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Nonlinear IS-LM Model with Tax Collection*

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1 Introduction

Since the pioneering work of Kalecki (1933) and the seminal work of Goodwin (1951), it has been recognized that economic dynamic systems usually incorporate delays in their actions and delay is one of the essentials for macroeconomic fluctuation. Nevertheless, little attention has been given to studies on delay in economic variables over the past few decades. After a long "gestation" period of time, the number of studies on delay gradually increases and various attempts have been done on the impact of delays on macro dynamics. Among others, we draw attention to the papers of De Cesare and Sportelli (2005) and Fanti and Manfredi (2007). Both papers introduce time delay into a simple IS-M model with a pure money financing deficit, which are used later to show the existence of cyclic fluctuations of the macro variables in the 1980s (Schinasi (1981, 1982) and Sasakura (1984)). Noticing the established fact that there are delays in collecting tax, De Cesare and Sportelli (2005) concern "economic situations where a finite time delay cannot ignored" and investigate how the fixed time delay in tax collection affects the fiscal policy outcomes. Two main results are shown: the emergence of limit cycle through a Hopf bifurcation when the length of the delay becomes longer and the co-existence of multiple stable and unstable limit cycles when the steady (equilibrium) point is locally stable. On the other hand, Fanti and Manfredi (2007) replace the fixed time delay with the distributed time delay, emphasizing the evidence that there is "a wide variation in collection lag" and demonstrate the possibility that in the same IS-LM framework, complex dynamics involving chaos is born though an *ála* period-doubling bifurcation with respect to the length of the delay. Recently Matsumoto and Szidarovszky (2013) reconsider the delay IS-LM model.. Stability conditions are derived and the destabilizing effect of the delay are numerically examined. Emergence of wide spectrum of dynamics ranging from simple cyclic oscillations to complex dynamics involving chaos is described through Hopf bifurcations.

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This short note is a complement of Matsumoto and Szidarovsky (2013) and aims to provide the basic structure of the non-delay IS-LM model, which could be useful to anlyse the delay IS-LM model. The followings are shown:

1) Stability condition;

2) Parametric conditions for stability switch;

3) Emergence of periodic and aperiodi oscillations via Hopf bifurcation;

4) Stability regain has initial point dependency.

This short note develops as follows. In Section 2, the non-delay IS-LM model is formulated and its steady state is obtained. In Section 3, the local stability condition is determined. In Section 4, the tax effect on stability is examined and the critial value of the tax rate is derived at which the stability switch occurs.

2 Non-delay IS-LM Model

We construct the fixed-price IS-LM model with a pure money financing deficit:

$$(M_I): \begin{cases} \dot{Y}(t) = \alpha \left[I(Y(t), R(t)) - s\left(Y(t) - T(t)\right) + g - T(t) \right], \\ \dot{R}(t) = \beta \left[L(Y(t), R(t)) - M(t) \right], \\ \dot{M}(t) = g - T(t), \end{cases}$$

where the three state variables, Y, R and M, respectively represent income, interest rate and real money supply, the parameters, a, β, g and s are positive adjustment coefficients in the markets of income and money, constant government expenditure and the constant marginal propensity to save and $I(\cdot)$ and $L(\cdot)$ denote the investment and liquidity preference functions. Tax revenue is denoted by T and is collected as a lump sum with a constant rate, $0 < \tau < 1$,

$$T(t) = \tau Y(t). \tag{1}$$

Following De Cesare and Sportelli (2005), we specify the investment and money demand functions as

$$I(Y,R) = A\frac{Y}{R} \text{ and } L(Y,R) = \gamma Y + \frac{\mu}{R}$$

with positive parameters A, γ and μ. The conditions, $\dot{Y}(t) = \dot{R}(t) = \dot{M}(t) = 0$ determine the unique steady state (Y^*, R^*, M^*) such that

$$Y^* = \frac{g}{\tau}, \; R^* = \frac{A}{s(1-\tau)} \text{ and } M^* = \gamma Y^* + \frac{\mu}{R^*}. \tag{2}$$

3 Local Stability

In this section, we investigate the local stability of the stationary state. To this end, we first expand the nonlinear model (M_I) in a Taylor's series around a neighborhood of the steady state and then discard all nonlinear terms to obtain the linearly approximated system,

$$\begin{pmatrix} \dot{Y}_\delta(t) \\ \dot{R}_\delta(t) \\ \dot{M}_\delta(t) \end{pmatrix} = \begin{pmatrix} -\alpha\tau & -\alpha\frac{s^2(1-\tau)^2}{\tau A}g & 0 \\ \beta\gamma & -\beta\frac{\mu s^2(1-\tau)^2}{A^2} & -\beta \\ -\tau & 0 & 0 \end{pmatrix} \begin{pmatrix} Y_\delta(t) \\ R_\delta(t) \\ M_\delta(t) \end{pmatrix} \tag{3}$$

where we define new variables $Y_\delta(t) = Y(t) - Y^*$, $R_\delta(t) = R(t) - R^*$ and $M_\delta(t) = M(t) - M^*$. To check whether the linear system (3) has solutions approaching the steady state, we look at the corresponding characteristic equation,

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0 \tag{4}$$

with

$$\begin{aligned} a_0 &= \alpha\beta\frac{s^2(1-\tau)^2}{A}g > 0, \\ a_1 &= \alpha\beta\frac{s^2(1-\tau)^2}{\tau A^2}(\gamma A g + \mu\tau^2) > 0, \\ a_2 &= \alpha\tau + \beta\mu\frac{s^2(1-\tau)^2}{A^2} > 0. \end{aligned}$$

We now determine the parametric conditions for which all roots of the characteristic equation satisfy $\mathrm{Re}(\lambda) < 0$. Since all coefficients are positive, according to the Routh-Hurwitz stability criterion,[1] the following inequality ensures local stability of the steady state,

$$a_1 a_2 - a_0 > 0$$

where

$$\begin{aligned} a_1 a_2 - a_0 &= \alpha\beta\frac{s^2(1-\tau)^2}{\tau A^4}\{\mu\tau^2\left[\beta\mu s^2(1-\tau)^2 + \alpha\tau A^2\right] \\ &+ A\left[\beta\gamma\mu s^2(1-\tau)^2 - (1-\alpha\gamma)\tau A^2\right]g\}. \end{aligned} \tag{5}$$

Apparently the inequality $1 - \alpha\gamma \leq 0$ leads to $a_2 a_1 - a_0 > 0$. To consider the complementary case of $1 - \alpha\gamma > 0$, we rewrite the right hand side of equation (5),

$$a_1 a_2 - a_0 = \alpha\beta\frac{s^2(1-\tau)^2}{\tau A^4}A\left[\beta\gamma\mu s^2(1-\tau)^2 - (1-\alpha\gamma)\tau A^2\right](g - \varphi(\tau))$$

[1] See, for example, Gandolfo (2010) for the Routh-Hurwitz stability theorem, according to which, in the case of cubic equation (4), $a_i > 0$ for $i = 0, 1, 2$ and $a_1 a_2 - a_0 > 0$ are the stability conditions.

where

$$\varphi(\tau) = \frac{\mu\tau^2\left[\beta\mu s^2(1-\tau)^2 + \alpha\tau A^2\right]}{A\left[(1-\alpha\gamma)\tau A^2 - \beta\gamma\mu s^2(1-\tau)^2\right]}. \tag{6}$$

The numerator of $\varphi(\tau)$ is definitely positive, however, the sign of the denominator is ambiguous. Provided that $1-\alpha\gamma > 0$, solving $(1-\alpha\gamma)\tau A^2 - \beta\gamma\mu s^2(1-\tau)^2 = 0$ for τ yields two real solutions, one is greater than unity and the other is less than unity. Let τ_- be a smaller solution, then the denominator is positive if $\tau_- < \tau < 1$ and negative if $\tau < \tau_-$ where

$$\tau_- = 1 + \frac{A^2(1-\alpha\gamma) - A\sqrt{1-\alpha\gamma}\sqrt{A^2(1-\alpha\gamma)+4\beta\gamma\mu s^2}}{2\beta\gamma\mu s^2} < 1.$$

Thus we have $a_1 a_2 - a_0 > 0$ if either $\tau \leq \tau_-$ or $\tau_- < \tau < 1$ and $\varphi(\tau) > g$. The local stability conditions of the undelay IS-LM model, (M_I), is summarized:

Theorem 1 *If one of the three exclusive conditions is satisfied, then the steady state is locally asymptotically stable;*

$$\begin{array}{ll} (i) & 1-\alpha\gamma \leq 0, \\ (ii) & 1-\alpha\gamma > 0 \text{ and } \tau \leq \tau_-, \\ (iii) & 1-\alpha\gamma > 0,\ \tau_- < \tau < 1 \text{ and } g < \varphi(\tau). \end{array}$$

The conditions (ii) and (iii) are visualized in Figure 1. The steady state is locally stable for all values of the parameters τ and g lying in the light-gray region and locally unstable in the dark-gray region where $1-\alpha\gamma > 0$, $\tau_- < \tau < 1$ and $g \geq \varphi(\tau)$. The division between these two areas is indicated by the distorted U-shaped boundary curve, the locus of $a_1a_2 - a_0 = 0$ or $g = \varphi(\tau)$. This curve separates the stable region from the unstable region in the (τ, g) plane and thus often called the *partition curve.* The light-gray region is further divided into two subregions by the vertical real line $\tau = \tau_-$. The condition (ii) holds in the subregion to the left of the line and the steady state is locally stable irrespective of the value of g. The condition (iii) holds in the subregion to the right. The boundary curve, $g = \varphi(\tau)$, is asymptotic to the vertical line as τ tends to τ_- from above.[2] The minimum value of $\varphi(\tau)$ is attained for $\tau = \tau_m$. The maximum value of tax rate τ is unity by definition and the corresponding value of $\varphi(\tau)$ is $\varphi(1)$. It is then apparent that the horizontal line at $g = \bar{g}$ has no intersection with the partition curve if $\bar{g} < \varphi(\tau_m)$, one intersection if $\bar{g} > \varphi(1)$ and two intersections including the equal roots otherwise. Notice that $\bar{g}$ is selected so as to satisfy $\varphi(\tau_m) < \bar{g} < \varphi(1)$ in Figure 1 and thus the horizontal line crosses twice the $g = \varphi(\tau)$ curve at points A and B, yielding the corresponding tax

[2] Note that in Figure 1, "τ_-" is not labeled to avoid overlapping τ_A.

rates τ_A and τ_B, respectively.[3]

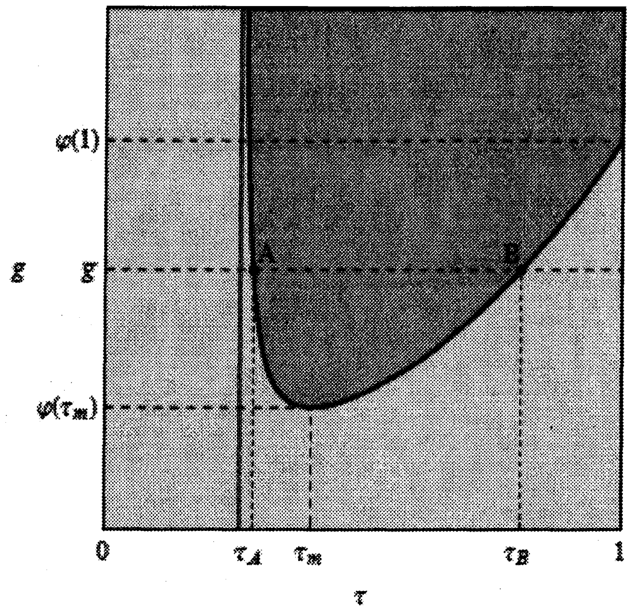


Figure 1. Stable and instable regions

4 Tax Effect on Stability

The local stability conditions of the steady state are analytically obtained. Now our concern is on global behavior of locally unstable trajectories. The nonlinearities of the dynamical system (M_I) indicates the emergence of limit cycle or other more complex behavior through a Hopf bifurcation when loss of stability occurs on the partition curve. Its conditions are as follows:

(H1) The characteristic equation at the critical point has a pair of purely imaginary roots and no other roots with zero real parts;

(H2) The real part of these imaginary roots change sign at the critical point.

Substituting $a_0 = a_1 a_2$ into equation (4) gives the factored form,

$$(\lambda^2 + a_1)(\lambda + a_2) = 0.$$

On this curve, the characteristic equation has a conjugate pair of purely imaginary roots and one real negative root,

$$\lambda_{1,2} = \pm i\sqrt{a_1} \text{ and } \lambda_3 = -a_2 < 0.$$

[3]The particular values of the paramters to depict Figure 1 are given in the Assumption below.

So the first condition (H1) is satisfied. To check the second condition, we select the tax rate τ as the bifurcation parameter and treat the root of the characteristic equation as a continuous function of τ:

$$\lambda(\tau)^3 + a_2\lambda(\tau)^2 + a_1\lambda(\tau) + a_0 = 0.$$

Differentiating it with respect to τ yields

$$\frac{d\lambda}{d\tau} = -\frac{\lambda^2 \frac{da_2}{d\tau} + \lambda \frac{da_1}{d\tau} + \frac{da_0}{d\tau}}{3\lambda^2 + 2a_2\lambda + a_1}.$$

Substituting $\lambda = i\sqrt{a_1}$ and rationalizing the right hand side yield the following form of the real part of this derivative,

$$\mathrm{Re}\left(\left.\frac{d\lambda}{d\tau}\right|_{\lambda=i\sqrt{a_1}}\right) = -\frac{\left(\frac{da_0}{d\tau} - a_1\frac{da_2}{d\tau}\right)(-2a_1) + 2a_2a_1\frac{da_1}{d\tau}}{4a_1(a_1+a_2^2)}.$$

The denominator is definitely positive. Let us denote the numerator as Ω and confirm its sign. Notice first that

$$\Omega = -\frac{2\alpha^2\beta^2 s^4(1-\tau)^2(Ag\gamma + \mu\tau^2)}{\tau^3 A^6}\Delta(\tau)$$

where

$$\Delta(\tau) = -\frac{\mu(1-\tau)\tau^2}{\beta\gamma\mu s^2(1-\tau)^2 - (1-\alpha\gamma)\tau A^2}\phi(\tau)$$

and

$$\begin{aligned}\phi(\tau) &= 2\beta^2\gamma\mu^2 s^4(1-\tau)^4 - \beta\mu A^2 s^2\left[1 - 4\alpha\gamma(1-\tau) - 3\tau\right](1-\tau)\tau \\ &\quad -2\alpha(1-\alpha\gamma)A^4\tau^2.\end{aligned}$$

Further we have

$$\frac{d\varphi}{d\tau} = -\frac{\mu\tau}{Af(\tau)^2}\phi(\tau)$$

with

$$f(\tau) = \beta\gamma\mu s^2(1-\tau)^2 - (1-\alpha\gamma)\tau A^2.$$

Notice that $-f(\tau)$ is the second factor of the denominator of $\varphi(\tau)$. Finally Ω can be expressed as

$$\Omega = kf(\tau)\frac{d\varphi}{d\tau}$$

with

$$k = \frac{2\alpha^2\beta^2 s^4(1-\tau)^2(Ag\gamma + \mu\tau^2)(1-\tau)\tau}{\tau^3 A^6} > 0.$$

Since $\varphi(\tau)$ is defined on the interval $(\tau_-, 1)$, we check the sign of Ω on that interval. Since,

$$f(\tau) > 0 \text{ for } \tau_- < \tau < 1,$$

we have

$$sign\left[\mathrm{Re}\left(\left.\frac{d\lambda}{d\tau}\right|_{\lambda=i\sqrt{a_1}}\right)\right]=-sign\left[\frac{d\varphi}{d\tau}\right]. \tag{7}$$

In Figure 1, the $g=\varphi(\tau)$ curve is downward-sloping at point A so due to equation (7),

$$\mathrm{Re}\left(\left.\frac{d\lambda}{d\tau}\right|_{\lambda=i\sqrt{a_1}}\right)>0$$

implying that all roots cross the imaginary axis at $i\sqrt{a_1}$ from left to right as τ increases, that is, the steady state loses stability. On the other hand, at point B, it is upward-sloping so due to equation (7),

$$\mathrm{Re}\left(\left.\frac{d\lambda}{d\tau}\right|_{\lambda=i\sqrt{a_1}}\right)<0$$

implying that the steady state regains stability. The effect caused by a change in the tax rate depends on constellations of τ and g and summarized as follows:

Theorem 2 *Given $g=\bar{g}$, stability switch occurs twice, to instability from stability for $\tau=\tau_A$ at point A and to stability from instability for $\tau=\tau_B$ at point B if $\varphi(\tau_m)\leq\bar{g}\leq\varphi(1)$, once from stability to instability if $\bar{g}>\varphi(1)$ and no stability switch occurs if $\bar{g}<\varphi(\tau_m)$ where $\bar{g}=\varphi(\tau_A)$ holds at point A and $\bar{g}=\varphi(\tau_B)$ holds at point B.*

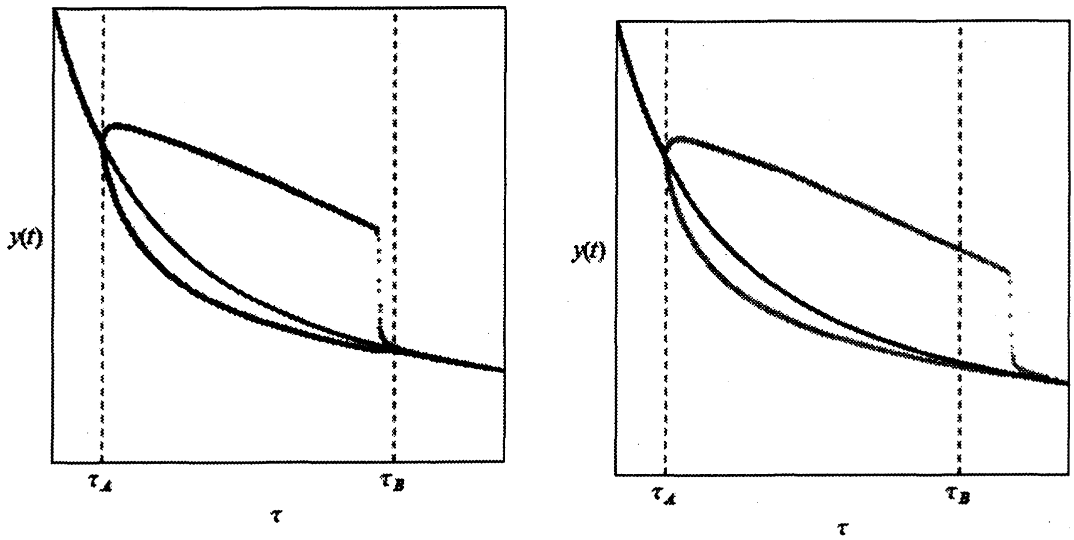
We numerically examine the analytical results just obtained. Before proceeding, we specify the parameter values as follows and formulate this selection as an assumption since we repeatedly use this set of the parameters in further numerical studies.

Assumption : $\alpha=\beta=A=1$, $\gamma=4/5$, $\mu=3$, $s=1/5$ and $\bar{g}=10$.

Figure 1 is actually illustrated under Assumption and takes the following parameter values, $\tau_A\simeq 0.29$, $\tau_B\simeq 0.8$, $\tau_m\simeq 0.4$, $\varphi(\tau_m)\simeq 4.7$ and $\varphi(1)=15$. Thus $1-\alpha\gamma>0$ and $g(\tau_m)<\bar{g}<g(1)$. According to Theorem 2, the stationary state of the 3D system (M_I) loses local stability at point A and regains it at point B. Local stability does not necessarily mean global stability in a nonlinear system. To find how nonlinearities in system (M_I) affect global dynamics, we numerically detect the effects caused by a change in the tax rate on global dynamics between τ_A and τ_B. In performing simulations, we take the same initial values for $Y(0)=Y^*$ and $R(0)=R^*$ and the different initial values of $M(0)$, $M(0)=M^*+1$ in the first simulation and $M(0)=M^*+5$ in the second simulation. The resultant bifurcation diagrams are presented in Figures 2(A) and (B), in each of which the downward sloping black curve depicts the equilibrium value of output, $Y^*=g/\tau$. In the simulations, the bifurcation parameter τ is increased from 0.2 to 1 with an increment of 1/1000, the iterations are repeated 5000 times and the local maximum and minimum of $Y(t)$ for the

last 100 iterations are plotted against each value of τ. In Figure 2(A), the bifurcation diagram of the first simulation is depicted. It is observed that the stationary state loses stability when τ arrives at τ_A and bifurcates to a periodic cycle having one maximum and one minimum for $\tau > \tau_A$. It is also observed that an oscillation disappears at $\tau = \tau_B$ and stability is regained for $\tau > \tau_B$. In Figure 2(B), the bifurcation diagram in the second simulation is illustrated. It is seen that stability is lost at $\tau = \tau_A$ as in Figure 2(A) but regained at some value larger than τ_B. Further simulations with different initial points have been conducted and then lead to the fact that stability is regained not necessarily at $\tau = \tau_B$ but at some larger value although stability is always lost at $\tau = \tau_A$. This difference implies that it depends on a selection of the initial values of the variables when it regains stability.[4] These numerical results are summarized as follows:

Proposition 3 *The nonlinear IS-LM model (M_I) generates periodic oscillations when the steady state is destabilized and has initial point dependency to regain stability.*



(A) $M(0) = M^* + 1$ (B) $M(0) = M^* + 5$

Figure 2. Bifurcation diagrams with different initial values

De Cesare and Sportelli (2005) and Fanti and Manfredi (2007) also examine the local stability and arrive at the same result as in Theorem 1. However, the former does not consider a Hopf bifurcation in the undelay model whereas the latter discusses the Hopf bifurcation with respect to the government expenditure, the other fiscal policy parameter, but not with respect to the tax rate. As

[4]Notice that Theorem 2 concerns only local stability of the stationary state.

can be seen in Figure 1, given $\tau \in (\tau_-, 1)$, increasing g destabilizes the steady state when it crosses the partition curve from below. As a natural consequence, neither authors mention a possibility of stability regain with respect to the tax rate.

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