer n. The transcendence of such values is known if $f(n)$ $(n = 0, 1, \ldots)$ is a lacunary sequence. In fact, Corvaja and Zannier [5] showed that if

$$\liminf_{n\to\infty} \frac{f(n+1)}{f(n)} > 1,$$

then, for any algebraic number α with $0 < |\alpha| < 1$, the value $\sum_{n=0}^{\infty} \alpha^{\lfloor f(n) \rfloor}$ is transcendental. For instance, let h be a real number with $h > 1$. Then, for any algebraic number α with $0 < |\alpha| < 1$, the value

$$\sum_{n=0}^{\infty} \alpha^{\lfloor h^n \rfloor} \tag{1.2}$$

is transcendental. Note that if h is an integer, then (1.2) is called a Fredholm series.

However, it is generally difficult to study the transcendence in the case where $f(n)$ $(n = 0, 1, \ldots)$ is not lacunary. Theorem 1.1 is applicable for certain classes of functions f which are not lacunary; assume for an arbitrary positive number A that

$$\limsup_{n\to\infty} \frac{f(n)}{n^A} = \infty, \tag{1.3}$$

then, for any Pisot or Salem number β, the value $\sum_{n=0}^{\infty} \beta^{-\lfloor f(n) \rfloor}$ is transcendental. We give examples of f satisfying (1.3). For convenience, we denote

$$\log^+ x := \log \max\{e, x\}$$

for a real number $x \geq 0$. For any real numbers ζ and η with $\zeta > 0$, or $\zeta = 0$ and $\eta > 0$, put

$$\begin{aligned}\psi(\zeta, \eta; x) &:= x^{(\log^+ x)^\zeta (\log^+ \log^+ x)^\eta} \\ &= \exp\left((\log^+ x)^{1+\zeta}(\log^+ \log^+ x)^\eta\right).\end{aligned}$$

In particular, put

$$\begin{aligned}\varphi(x) &:= \psi(1, 0; x) = x^{\log^+ x}, \\ \psi(x) &:= \psi(0, 1; x) = x^{\log^+ \log^+ x}.\end{aligned}$$

If β is a Pisot or Salem number, then the number

$$\sum_{n=0}^{\infty} \beta^{-\lfloor \psi(\zeta,\eta;n) \rfloor}$$

is transcendental for any real numbers ζ and η with $\zeta > 0$, or $\zeta = 0$ and $\eta > 0$. In fact,

$$\limsup_{n\to\infty} \frac{\psi(\zeta, \eta; n)}{n^A} = \infty$$

for any positive real number A. Note that $\psi(\zeta, \eta; n)$ $(n = 0, 1, \ldots)$ is not lacunary because

$$\lim_{n\to\infty} \frac{\psi(\zeta, \eta; n+1)}{\psi(\zeta, \eta; n)} = 1.$$

In Section 2 we investigate the algebraic independence of real numbers in the case where $\beta = b > 1$ is an integer. In particular, Corollary 2.4 implies that

$$\sum_{n=0}^{\infty} b^{-\lfloor \psi(n) \rfloor}, \ \sum_{n=0}^{\infty} b^{-\lfloor \varphi(n) \rfloor}$$

are algebraically independent.

2 Main results

We introduce the criteria for algebraic independence in [6]. Let S be a nonempty subset of $\mathbb{N}$ and k a nonnegative integer. Put

$$kS := \begin{cases} \{0\} & (k = 0), \\ \{s_1 + \cdots + s_k \mid s_i \in S \text{ for any } i = 1, \ldots, k\} & (k \geq 1). \end{cases}$$

For any real number x with $x > \min\{n \in S\}$, let

$$\theta(x; S) := \max\{n \in S \mid n < x\}.$$

Moreover, let r be a positive integer. Then, for any nonempty subsets $S_1, \ldots, S_r$ of $\mathbb{N}$ and $k_1, \ldots, k_r \in \mathbb{N}$, we set

$$k_1 S_1 + \cdots + k_r S_r := \{t_1 + \cdots + t_r \mid t_i \in k_i S_i \text{ for any } i = 1, \ldots, r\}.$$

THEOREM 2.1 (Theorem 2.1 in [6]). *Let $r \geq 2$ be an integer and $\xi_1, \ldots, \xi_r$ positive real numbers satisfying the following three assumptions:*

1. *For any $\varepsilon > 0$, we have, as x tends to infinity,*

$$\begin{aligned} \lambda_b(\xi_1; x) &= o(x^{\varepsilon}), & (2.1)\\ \lambda_b(\xi_i; x) &= o(\lambda_b(\xi_{i-1}; x)^{\varepsilon}) \text{ for } i = 2, \ldots, r. & (2.2) \end{aligned}$$

2. *There exists a positive constant C_7 such that*

$$S_b(\xi_r) \cap [C_7 x, x] \neq \emptyset \qquad (2.3)$$

for any sufficiently large $x \in \mathbb{R}$.

3. *Let $k_1, \ldots, k_{r-1}, k_r$ be nonnegative integers. Then there exist a positive integer $\tau = \tau(k_1, \ldots, k_{r-1})$ and a positive constant $C_8 = C_8(k_1, \ldots, k_{r-1}, k_r)$, both depending only on the indicated parameters, such that*

$$\begin{aligned} &x \prod_{i=1}^{r} \lambda_b(\xi_i; x)^{-k_i} \\ &> x - \theta(x; k_1 S_b(\xi_1) + \cdots + k_{r-2} S_b(\xi_{r-2}) + \tau S_b(\xi_{r-1})) \end{aligned}$$

for any $x \in \mathbb{R}$ with $x \geq C_8$.

The first assumption of Theorem 2.1 implies that, for any $\varepsilon > 0$, we have,

$$\lambda_b(\xi_h; x) = o(x^{\varepsilon}) \text{ for } i = 1, \ldots, r$$

as x tends to infinity. Thus, the transcendence of $\xi_1, \ldots, \xi_r$ follows from Theorem 1.1. Using Theorem 2.1, we deduce the following:

THEOREM 2.2 (Theorems 1.3 and 1.4 in [6]). *Let b be an integer greater than 1.*
(1) *The continuum set*

$$\left\{ \sum_{n=0}^{\infty} b^{-\psi(\zeta,0;n)} \,\middle|\, \zeta \geq 1, \zeta \in \mathbb{R} \right\}$$

is algebraically independent.
(2) *For any distinct positive real numbers ζ and ζ', the numbers*

$$\sum_{n=0}^{\infty} b^{-\psi(\zeta,0;n)} \text{ and } \sum_{n=0}^{\infty} b^{-\psi(\zeta',0;n)}$$

are algebraically independent.

In the rest of this section we consider algebraically independence of two real numbers. We call the third assumption of Theorem 2.1 Assumption A. We give another condition for Assumption A as follows: Let k be any nonnegative integer. Then there exists a positive integer $\sigma = \sigma(k)$, depending only on k, such that

$$x\lambda_b(\xi_1; x)^{-k} > x - \theta(x; \sigma S(\xi_1))$$

for any sufficiently large x. We call the condition above Condition B. We show that if the first assumption of Theorem 2.1 holds, then assumption A is equivalent to Condition B. First, Assumption A implies Condition B, by taking $k_1 = k$ and $k_2 = 0$. Conversely, we assume that Condition B holds. Let k_1 and k_2 be nonnegative integers. Then the first assumption of Theorem 2.1 implies that

$$x\lambda_b(\xi_1; x)^{-k_1}\lambda_b(\xi_2; x)^{-k_2} > x\lambda_b(\xi_1; x)^{-1-k_1}$$

for any sufficiently large $x \in \mathbb{R}$. Thus, using Condition B with $k = 1 + k_1$, we get

$$x\lambda_b(\xi_1; x)^{-k_1}\lambda_b(\xi_2; x)^{-k_2} > x - \theta(x; \sigma S(\xi_1))$$

for any sufficiently large $x \in \mathbb{R}$, where $\sigma = \sigma(1 + k_1)$. Hence, we checked Assumption A.

We give criteria for algebraic independence of two real numbers. Let f be a nonnegative valued function defined on $[0, \infty)$. We call f ultimately increasing if there exists a positive M such that f is strictly increasing on $[M, \infty)$.

THEOREM 2.3. *Let $f(x)$ and $u(x)$ be ultimately increasing nonnegative valued functions defined on $[0, \infty)$. Let $g(x)$ and $v(x)$ be the inverse functions of $f(x)$ and $u(x)$, respectively. Suppose that*

$$\lfloor f(n+1)\rfloor > \lfloor f(n)\rfloor, \ \lfloor u(n+1)\rfloor > \lfloor u(n)\rfloor \tag{2.4}$$

for any sufficiently large integer n. Assume that f satisfies the following two assumptions:

1. *The function $(\log f(x))/(\log x)$ is ultimately increasing. Moreover,*

$$\lim_{x\to\infty} \frac{\log f(x)}{\log x} = \infty. \tag{2.5}$$

2. *The function $f(x)$ is differentiable. Moreover, there exists a positive real number δ such that*

$$(\log f(x))' < x^{-\delta} \tag{2.6}$$

for any sufficiently large $x \in \mathbb{R}$.

Moreover, suppose that $u(x)$ fulfills the following two assumptions:

1. *There exists a positive constant C_9 such that*

$$\frac{u(x+1)}{u(x)} < C_9 \tag{2.7}$$

for any sufficiently large $x \in \mathbb{R}$.

2.

$$\lim_{x\to\infty}\frac{\log g(x)}{\log v(x)}=\infty. \tag{2.8}$$

Then, for any integer $b \geq 2$, the numbers

$$\sum_{n=0}^{\infty} b^{-\lfloor f(n)\rfloor},\ \sum_{n=0}^{\infty} b^{-\lfloor u(n)\rfloor}$$

are algebraically independent.

The assumptions on f in Theorem 2.3 give a sufficient condition for Condition B. Note that (2.5) and (2.6) are easy to check because these depend only on the asymptotic behavior of $\log f(x)$. We deduce examples of algebraic independent real numbers as follows:

COROLLARY 2.4. *For any integer $b \geq 2$, the numbers*

$$\sum_{n=0}^{\infty} b^{-\lfloor \psi(n)\rfloor},\ \sum_{n=0}^{\infty} b^{-\lfloor \varphi(n)\rfloor}$$

are algebraically independent.

The following corollary is a generalization of the second statement of Theorem 2.2 and Corollary 2.4.

COROLLARY 2.5. *Let $\zeta, \zeta', \eta, \eta'$ be real numbers. Suppose that $\zeta > 0$, or $\zeta = 0, \eta > 0$ and that $\zeta' > 0$, or $\zeta' = 0, \eta' > 0$. If $(\zeta, \eta) \neq (\zeta', \eta')$, then for any integer $b \geq 2$, the numbers*

$$\sum_{n=0}^{\infty} b^{-\lfloor \psi(\zeta,\eta;n)\rfloor},\ \sum_{n=0}^{\infty} b^{-\lfloor \psi(\zeta',\eta';n)\rfloor}$$

are algebraically independent.

Theorem 2.3 is applicable to the algebraic independence of two real numbers including Fredholm series.

COROLLARY 2.6. *Let ζ, η, be real numbers with $\zeta > 0$, or $\zeta = 0, \eta > 0$. Let h be a real number with $h > 1$. Then, for any integer $b \geq 2$, the numbers*

$$\sum_{n=0}^{\infty} b^{-\lfloor \psi(\zeta,\eta;n)\rfloor},\ \sum_{n=0}^{\infty} b^{-\lfloor h^n\rfloor}$$

are algebraically independent.

3 Proof of main results

In this section we verify Theorem 2.3, using Theorem 2.1. We also show the corollaries of Theorem 2.3.

Proof of Theorem 2.3. Put

$$\xi_1 := \sum_{n=0}^{\infty} b^{-\lfloor f(n) \rfloor}, \quad \xi_2 := \sum_{n=0}^{\infty} b^{-\lfloor u(n) \rfloor}.$$

We verify that ξ_1 and ξ_2 satisfy the assumptions of Theorem 2.1. If necessary, changing finite terms of $f(n)$, we may assume that $S_b(\xi_1) \ni 0$. First, (2.1) and (2.2) follow from (2.5) and (2.8), respectively. In fact, we see

$$\lim_{x\to\infty} \frac{\log \lambda_b(\xi_1; x)}{\log x} = \lim_{x\to\infty} \frac{\log g(x)}{\log x} = 0$$

and

$$\lim_{x\to\infty} \frac{\log \lambda_b(\xi_2; x)}{\log \lambda_b(\xi_1; x)} = \lim_{x\to\infty} \frac{\log v(x)}{\log g(x)} = 0.$$

Moreover, we see (2.3) by (2.7). Thus, we checked the first and second assumptions of Theorem 2.1. In what follows, we prove the third assumption. As we mentioned after Theorem 2.1, it suffices to check Condition B.

LEMMA 3.1. *For any positive integer l, we have*

$$R - \theta(R; lS_b(\xi_1)) \ll Rg(R)^{-l\delta/2} \tag{3.1}$$

for any $R \geq 1$.

Proof. We show (3.1) by induction on l. First we consider the case of $l = 1$. By (2.6) and the mean value theorem, there exists $\iota = \iota(x) \in (0,1)$ such that

$$\log\left(\frac{f(x+1)}{f(x)}\right) < (x+\iota)^{-\delta} < 1.$$

Thus,

$$f(x+1) < ef(x) \tag{3.2}$$

for any sufficiently large x. Using (2.6) and the mean value theorem again, we see for any sufficiently large x that there exists $\rho = \rho(x) \in (0,1)$ such that

$$f(x+1) - f(x) = f'(x+\rho) < \frac{f(x+\rho)}{(x+\rho)^\delta}. \tag{3.3}$$

Combining (3.2) and (3.3), we get

$$f(x+1) - f(x) \ll \frac{f(x+1)}{(x+1)^\delta} \ll \frac{f(x)}{(x+1)^\delta}. \tag{3.4}$$

By (2.4), if R is sufficiently large, then there exists a unique integer $m \geq 0$ such that

$$\lfloor f(m) \rfloor < R \leq \lfloor f(m+1) \rfloor. \tag{3.5}$$

Hence, using (3.4), we obtain

$$\begin{aligned} R - \theta(R; S_b(\xi_1)) &= R - \lfloor f(m) \rfloor \\ &\leq f(m+1) - f(m) + 1 \ll \frac{f(m)}{(m+1)^\delta}, \end{aligned}$$

where we use $f(m) > (m+1)^\delta$ for the last inequality. By (3.5), we deduce for any sufficiently large R that

$$R - \theta(R; S_b(\xi_1)) \ll \frac{R}{g(f(m+1))^\delta} \leq \frac{R}{g(R)^\delta}, \tag{3.6}$$

which implies (3.1) with $l = 1$.

Next, we assume that $l \geq 2$. Put

$$R' := R - \theta(R; (l-1)S_b(\xi_1)).$$

Since $S_b(\xi_1) \ni 0$, we have $(l-1)S_b(\xi_1) \subset lS_b(\xi_1)$ and so

$$R - \theta(R; lS_b(\xi_1)) \leq R'.$$

Hence, for the proof of (3.1), we may assume that

$$R' \geq Rg(R)^{-l\delta/2}.$$

In particular, (2.5) implies that if R is sufficiently large, then

$$R' \geq R^{1/2}. \tag{3.7}$$

The inductive hypothesis implies that

$$R' \ll Rg(R)^{-(l-1)\delta/2}. \tag{3.8}$$

Observe that

$$\theta(R; (l-1)S_b(\xi_1)) + \theta(R'; S_b(\xi_1)) \in lS_b(\xi_1)$$

and that

$$\begin{aligned} &\theta(R; (l-1)S_b(\xi_1)) + \theta(R'; S_b(\xi_1)) \\ &< \theta(R; (l-1)S_b(\xi_1)) + R' = R \end{aligned}$$

by the definition of R'. Thus, we see

$$\begin{aligned} &R - \theta(R; lS_b(\xi_1)) \\ &\leq R - \theta(R; (l-1)S_b(\xi_1)) - \theta(R'; S_b(\xi_1)) \\ &= R' - \theta(R'; S_b(\xi_1)) \ll \frac{R'}{g(R')^\delta} \end{aligned} \tag{3.9}$$

by (3.6). Combining (3.7), (3.8), and (3.9), we obtain for any sufficiently large R that

$$\begin{aligned} R-\theta(R; lS_b(\xi_1)) &\ll R' g(R^{1/2})^{-\delta} \\ &\ll Rg(R)^{-(l-1)\delta/2} g(R^{1/2})^{-\delta}. \end{aligned} \tag{3.10}$$

We use the assumption that the function $(\log f(x))/(\log x)$ is ultimately increasing. Considering the cases of $x = g(R)$ and $x = g(R^{1/2})$, we see

$$\frac{\log R}{\log g(R)} \geq \frac{\log R^{1/2}}{\log g(R^{1/2})} = \frac{1}{2}\frac{\log R}{\log g(R^{1/2})}.$$

Thus, we obtain

$$\log g(R^{1/2}) \geq \frac{1}{2}\log g(R),$$

and so

$$g(R^{1/2}) \geq g(R)^{1/2} \tag{3.11}$$

for any sufficiently large R. Hence, combining (3.10) and (3.11), we deduce for any sufficiently large R that

$$R-\theta(R; lS_b(\xi_1)) \ll Rg(R)^{-l\delta/2},$$

which implies (3.1) □

Lemma 3.1 implies that ξ_1 satisfies Condition B. Finally, we proved Theorem 2.3. □

In what follows, we prove the corollaries of Theorem 2.3. Since Corollary 2.4 follows from Corollary 2.5, we only verify Corollaries 2.5 and 2.6.

Proof of Corollary 2.5. Without loss of generality, we may assume that $\zeta < \zeta'$, or $\zeta = \zeta'$ and $\eta < \eta'$. Put

$$f(x) := \psi(\zeta, \eta; x),\ u(x) := \psi(\zeta', \eta'; x).$$

For any sufficiently large x, we have

$$\frac{\log f(x)}{\log x} = (\log x)^{\zeta}(\log\log x)^{\eta},$$

which implies that the first assumption on f in Theorem 2.3 holds since $\zeta > 0$, or $\zeta = 0$ and $\eta > 0$. Moreover, using

$$(\log f(x))' = \begin{cases} (1+\zeta)(\log x)^{\zeta}/x & (\eta = 0), \\ (\log x)^{\zeta}(\log\log x)^{\eta-1}\cdot(\eta + (1+\zeta)\log\log x)/x & (\eta \neq 0). \end{cases}$$

Thus, we checked the second assumption on f in Theorem 2.3. Similarly, since

$$(\log u(x))' < 1$$

for any sufficiently large $x \in \mathbb{R}$, we see (2.7).

In what follows, we prove (2.8). Since $g(x)$ and $v(x)$ are inverse functions of $f(x)$ and $u(x)$, respectively, we have

$$\begin{aligned}(\log g(x))^{1+\zeta}(\log\log g(x))^{\eta} &= \log x &(3.12)\\ (\log v(x))^{1+\zeta'}(\log\log v(x))^{\eta'} &= \log x &(3.13)\end{aligned}$$

First we assume that $\zeta < \zeta'$. Let $d := \zeta' - \zeta > 0$. Using (3.12) and (3.13), we get, for any sufficiently large x,

$$(\log v(x))^{1+\zeta+(2d)/3} < \log x < (\log g(x))^{1+\zeta+d/3},$$

and so

$$\lim_{x\to\infty} \frac{\log g(x)}{\log v(x)} = \infty.$$

Next we consider the case of $\zeta = \zeta'$ and $\eta < \eta'$. We see by (3.12) and (3.13) that

$$\frac{(\log\log v(x))^{\eta'}}{(\log\log g(x))^{\eta}} = \left(\frac{\log g(x)}{\log v(x)}\right)^{1+\zeta}. \tag{3.14}$$

Taking the logarithms of both sides of (3.14), we get

$$\begin{aligned}&\eta' \log\log\log v(x) - \eta \log\log\log g(x)\\ &= (1+\zeta)\log\log g(x) - (1+\zeta)\log\log v(x) \qquad (3.15)\end{aligned}$$

Since $g(x) \geq v(x)$ for any sufficiently large x, dividing both sides of (3.15) by $\log\log g(x)$, we get

$$\lim_{x\to\infty} \frac{\log\log v(x)}{\log\log g(x)} = 1. \tag{3.16}$$

Thus, by $\eta' > \eta$, we obtain by (3.14) and (3.16) that

$$\lim_{x\to\infty} \frac{\log g(x)}{\log v(x)} = \infty.$$

Finally, we verified (2.8). □

Proof of Corollary 2.6. In the proof of Corollary 2.5, We checked that (2.5) and (2.6) are satisfied. Moreover, (2.7) and (2.8) are easily seen because $u(x) = h^x$ and

$$v(x) = \log_h x = \frac{\log x}{\log h}. \tag{3.17}$$

In fact, comparing (3.12) and (3.17), we see

$$\lim_{x\to\infty} \frac{\log g(x)}{\log v(x)} = \infty.$$

□

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