

# Convolution identities for Cauchy numbers of the first kind and of the second kind

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## 1 Introduction

The Cauchy numbers of the first kind  $c_n$  ( $n \geq 0$ ) are defined by

$$c_n = \int_0^1 x(x-1)\dots(x-n+1)dx$$

and the generating function of  $c_n$  is given by

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!} \quad (|x| < 1)$$

([4, 11]). The first few values are

$$c_0 = 1, \quad c_1 = \frac{1}{2}, \quad c_2 = -\frac{1}{6}, \quad c_3 = \frac{1}{4}, \quad c_4 = -\frac{19}{30}, \quad c_5 = \frac{9}{4}, \quad c_6 = -\frac{863}{84}.$$

The Cauchy numbers are almost identical with the Bernoulli numbers of the second kind,  $b_n$ , as introduced in [8] and later studied by various authors. In fact, we have  $c_n = n!b_n$ . Notice that the Cauchy numbers  $c_n$  can be expressed in terms of the unsigned Stirling numbers of the first kind and  $\left[ \begin{smallmatrix} n \\ i \end{smallmatrix} \right]$ :

$$c_n = \sum_{i=0}^n \frac{(-1)^{n-i}}{i+1} \left[ \begin{smallmatrix} n \\ i \end{smallmatrix} \right] \tag{1}$$

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([9, 11]), where the unsigned Stirling numbers of the first kind  $\left[ \begin{smallmatrix} n \\ i \end{smallmatrix} \right]$  arise as coefficients of the rising factorial

$$x(x+1)\dots(x+n-1) = \sum_{i=0}^n \left[ \begin{smallmatrix} n \\ i \end{smallmatrix} \right] x^i$$

(see e.g. [7]).

The Cauchy numbers of the second kind  $\widehat{c}_n$  ( $n \geq 0$ ) are defined by

$$\widehat{c}_n = \int_0^1 (-x)(-x-1)\dots(-x-n+1)dx$$

and the generating function of  $\widehat{c}_n$  is given by

$$\frac{x}{(1+x)\ln(1+x)} = \sum_{n=0}^{\infty} \widehat{c}_n \frac{x^n}{n!} \quad (|x| < 1)$$

([4, 11]). The first few values are

$$\widehat{c}_0 = 1, \widehat{c}_1 = -\frac{1}{2}, \widehat{c}_2 = \frac{5}{6}, \widehat{c}_3 = -\frac{9}{4}, \widehat{c}_4 = \frac{251}{30}, \widehat{c}_5 = -\frac{475}{12}, \widehat{c}_6 = \frac{19087}{84}.$$

With the classical umbral calculus notation (see e.g. [12]), define  $(c_l + \widehat{c}_m)^n$  and  $(\widehat{c}_l + \widehat{c}_m)^n$  for  $l, m, n \geq 0$  by

$$(c_l + c_m)^n := \sum_{j=0}^n \binom{n}{j} c_{l+j} c_{m+n-j},$$

$$(\widehat{c}_l + \widehat{c}_m)^n := \sum_{j=0}^n \binom{n}{j} \widehat{c}_{l+j} \widehat{c}_{m+n-j},$$

respectively. The analogous concept for the Bernoulli numbers  $B_n$ , defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \quad (|x| < 2\pi),$$

has been extensively studied by many authors, including Agoh and Dilcher ([1, 2, 5] and references there). Define

$$(B_l + B_m)^n := \sum_{j=0}^n \binom{n}{j} B_{l+j} B_{m+n-j}.$$

Then Euler's famous formula can be written as

$$(B_0 + B_0)^n = -nB_{n-1} - (n-1)B_n \quad (n \geq 1). \quad (2)$$

In [1] an expression for  $(B_l + B_m)^n$  was found. In addition, some initial cases were listed, including e.g.,

$$\begin{aligned} (B_0 + B_1)^n &= -\frac{1}{2}(n+1)B_n - \frac{1}{2}nB_{n+1}, \\ (B_0 + B_2)^n &= -\frac{1}{6}(n-1)B_n - \frac{1}{2}nB_{n+1} - \frac{1}{3}nB_{n+2}, \\ (B_1 + B_1)^n &= \frac{1}{6}(n-1)B_n - B_{n+1} - \frac{1}{6}(n+3)B_{n+2}. \end{aligned}$$

On the other hand, an analogous formula to (2) is given by

$$(c_0 + c_0)^n = -n(n-2)c_{n-1} - (n-1)c_n \quad (n \geq 0) \quad (3)$$

([13]). A different direction of extended convolution identities for Cauchy numbers is discussed in [10].

In this paper, we give an explicit formula for  $(c_l + c_m)^n$  and  $(\widehat{c}_l + \widehat{c}_m)^n$  ( $n \geq 0$ ) together with some initial cases, including

$$(c_0 + c_1)^n = -\frac{1}{2}(n+1)(n-1)c_n - \frac{1}{2}nc_{n+1}, \quad (4)$$

$$(c_0 + c_2)^n = \frac{n!}{6} \sum_{k=0}^n \frac{(-1)^{n-k}(k-1)c_k}{k!} - \frac{1}{6}n(2n+1)c_{n+1} - \frac{1}{3}nc_{n+2}, \quad (5)$$

$$(c_1 + c_1)^n = -\frac{n!}{6} \sum_{k=0}^n \frac{(-1)^{n-k}(k-1)c_k}{k!} - \frac{1}{6}n(n+5)c_{n+1} - \frac{1}{6}(n+3)c_{n+2} \quad (6)$$

and

$$(\widehat{c}_0 + \widehat{c}_0)^n = n! \sum_{k=0}^n (-1)^{n-k} \frac{\widehat{c}_k}{k!} - n\widehat{c}_n, \quad (7)$$

$$(\widehat{c}_0 + \widehat{c}_1)^n = -\frac{(n+1)!}{2} \sum_{k=0}^n (-1)^{n-k} \frac{\widehat{c}_k}{k!} - \frac{1}{2}n\widehat{c}_{n+1}, \quad (8)$$

$$\begin{aligned} (\widehat{c}_0 + \widehat{c}_2)^n &= \frac{n!}{12} \sum_{k=0}^n \frac{(-1)^{n-k}}{k!} (2k(n+2k-2) + 5(n-k+2)(n-k+1)) \widehat{c}_k \\ &\quad - \frac{n}{3}\widehat{c}_{n+2} - \frac{n}{2}\widehat{c}_{n+1}, \end{aligned} \quad (9)$$

$$\begin{aligned} (\widehat{c}_1 + \widehat{c}_1)^n &= \frac{n!}{12} \sum_{k=0}^n \frac{(-1)^{n-k}}{k!} ((n+1)(n+2) + k(8n-9k+19)) \widehat{c}_k \\ &\quad - \widehat{c}_{n+1} - \frac{n+3}{6}\widehat{c}_{n+2}. \end{aligned} \quad (10)$$

## 2 Fundamental results

Euler's identity (2) is an easy consequence of the formula

$$b(x)^2 = (1-x)b(x) - xb'(x),$$

where  $b(x) = x/(e^x - 1)$  (see e.g. [6]). Similarly, the identity (3) is obtained because  $c(x) = x/\ln(1+x)$  satisfies the formula

$$c(x)^2 = (1+x)c(x) - (1+x)xc'(x). \quad (11)$$

From the generating function for the  $c_n$  we obtain for  $i, \nu \geq 0$ ,

$$x^i c^{(\nu)}(x) = \sum_{n=0}^{\infty} \frac{n!}{(n-i)!} c_{n+\nu-i} \frac{x^n}{n!}. \quad (12)$$

Therefore the identity (11) immediately leads to the formula (3). Differentiating both sides of (11) with respect to  $x$  and dividing them by 2, we obtain

$$c(x)c'(x) = -\frac{1}{2}x(x+1)c''(x) - \frac{1}{2}xc'(x) + \frac{1}{2}c(x). \quad (13)$$

Then by applying (12), we have the identity (4).

In general, by differentiating both sides of (11)  $\mu$  times with respect to  $x$ , we have the following. The left-hand side is due to the General Leibniz's rule. The right-hand side can be proved by induction.

**Proposition 1.** For  $\mu \geq 0$

$$\begin{aligned} & \sum_{\kappa=0}^{\mu} \binom{\mu}{\kappa} c^{(\kappa)}(x) c^{(\mu-\kappa)}(x) \\ &= -(\mu-2)\mu c^{(\mu-1)}(x) - ((2\mu-1)x + \mu-1)c^{(\mu)}(x) - x(1+x)c^{(\mu+1)}(x). \end{aligned} \quad (14)$$

By applying (12) in Proposition 1, we obtain the following result.

**Theorem 1.** For  $\mu \geq 0$  and  $n \geq 0$  we have

$$\sum_{\kappa=0}^{\mu} \binom{\mu}{\kappa} (c_{\kappa} + c_{\mu-\kappa})^n = -(n+\mu-2)(n+\mu)c_{n+\mu-1} - (n+\mu-1)c_{n+\mu}.$$

**Examples.** If we put  $\mu = 0, 1$  in Theorem 1, we have the identities (3) and (4), respectively. If we put  $\mu = 2, 3, 4$  in Theorem 1, we obtain

$$(c_0 + c_2)^n + (c_1 + c_1)^n = -\frac{1}{2}n(n+2)c_{n+1} - \frac{1}{2}(n+1)c_{n+2}, \quad (15)$$

$$(c_0 + c_3)^n + 3(c_1 + c_2)^n = -\frac{1}{2}(n+1)(n+3)c_{n+2} - \frac{1}{2}(n+2)c_{n+3}, \quad (16)$$

$$\begin{aligned} & (c_0 + c_4)^n + 4(c_1 + c_3)^n + 3(c_2 + c_2)^n \\ &= -\frac{1}{2}(n+2)(n+4)c_{n+3} - \frac{1}{2}(n+3)c_{n+4}. \end{aligned} \quad (17)$$

Since  $\widehat{c}(x) = x/(1+x) \ln(1+x)$  satisfies the formula

$$\widehat{c}(x)^2 = -x\widehat{c}'(x) + \frac{1}{1+x}\widehat{c}(x), \quad (18)$$

by  $1/(1+x) = \sum_{i=0}^{\infty} (-1)^i x^i$  and the fact

$$x^i \widehat{c}^{(\nu)}(x) = \sum_{n=0}^{\infty} \frac{n!}{(n-i)!} \widehat{c}_{n+\nu-i} \frac{x^n}{n!} \quad (i, \nu \geq 0), \quad (19)$$

we have

$$(\widehat{c}_0 + \widehat{c}_0)^n = -n\widehat{c}_n + n! \sum_{k=0}^n (-1)^{n-k} \frac{\widehat{c}_k}{k!} \quad (n \geq 0),$$

which is the formula (7). Differentiating both sides of (18) by  $x$  and dividing them by 2, we obtain

$$\widehat{c}(x)\widehat{c}'(x) = -\frac{1}{2}x\widehat{c}''(x) - \frac{x}{2(1+x)}\widehat{c}'(x) - \frac{1}{2(1+x)^2}\widehat{c}(x). \quad (20)$$

Then by (19) and for  $k \geq 1$

$$\frac{1}{(1+x)^k} = \sum_{i=0}^{\infty} (-1)^i \binom{k+i-1}{i} x^i, \quad (21)$$

we have

$$\begin{aligned} \sum_{n=0}^{\infty} (\widehat{c}_0 + \widehat{c}_1)^n \frac{x^n}{n!} &= -\frac{1}{2} \sum_{n=0}^{\infty} n\widehat{c}_{n+1} \frac{x^n}{n!} - \frac{1}{2} \sum_{i=0}^{\infty} (-1)^i \sum_{n=0}^{\infty} \frac{n!}{(n-i-1)!} \widehat{c}_{n-i} \frac{x^n}{n!} \\ &\quad - \frac{1}{2} \sum_{i=0}^{\infty} (-1)^i (i+1) \sum_{n=0}^{\infty} \frac{n!}{(n-i)!} \widehat{c}_{n-i} \frac{x^n}{n!} \\ &= -\frac{1}{2} \sum_{n=0}^{\infty} n\widehat{c}_{n+1} \frac{x^n}{n!} - \frac{1}{2} \sum_{i=0}^{\infty} (-1)^i \sum_{n=0}^{\infty} \frac{(n+1)!}{(n-i)!} \widehat{c}_{n-i} \frac{x^n}{n!}, \end{aligned}$$

yielding the formula (8).

In general, by differentiating both sides of (18) by  $x$  at  $\mu$  times, we have the following.

**Proposition 2.** For  $\mu \geq 0$

$$\sum_{\kappa=0}^{\mu} \binom{\mu}{\kappa} \widehat{c}^{(\kappa)}(x) \widehat{c}^{(\mu-\kappa)}(x) = \sum_{\nu=0}^{\mu} (-1)^{\mu-\nu} \frac{\mu! \widehat{c}^{(\nu)}(x)}{\nu! (1+x)^{\mu-\nu+1}} - \mu \widehat{c}^{(\mu)}(x) - x \widehat{c}^{(\mu+1)}(x).$$

By applying (21) and (19) in Proposition 2, we obtain the following result.

**Theorem 2.** For  $\mu \geq 0$  and  $n \geq 0$  we have

$$\sum_{\kappa=0}^{\mu} \binom{\mu}{\kappa} (\widehat{c}_\kappa + \widehat{c}_{\mu-\kappa})^n = -(n+\mu)\widehat{c}_{n+\mu} + \sum_{\nu=0}^{\mu} \frac{\mu! n!}{\nu!} \sum_{i=0}^n (-1)^{\mu-\nu+i} \binom{\mu-\nu+i}{i} \frac{\widehat{c}_{n+\nu-i}}{(n-i)!}.$$

**Examples.** If we put  $\mu = 0, 1$  in Theorem 2, we have the identities (7) and (8), respectively. If we put  $\mu = 2, 3, 4$  in Theorem 2, we obtain

$$(\widehat{c}_0 + \widehat{c}_2)^n + (\widehat{c}_1 + \widehat{c}_1)^n = \frac{(n+2)!}{2} \sum_{k=0}^n (-1)^{n-k} \frac{\widehat{c}_k}{k!} - \frac{n+2}{2} \widehat{c}_{n+1} - \frac{n+1}{2} \widehat{c}_{n+2}, \quad (22)$$

$$\begin{aligned} & (\widehat{c}_0 + \widehat{c}_3)^n + 3(\widehat{c}_1 + \widehat{c}_2)^n \\ &= -\frac{(n+3)!}{2} \sum_{k=0}^n (-1)^{n-k} \frac{\widehat{c}_k}{k!} + \frac{(n+2)(n+3)}{2} \widehat{c}_{n+1} - \frac{n+3}{2} \widehat{c}_{n+2} - \frac{n+2}{2} \widehat{c}_{n+3}, \end{aligned} \quad (23)$$

$$\begin{aligned} & (\widehat{c}_0 + \widehat{c}_4)^n + 4(\widehat{c}_1 + \widehat{c}_3)^n + 3(\widehat{c}_2 + \widehat{c}_2)^n \\ &= \frac{(n+4)!}{2} \sum_{k=0}^n (-1)^{n-k} \frac{\widehat{c}_k}{k!} - \frac{(n+2)(n+3)(n+4)}{2} \widehat{c}_{n+1} + \frac{(n+3)(n+4)}{2} \widehat{c}_{n+2} \\ & \quad - \frac{n+4}{2} \widehat{c}_{n+3} - \frac{n+3}{2} \widehat{c}_{n+4}. \end{aligned} \quad (24)$$

Next, notice that  $c(x)$  satisfies the identity

$$3c(x)c''(x) = -(1+x)xc^{(3)}(x) - \frac{3}{2}xc''(x) + \frac{x}{2(1+x)}c'(x) - \frac{1}{2(1+x)}c(x).$$

Since by (12) we have

$$\begin{aligned} x^2c^{(3)}(x) &= \sum_{n=0}^{\infty} n(n-1)c_{n+1} \frac{x^n}{n!}, \\ xc^{(3)}(x) &= \sum_{n=0}^{\infty} nc_{n+2} \frac{x^n}{n!}, \\ xc''(x) &= \sum_{n=0}^{\infty} nc_{n+1} \frac{x^n}{n!} \end{aligned}$$

and

$$\begin{aligned} \frac{x}{1+x}c'(x) &= \left( \sum_{\nu=0}^{\infty} (-x)^\nu \right) \left( \sum_{n=0}^{\infty} nc_n \frac{x^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \left( n! \sum_{k=0}^n \frac{(-1)^{n-k} kc_k}{k!} \right) \frac{x^n}{n!} \end{aligned}$$

and

$$\frac{1}{1+x}c(x) = \sum_{n=0}^{\infty} \left( n! \sum_{k=0}^n \frac{(-1)^{n-k}c_k}{k!} \right) \frac{x^n}{n!},$$

we get

$$3(c_0 + c_2)^n = -nc_{n+2} - n \left( n + \frac{1}{2} \right) c_{n+1} + \frac{n!}{2} \sum_{k=0}^n \frac{(-1)^{n-k}(k-1)c_k}{k!},$$

yielding the identity (5). In addition, by the identities (5) and (15), we have

$$(c_1 + c_1)^n = -\frac{n!}{6} \sum_{k=0}^n \frac{(-1)^{n-k}(k-1)c_k}{k!} - \frac{1}{6}n(n+5)c_{n+1} - \frac{1}{6}(n+3)c_{n+2},$$

which is the identity (6).

Next, notice that  $\widehat{c}(x)$  satisfies the identity

$$\widehat{c}(x)\widehat{c}'(x) = -\frac{x}{3}\widehat{c}^{(3)}(x) - \frac{x}{2(1+x)}\widehat{c}''(x) + \frac{x}{6(1+x)^2}\widehat{c}'(x) + \frac{5}{6(1+x)^3}\widehat{c}(x).$$

By (19) and (21)

$$\begin{aligned} & (\widehat{c}_0 + \widehat{c}_2)^n \\ &= -\frac{n}{3}\widehat{c}_{n+2} - \frac{1}{2} \sum_{i=0}^{\infty} (-1)^i \frac{n!}{(n-i-1)!} \widehat{c}_{n-i+1} \\ & \quad + \frac{1}{6} \sum_{i=0}^{\infty} (-1)^i \frac{(i+1)n!}{(n-i-1)!} \widehat{c}_{n-i} + \frac{5}{6} \sum_{i=0}^{\infty} (-1)^i \frac{(i+2)(i+1)}{2} \frac{n!}{(n-i)!} \widehat{c}_{n-i} \\ &= -\frac{n}{3}\widehat{c}_{n+2} - \frac{n}{2}\widehat{c}_{n+1} + \frac{n!}{2} \sum_{i=0}^{n-2} \frac{(-1)^i}{(n-i-1)!} \widehat{c}_{n-i} \\ & \quad + \frac{n!}{6} \sum_{i=0}^{n-1} \frac{(-1)^i(i+1)}{(n-i-1)!} \widehat{c}_{n-i} + \frac{5n!}{12} \sum_{i=0}^{n-2} \frac{(-1)^i(i+2)(i+1)}{(n-i)!} \widehat{c}_{n-i} \\ &= -\frac{n}{3}\widehat{c}_{n+2} - \frac{n}{2}\widehat{c}_{n+1} \\ & \quad + \frac{n!}{12} \sum_{i=0}^n \frac{(-1)^i}{(n-i)!} (6(n-i)(n-i-1) + 2(i+1)(n-i) + 5(i+2)(i+1)) \widehat{c}_{n-i}, \end{aligned}$$



yielding the identity (9). In addition, by the identities (9) and (22), we have

$$(\widehat{c}_1 + \widehat{c}_1)^n = \frac{n!}{12} \sum_{k=0}^n \frac{(-1)^{n-k}}{k!} ((n+1)(n+2) + k(8n-9k+19)) \widehat{c}_k - \widehat{c}_{n+1} - \frac{n+3}{6} \widehat{c}_{n+2},$$

which is the identity (10).

**Proposition 3.** For  $m \geq 0$  we have

$$\begin{aligned} & \widehat{c}(x) \widehat{c}^{(m)}(x) \\ &= -\frac{x}{m+1} \widehat{c}^{(m+1)}(x) - \sum_{k=0}^{m-1} \binom{m}{k} \frac{c_{k+1}}{k+1} \frac{x}{(1+x)^{k+1}} \widehat{c}^{(m-k)}(x) + \frac{\widehat{c}_m}{(1+x)^{m+1}} \widehat{c}(x). \end{aligned} \tag{25}$$

**Lemma 1.** For  $n \geq 0$  we have

$$\widehat{c}^{(n)}(x) = \left( \frac{-1}{1+x} \right)^{n+1} \sum_{j=0}^n \frac{\widehat{g}_{n,j}}{(\ln(1+x))^{j+1}},$$

where

$$\widehat{g}_{n,j} = j! \left( n \begin{bmatrix} n \\ j+1 \end{bmatrix} - \begin{bmatrix} n \\ j \end{bmatrix} x \right).$$

*Proof.* Since  $\widehat{g}_{1,0} = 1$  and  $\widehat{g}_{1,1} = -x$ , for  $n = 1$  we have

$$\begin{aligned} \frac{d}{dx} \frac{x}{(1+x) \ln(1+x)} &= \frac{\ln(1+x) - x}{(1+x)^2 (\ln(1+x))^2} \\ &= \frac{1}{(1+x)^2} \left( \frac{\widehat{g}_{1,0}}{\ln(1+x)} + \frac{\widehat{g}_{1,1}}{(\ln(1+x))^2} \right). \end{aligned}$$

Assume that the result holds for  $n$ . Then

$$\begin{aligned}
& \frac{d^{n+1}}{dx^{n+1}} \frac{x}{(1+x)\ln(1+x)} \\
&= (n+1) \left( \frac{-1}{1+x} \right)^{n+2} \sum_{j=0}^n \frac{\widehat{g}_{n,j}}{(\ln(1+x))^{j+1}} \\
&\quad + \left( \frac{-1}{1+x} \right)^{n+1} \sum_{j=0}^n \frac{-j! \binom{n}{j} (\ln(1+x))^{j+1} - \widehat{g}_{n,j}(j+1)(\ln(1+x))^j \frac{1}{1+x}}{(\ln(1+x))^{2j+2}} \\
&= n \left( \frac{-1}{1+x} \right)^{n+2} \sum_{j=0}^n \frac{\widehat{g}_{n,j}}{(\ln(1+x))^{j+1}} + \left( \frac{-1}{1+x} \right)^{n+2} \sum_{j=0}^n \frac{\widehat{g}_{n,j}}{(\ln(1+x))^{j+1}} \\
&\quad + \left( \frac{-1}{1+x} \right)^{n+2} \sum_{j=0}^n \frac{j! \binom{n}{j} (1+x)}{(\ln(1+x))^{j+1}} + \left( \frac{-1}{1+x} \right)^{n+2} \sum_{j=0}^n \frac{\widehat{g}_{n,j}(j+1)}{(\ln(1+x))^{j+2}} \\
&= \left( \frac{-1}{1+x} \right)^{n+2} \sum_{j=0}^n \frac{n\widehat{g}_{n,j} + j! \binom{n+1}{j+1}}{(\ln(1+x))^{j+1}} + \left( \frac{-1}{1+x} \right)^{n+2} \sum_{j=1}^{n+1} \frac{j\widehat{g}_{n,j-1}}{(\ln(1+x))^{j+1}} \\
&= \left( \frac{-1}{1+x} \right)^{n+2} \sum_{j=0}^{n+1} \frac{\widehat{g}_{n+1,j}}{(\ln(1+x))^{j+1}}.
\end{aligned}$$

□

**Lemma 2.** For integers  $m$  and  $j$  with  $m \geq j \geq 0$  we have

$$\widehat{g}_{m+1,j} - (m+1)\widehat{g}_{m,j-1} = \sum_{k=0}^{m-j} \binom{m+1}{k+1} (-1)^k c_{k+1} \widehat{g}_{m-k,j}.$$

*Proof.* It is enough to prove

$$\binom{m+1}{j+1} - \frac{m}{j} \binom{m}{j} = \sum_{k=0}^{m-j} \binom{m+1}{k+1} (-1)^k c_{k+1} \frac{m-k}{m+1} \binom{m-k}{j+1} \quad (26)$$

and

$$\binom{m+1}{j} - \frac{m+1}{j} \binom{m}{j-1} = \sum_{k=0}^{m-j} \binom{m+1}{k+1} (-1)^k c_{k+1} \binom{m-k}{j}. \quad (27)$$

The identity (26) holds because by (1) formulae in Table 241 in [7]

$$\begin{aligned}
& \sum_{k=0}^{m-j+1} \binom{m+1}{k} (-1)^k c_k \frac{m-k+1}{m+1} \begin{bmatrix} m-k+1 \\ j+1 \end{bmatrix} \\
&= \sum_{k=0}^{m-j+1} \binom{m+1}{k} (-1)^k \sum_{i=0}^k \frac{(-1)^{k-i}}{i+1} \begin{bmatrix} k \\ i \end{bmatrix} \frac{m-k+1}{m+1} \begin{bmatrix} m-k+1 \\ j+1 \end{bmatrix} \\
&= \sum_{i=0}^{m-j+1} \frac{(-1)^i}{i+1} \sum_{k=i}^{m-j+1} \binom{m+1}{k} \begin{bmatrix} k \\ i \end{bmatrix} \frac{m-k+1}{m+1} \begin{bmatrix} m-k+1 \\ j+1 \end{bmatrix} \\
&= \sum_{i=0}^{m-j+1} \frac{(-1)^i}{i+1} \binom{i+j}{j} \begin{bmatrix} m+1 \\ i+j+1 \end{bmatrix} \\
&= \frac{m}{j} \begin{bmatrix} m \\ j \end{bmatrix}.
\end{aligned}$$

Similarly, the identity (27) holds because

$$\begin{aligned}
& \sum_{k=0}^{m-j+1} \binom{m+1}{k} (-1)^k c_k \begin{bmatrix} m-k+1 \\ j \end{bmatrix} \\
&= \sum_{k=0}^{m-j+1} \binom{m+1}{k} (-1)^k \sum_{i=0}^k \frac{(-1)^{k-i}}{i+1} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} m-k+1 \\ j \end{bmatrix} \\
&= \sum_{i=0}^{m-j+1} \frac{(-1)^i}{i+1} \sum_{k=i}^{m-j+1} \binom{m+1}{k} \begin{bmatrix} k \\ i \end{bmatrix} \begin{bmatrix} m-k+1 \\ j \end{bmatrix} \\
&= \sum_{i=0}^{m-j+1} \frac{(-1)^i}{i+1} \binom{i+j}{j} \begin{bmatrix} m+1 \\ i+j \end{bmatrix} \\
&= \frac{m+1}{j} \begin{bmatrix} m \\ j-1 \end{bmatrix}.
\end{aligned}$$

□

We also need the following relation ([11, Theorem 2.4 (2.1)]) in order to prove Proposition 3.

**Lemma 3.** For  $m \geq 0$

$$(-1)^m \frac{\widehat{C}_m}{m!} = \sum_{k=0}^m (-1)^k \frac{C_k}{k!}.$$

*Proof of Proposition 3.* The identity (25) holds for  $m = 0$  and  $m = 1$  because of (18) and (20), respectively. Let  $m \geq 2$ . By Lemma 1 with  $\widehat{g}_{n,n} = -n!x$  ( $n \geq 0$ ),  $\widehat{g}_{n,0} = n!$  ( $n \geq 1$ ) and  $\widehat{g}_{0,0} = -x$  together with Lemmata 2 and 3

$$\begin{aligned}
& \widehat{c}(x)\widehat{c}^{(m)}(x) + \frac{x}{m+1}\widehat{c}^{(m+1)}(x) \\
&= \frac{x}{m+1} \left(\frac{-1}{1+x}\right)^{m+2} \left(\sum_{j=0}^{m-1} \frac{\widehat{g}_{m+1,j+1} - (m+1)\widehat{g}_{m,j}}{(\ln(1+x))^{j+2}} + \frac{(m+1)!}{\ln(1+x)}\right) \\
&= \frac{x}{m+1} \left(\frac{-1}{1+x}\right)^{m+2} \left(\sum_{j=0}^{m-1} \frac{1}{(\ln(1+x))^{j+2}} \sum_{k=0}^{m-j-1} \binom{m+1}{k+1} (-1)^k c_{k+1} \widehat{g}_{m-k,j+1}\right. \\
&\quad \left.+ \frac{(m+1)!}{\ln(1+x)} \sum_{k=0}^{m-1} (-1)^k \frac{c_{k+1}}{(k+1)!} + \frac{(-1)^m (m+1)\widehat{c}_m}{\ln(1+x)}\right) \\
&= \frac{x}{m+1} \left(\frac{-1}{1+x}\right)^{m+2} \left(\sum_{j=0}^m \frac{m+1}{(\ln(1+x))^{j+1}} \sum_{k=0}^{m-j} \frac{c_{k+1}}{k+1} \binom{m}{k} (-1)^k \widehat{g}_{m-k,j}\right. \\
&\quad \left.+ \frac{(-1)^m}{\ln(1+x)} (c_{m+1}x + (m+1)\widehat{c}_m)\right) \\
&= \left(\frac{-1}{1+x}\right)^{m+2} \sum_{k=0}^m \binom{m}{k} \frac{c_{k+1}}{k+1} (-1)^k \sum_{j=0}^{m-k} \frac{x\widehat{g}_{m-k,j}}{(\ln(1+x))^{j+1}} \\
&\quad + \left(\frac{-1}{1+x}\right)^{m+2} \frac{x}{\ln(1+x)} (-1)^m \left(\frac{c_{m+1}}{m+1}x + \widehat{c}_m\right) \\
&= -\sum_{k=0}^{m-1} \binom{m}{k} \frac{c_{k+1}}{k+1} \frac{x}{(1+x)^{k+1}} \widehat{c}^{(m-k)}(x) + \frac{\widehat{c}_m}{(1+x)^{m+1}} \widehat{c}(x).
\end{aligned}$$

□

By Proposition 3, we obtain the following theorem.

**Theorem 3.** For  $m \geq 0$  and  $n \geq 0$  we have

$$\begin{aligned} & (\widehat{c}_0 + \widehat{c}_m)^n \\ &= -\frac{n}{m+1} \widehat{c}_{n+m} \\ & \quad - \frac{1}{m+1} \sum_{k=0}^{m-1} \binom{m+1}{k+1} c_{k+1} \sum_{i=0}^{n-1} (-1)^i \binom{k+i}{i} \frac{n!}{(n-i-1)!} \widehat{c}_{n+m-k-i-1} \\ & \quad + \widehat{c}_m \sum_{i=0}^n (-1)^i \binom{m+i}{i} \frac{n!}{(n-i)!} \widehat{c}_{n-i}. \end{aligned}$$

**Examples.** When  $m = 0$ ,  $m = 1$  and  $m = 2$ , we have the formulae (7), (8) and (9), respectively. If  $m = 3$ , we have

$$\begin{aligned} (\widehat{c}_0 + \widehat{c}_3)^n &= -\frac{(n+1)!}{8} \sum_{k=0}^n \frac{(-1)^{n-k} (2(2n^2 - 6n + 9) - (n-k)(n+9k-27)) \widehat{c}_k}{k!} \\ & \quad + \frac{n(2n-1)\widehat{c}_{n+1}}{4} - \frac{n\widehat{c}_{n+2}}{2} - \frac{n\widehat{c}_{n+3}}{4}. \end{aligned}$$

*Proof of Theorem 3.* By (19)

$$x\widehat{c}^{(m+1)}(x) = \sum_{n=0}^{\infty} n\widehat{c}_{n+m} \frac{x^n}{n!}.$$

By (21) we get

$$\begin{aligned} & \sum_{k=0}^{m-1} \binom{m+1}{k+1} \frac{c_{k+1}x}{(1+x)^{k+1}} \widehat{c}^{(m-k)}(x) \\ &= \sum_{k=0}^{m-1} \binom{m+1}{k+1} c_{k+1} \sum_{i=0}^{\infty} (-1)^i \binom{k+i}{i} x^{i+1} \widehat{c}^{(m-k)}(x) \\ &= \sum_{k=0}^{m-1} \binom{m+1}{k+1} c_{k+1} \sum_{i=0}^{n-1} (-1)^i \binom{k+i}{i} \sum_{n=0}^{\infty} \frac{n!}{(n-i-1)!} \widehat{c}_{n+m-k-i-1} \frac{x^n}{n!}. \end{aligned}$$

Finally, for  $m \geq 0$

$$\frac{\widehat{c}_m}{(1+x)^{m+1}} \widehat{c}(x) = \widehat{c}_m \sum_{i=0}^n (-1)^i \binom{m+i}{i} \sum_{n=0}^{\infty} \frac{n!}{(n-i)!} \widehat{c}_{n-i} \frac{x^n}{n!}.$$

□

Similarly to Theorem 3 for Cauchy numbers of the second kind, we have the following for Cauchy numbers of the first kind.

**Theorem 4.** For  $m \geq 2$  and  $n \geq 0$  we have

$$\begin{aligned} (c_0 + c_m)^n = & \\ & - \frac{n}{m+1} \left( \frac{2n+m-1}{2} c_{n+m-1} + c_{n+m} \right) \\ & - \sum_{k=1}^{m-1} \frac{c_{k+1}}{k+1} \binom{m}{k} \sum_{i=0}^{n-1} (-1)^i \binom{k+i-1}{i} \frac{n!}{(n-i-1)!} c_{n+m-k-i-1} \\ & + c_m \sum_{i=0}^n (-1)^i \binom{m+i-2}{i} \frac{n!}{(n-i)!} c_{n-i}. \end{aligned}$$

The expression of  $(c_0 + c_m)^n$  is based upon the following relation.

**Proposition 4.** For  $m \geq 0$  we have

$$\begin{aligned} c(x)c^{(m)}(x) & \\ & = -\frac{x(1+x)}{m+1} c^{(m+1)}(x) - \sum_{k=0}^{m-1} \frac{c_{k+1}}{k+1} \binom{m}{k} \frac{x}{(1+x)^k} c^{(m-k)}(x) \\ & \quad + \frac{c_m}{(1+x)^{m-1}} c(x). \end{aligned}$$

We need the following Lemma.

**Lemma 4.** For  $n \geq 0$  we have

$$c^{(n)}(x) = \left( \frac{-1}{1+x} \right)^n \sum_{j=0}^n \frac{g_{n,j}}{(\ln(1+x))^{j+1}},$$

where

$$g_{n,j} = j! \left( x \begin{bmatrix} n \\ j \end{bmatrix} - (x+1)n \begin{bmatrix} n-1 \\ j \end{bmatrix} \right).$$

*Proof.* It is proven inductively that

$$\frac{d^n}{dx^n} \frac{1}{\ln(1+x)} = \frac{(-1)^n}{(1+x)^n} \sum_{j=0}^n \begin{bmatrix} n \\ j \end{bmatrix} \frac{j!}{(\ln(1+x))^{j+1}} \quad (n \geq 0)$$

(Cf. [3, Lemma 4.1]). Using Leibniz's rule, we have

$$\begin{aligned}
\frac{d^n}{dx^n} \frac{x}{\ln(1+x)} &= x \frac{d^n}{dx^n} \frac{1}{\ln(1+x)} + n \frac{d^{n-1}}{dx^{n-1}} \frac{1}{\ln(1+x)} \\
&= x \frac{(-1)^n}{(1+x)^n} \sum_{j=0}^n \binom{n}{j} \frac{j!}{(\ln(1+x))^{j+1}} \\
&\quad + n \frac{(-1)^{n-1}}{(1+x)^{n-1}} \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{j!}{(\ln(1+x))^{j+1}} \\
&= \left( \frac{-1}{1+x} \right)^n \sum_{j=0}^n \frac{j!}{(\ln(1+x))^{j+1}} \left( x \binom{n}{j} - (x+1)n \binom{n-1}{j} \right).
\end{aligned}$$

□

### 3 Main result

We can show the following proposition in order to obtain the main result.

**Proposition 5.** *Let  $l, m$  be nonnegative integers with  $m \geq l \geq 1$ . Then*

$$\begin{aligned}
c^{(l)}(x)c^{(m)}(x) &= -\frac{l!m!}{(l+m+1)!}x(1+x)c^{(l+m+1)}(x) - \frac{l!m!}{(l+m)!}(2x+1)c^{(l+m)}(x) \\
&\quad - \frac{l!m!}{(l+m-1)!}c^{(l+m-1)}(x) + \sum_{k=0}^{m-l-1} \frac{a_{l,m,m-k}x + b_{l,m,m-k}}{(1+x)^{l+k}}c^{(m-k)}(x) \\
&\quad + \sum_{k=0}^{l-2} \frac{a_{l,m,l-k}x + b_{l,m,l-k}}{(1+x)^{m+k}}c^{(l-k)}(x) + \frac{a_{l,m,1}x}{(1+x)^{l+m-1}}c'(x) + \frac{b_{l,m,0}}{(1+x)^{l+m-1}}c(x),
\end{aligned}$$

where for  $l+1 \leq r = m-k \leq m$

$$\begin{aligned}
a_{l,m,r} &= (-1)^{l+1} \binom{m}{k} \sum_{i=0}^l i! \binom{l}{i} \binom{l+k-2}{i} \frac{c_{l+k-i+1}}{l+k-i+1}, \\
b_{l,m,r} &= (-1)^l l \binom{m}{k} \sum_{i=0}^{l-1} i! \binom{l-1}{i} \binom{l+k-2}{i} \frac{c_{l+k-i}}{l+k-i},
\end{aligned}$$

and for  $1 \leq r = l - k \leq l$

$$a_{l,m,r} = \sum_{j=0}^{r-2} \frac{(-1)^{l+j+1}}{r} \binom{l}{j} \binom{m}{r-j-1} (r-1)^{-1} \\ \times \sum_{i=0}^{l-j} i! \binom{l-j}{i} \binom{l+m-r-2}{i} c_{l+m-i-r+1} \\ + \frac{(-1)^{l+r}}{r} \binom{l}{r-1} \sum_{i=0}^{l-r+1} i! \binom{l-r+1}{i} \binom{l+m-r-1}{i} c_{l+m-i-r+1}$$

except

$$a_{l,m,3} = -\frac{m(m-1)}{3!} \sum_{i=0}^4 i! \binom{4}{i} \binom{m-1}{i} c_{m-i+2} \\ + \frac{4(m-3)}{3!} \sum_{i=0}^3 i! \binom{3}{i} \binom{m}{i} c_{m-i+2} \quad (l=4)$$

and

$$a_{l,m,2} = (-1)^{l+1} \frac{m-l}{2} \sum_{i=0}^l i! \binom{l}{i} \binom{l+m-3}{i} c_{l+m-i-1} \quad (l=2,3),$$

and for  $2 \leq r = l - k \leq l$

$$b_{l,m,r} = \sum_{j=1}^{r-1} (-1)^{l+j+1} \binom{l}{j} \binom{m}{r-j} \binom{r}{j}^{-1} \sum_{i=0}^{l-j} i! \binom{l-j}{i} \binom{l+m-r-2}{i} c_{l+m-i-r}$$

except

$$b_{l,m,0} = (-1)^l \sum_{i=0}^l i! \binom{l}{i} \binom{l+m-2}{i} c_{l+m-i} = -a_{l,m,1}.$$

With this proposition we now obtain our main result.



**Theorem 5.** Let  $l, m, n$  be integers with  $m \geq l \geq 1$  and  $n \geq 0$ . Then

$$\begin{aligned}
& (c_l + c_m)^n \\
&= -\frac{l!m!}{(l+m+1)!} \left( (n+l+m+1)c_{n+l+m} + (n+l+m+1)(n+l+m)c_{n+l+m-1} \right) \\
&+ \sum_{k=0}^{m-l-1} \sum_{i=0}^n \frac{(-1)^{i-1}n!}{(n-i)!} \\
&\quad \times \left( \binom{l+k+i-2}{i-1} a_{l,m,m-k} - \binom{l+k+i-1}{i} b_{l,m,m-k} \right) c_{n+m-k-i} \\
&+ \sum_{k=0}^{l-2} \sum_{i=0}^n \frac{(-1)^{i-1}n!}{(n-i)!} \\
&\quad \times \left( \binom{m+k+i-2}{i-1} a_{l,m,l-k} - \binom{m+k+i-1}{i} b_{l,m,l-k} \right) c_{n+l-k-i} \\
&+ \sum_{i=0}^n (-1)^i \binom{l+m+i-2}{i} \frac{n!}{(n-i)!} (n-i-1) a_{l,m,1} c_{n-i}.
\end{aligned}$$

where  $a_{l,m,r}$  and  $b_{l,m,r}$  are defined in Proposition 5.

Similarly, for general integers  $l$  and  $m$ , we obtain the following result.

**Proposition 6.** Let  $l, m$  be fixed nonnegative integers with  $m \geq l \geq 1$ . Then

$$\begin{aligned}
\widehat{c}^{(l)}(x)\widehat{c}^{(m)}(x) &= -\frac{l!m!}{(l+m+1)!} x\widehat{c}^{(l+m+1)}(x) - \frac{l!m!}{(l+m)!} \widehat{c}^{(l+m)}(x) \\
&+ \sum_{k=0}^{m-l-1} \frac{a_{l,m,m-k}x + b_{l,m,m-k}}{(1+x)^{l+k+1}} \widehat{c}^{(m-k)}(x) + \sum_{k=0}^l \frac{a_{l,m,l-k}x + b_{l,m,l-k}}{(1+x)^{m+k+1}} \widehat{c}^{(l-k)}(x),
\end{aligned}$$

where for  $l+1 \leq r = m-k \leq m$

$$\begin{aligned}
a_{l,m,r} &= (-1)^{l+1} \binom{m}{k} \sum_{i=0}^l i! \binom{l}{i} \binom{l+k-1}{i} \frac{c_{l+k-i+1}}{l+k-i+1}, \\
b_{l,m,r} &= (-1)^l \binom{m}{k} \sum_{i=0}^{l-1} i! \binom{l-1}{i} \binom{l+k-1}{i} \frac{c_{l+k-i}}{l+k-i},
\end{aligned}$$

and for  $1 \leq r = l - k \leq l$

$$\begin{aligned} a_{l,m,r} &= \sum_{j=0}^{r-2} \frac{(-1)^{l+j+1}}{r} \binom{l}{j} \binom{m}{r-j-1} \binom{r-1}{j}^{-1} \\ &\quad \times \sum_{i=0}^{l-j} i! \binom{l-j}{i} \binom{l+m-r-1}{i} c_{l+m-i-r+1} \\ &\quad + \frac{(-1)^{l+r}}{r} \binom{l}{r-1} \sum_{i=0}^{l-r+1} i! \binom{l-r+1}{i} \binom{l+m-r-1}{i} c_{l+m-i-r+1} \end{aligned}$$

with  $a_{l,m,0} = 0$  and for  $0 \leq r = l - k \leq l$

$$\begin{aligned} b_{l,m,r} &= \sum_{j=1}^r (-1)^{l+j+1} \binom{l}{j} \binom{m}{r-j} \binom{r}{j}^{-1} \sum_{i=0}^{l-j} i! \binom{l-j}{i} \binom{l+m-r-1}{i} c_{l+m-i-r} \\ &\quad + (-1)^{l+r} \binom{l}{r} \sum_{i=0}^{l-r} i! \binom{l-r}{i} \binom{l+m-r}{i} \widehat{c}_{l+m-i-r}. \end{aligned}$$

By using Proposition 6, we obtain explicit expressions for  $(\widehat{c}_l + \widehat{c}_m)^n$ .

**Theorem 6.** Let  $l, m, n$  be integers with  $m \geq l \geq 1$  and  $n \geq 0$ . Then

$$\begin{aligned} &(\widehat{c}_l + \widehat{c}_m)^n \\ &= -\frac{l!m!}{(l+m+1)!} (n+l+m+1) \widehat{c}_{n+l+m} \\ &\quad + \sum_{k=0}^{m-l-1} \sum_{i=0}^n \frac{(-1)^{i-1} n!}{(n-i)!} \\ &\quad \times \left( \binom{l+k+i-1}{i-1} a_{l,m,m-k} - \binom{l+k+i}{i} b_{l,m,m-k} \right) \widehat{c}_{n+m-k-i} \\ &\quad + \sum_{k=0}^l \sum_{i=0}^n \frac{(-1)^{i-1} n!}{(n-i)!} \\ &\quad \times \left( \binom{m+k+i-1}{i-1} a_{l,m,l-k} - \binom{m+k+i}{i} b_{l,m,l-k} \right) \widehat{c}_{n+l-k-i}. \end{aligned}$$

where  $a_{l,m,r}$  and  $b_{l,m,r}$  are defined in Proposition 6, and  $\binom{n}{-1} = 0$  ( $n \geq 0$ ).

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