数理解析研究所講究録
第 1901 巻 2014 年 1-25

Existence and Precise Asymptotic Behavior of Positive Intermediate Solutions of Perturbed Systems of Second Order Nonlinear Differential Equations

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Key words and phrases: systems of differential equations, positive solutions, asymptotic behavior, regularly varying functions

2010 Mathematics Subject Classifications: 34C11, 26A12

1 Introduction

We consider nonlinear differential systems of the form

(A) $$x'' + p_1(t)x^{\alpha_1} + q_1(t)y^{\beta_1} = 0, \quad y'' + p_2(t)x^{\alpha_2} + q_2(t)y^{\beta_2} = 0,$$

where α_i and β_i, $i = 1, 2$, are positive constants and $p_i(t)$ and $q_i(t), i = 1, 2$, are positive continuous functions on $[a, \infty), a > 0$.

By a positive solution of (A) we mean a vector function $(x(t), y(t))$ on an interval of the form $[t_0, \infty)$, $t_0 \geq a$, with positive components satisfying system (A) for $t \geq t_0$.

We are interested in the existence and precise asymptotic behavior of the so-called intermediate positive solutions of (A), i.e., solutions which satisfy

(1.1) $$\lim_{t\to\infty} \frac{x(t)}{t} = \lim_{t\to\infty} \frac{y(t)}{t} = 0, \quad \lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = \infty.$$

It is easy to see that such a solution of (A) satisfies the system of integral equations

(1.2) $$\begin{aligned} x(t) &= x_0 + \int_{t_0}^{t} \int_{s}^{\infty} \left[p_1(r)x(r)^{\alpha_1} + q_1(r)y(r)^{\beta_1}\right] dr ds, \\ y(t) &= y_0 + \int_{t_0}^{t} \int_{s}^{\infty} \left[p_2(r)x(r)^{\alpha_2} + q_2(r)y(r)^{\beta_2}\right] dr ds, \end{aligned}$$

for $t \geq t_0$ and some positive constants x_0 and y_0.

In this lecture (paper) we restrict our consideration to regularly varying intermediate solutions of (A). We recall that a measurable function $f : (0, \infty) \to (0, \infty)$ is said to be *regularly varying of index* $\rho \in \mathbf{R}$ if it satisfies

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\rho} \quad \text{for } \forall\lambda > 0.$$

The totality of regularly varying functions of index ρ will be denoted by $\mathrm{RV}(\rho)$. We often use the symbol SV instead of RV(0) and call members of SV *slowly varying functions.* By definition any function $f(t) \in \mathrm{RV}(\rho)$ is written as $f(t) = t^\rho g(t)$ with $g(t) \in \mathrm{SV}$. A function $f(t) \in \mathrm{RV}(\rho)$ is called a *trivial* regularly varying function of index ρ if it satisfies $\lim_{t\to\infty} f(t)/t^\rho = \text{const} > 0$ and a *nontrivial* regularly varying function of index ρ otherwise. The set of all trivial (resp. nontrivial) regularly varying functions of index ρ will be denoted by tr-RV(ρ) (resp. ntr-RV(ρ)).

If we represent regularly varying solutions $(x(t), y(t))$ of (A) by the expressions

$$x(t) = t^\rho \xi(t), \quad y(t) = t^\sigma \eta(t), \quad \xi(t), \eta(t) \in \mathrm{SV}, \tag{1.3}$$

then the requirement that $x(t)$ and $y(t)$ satisfy (1.1) restrict the values of ρ and σ and the behavior of $\xi(t)$ and $\eta(t)$ at infinity as follows:

$$\rho \in [0,1], \quad \lim_{t\to\infty} \xi(t) = \infty \quad \text{if } \rho = 0, \quad \lim_{t\to\infty} \xi(t) = 0 \quad \text{if } \rho = 1,$$
$$\sigma \in [0,1], \quad \lim_{t\to\infty} \eta(t) = \infty \quad \text{if } \sigma = 0, \quad \lim_{t\to\infty} \eta(t) = 0 \quad \text{if } \sigma = 1.$$

From this remark we see that there are six different types of the asymptotic behavior at infinity for possible regularly varying intermediate solutions $(x(t), y(t))$ of system (A):

(i) $(x(t), y(t)) \in \mathrm{RV}(\rho)\times\mathrm{RV}(\sigma)$, $\rho \in (0,1), \sigma \in (0,1)$;
(ii) $(x(t), y(t)) \in$ ntr-RV$(1)\times$RV(σ), $\sigma \in (0,1)$;
(iii) $(x(t), y(t)) \in$ ntr-RV$(0)\times$RV(σ), $\sigma \in (0,1)$;
(iv) $(x(t), y(t)) \in$ ntr-RV(1) $\times$ ntr-RV(1);
(v) $(x(t), y(t)) \in$ ntr-RV$(1)\times$ntr-RV(0);
(vi) $(x(t), y(t)) \in$ ntr-RV$(0)\times$ntr-RV(0).

In Section 3 an asymptotic analysis of regularly varying intermediate solutions will be made by regarding (A) as a small perturbation of the diagonal system

$$x'' + p_1(t)x^{\alpha_1} = 0, \quad y'' + q_2(t)y^{\beta_2} = 0,$$

where $\alpha_1 < 1, \beta_2 < 1$, and $p_1(t)$ and $q_2(t)$ are regularly varying functions of indices λ_1 and μ_2, respectively. The existence of all six types of intermediate solutions listed above will be established by combining the known information about regularly varying solutions of the diagonal system with fixed point techniques.

Section 4 is devoted to the study of (A) viewed as a perturbation of the cyclic system

$$x'' + q_1(t)y^{\beta_1} = 0, \quad y'' + p_2(t)x^{\alpha_2} = 0,$$

where $\alpha_2\beta_1 < 1$ and $p_2(t)$ and $q_1(t)$ are regularly varying functions of indices λ_2 and μ_1, respectively. It is shown that the existence and precise asymptotic behavior of intermediate solutions of the types (i)-(iii) of cyclic systems of the above form is preserved for (A) if the perturbations are small in the sense specified below.

Analogous results on the existence and precise asymptotic behavior of the so-called strongly decreasing regularly varying solutions of the system of two perturbed Thomas-Fermi equations

$$x'' = p_1(t)x^{\alpha_1} + q_1(t)y^{\beta_1}, \quad y'' = p_2(t)x^{\alpha_2} + q_2(t)y^{\beta_2} \tag{B}$$

have been established by the present authors in [4].

2 Regularly varying functions

For the reader's convenience we recall here the definition of regularly varying functions, basic terminologies and notations, and Karamata's integration theorem which will play a central role in establishing the main results of this paper.

Definition 2.1. A measurable function $f : (0, \infty) \to (0, \infty)$ is said to be *regularly varying of index* $\rho \in \mathbf{R}$ if it satisfies

$$\lim_{t \to \infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \text{for } \forall \lambda > 0,$$

or equivalently it is expressed in the form

$$f(t) = c(t) \exp\left\{ \int_{t_0}^{t} \frac{\delta(s)}{s} ds \right\}, \quad t \geq t_0,$$

for some $t_0 > 0$ and some measurable functions $c(t)$ and $\delta(t)$ such that

$$\lim_{t \to \infty} c(t) = c_0 \in (0, \infty) \quad \text{and} \quad \lim_{t \to \infty} \delta(t) = \rho.$$

The totality of regularly varying functions of index ρ is denoted by $\mathrm{RV}(\rho)$. We often use the symbol SV instead of $\mathrm{RV}(0)$ and call members of SV *slowly varying functions.* By definition any function $f(t) \in \mathrm{RV}(\rho)$ is written as $f(t) = t^\rho g(t)$ with $g(t) \in \mathrm{SV}$. So, the class SV of slowly varying functions is of fundamental importance in theory of regular variation. Typical examples of slowly varying functions are: all functions tending to positive constants as $t \to \infty$,

$$\prod_{n=1}^{N} (\log_n t)^{\alpha_n}, \quad \alpha_n \in \mathbf{R}, \quad \text{and} \quad \exp\left\{ \prod_{n=1}^{N} (\log_n t)^{\beta_n} \right\}, \quad \beta_n \in (0, 1),$$

where $\log_n t$ denotes the n-th iteration of the logarithm. It is known that the function

$$L(t) = \exp\left\{ (\log t)^{\frac{1}{3}} \cos\, (\log t)^{\frac{1}{3}} \right\}$$

is a slowly varying function which is oscillating in the sense that

$$\limsup_{t \to \infty} L(t) = \infty \quad \text{and} \quad \liminf_{t \to \infty} L(t) = 0.$$

A function $f(t) \in \mathrm{RV}(\rho)$ is called a *trivial* regularly varying function of index ρ if it is expressed in the form $f(t) = t^\rho L(t)$ with $L(t) \in \mathrm{SV}$ satisfying $\lim_{t \to \infty} L(t) = \text{const} > 0$. Otherwise $f(t)$ is called a *nontrivial* regularly varying function of index ρ. The symbol tr-$\mathrm{RV}(\rho)$ (or ntr-$\mathrm{RV}(\rho)$) is used to denote the set of all trivial $\mathrm{RV}(\rho)$-functions (or the set of all nontrivial $\mathrm{RV}(\rho)$-functions).

The following proposition, known as Karamata's integration theorem, is particularly useful in handling slowly and regularly varying functions analytically and is extensively used throughout the paper.

Proposition 2.1. *Let* $L(t) \in \mathrm{SV}$. *Then,*

(i) *if* $\alpha > -1$,

$$\int_a^t s^\alpha L(s)ds \sim \frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty;$$

(ii) *if* $\alpha < -1$,

$$\int_t^\infty s^\alpha L(s)ds \sim -\frac{1}{\alpha+1} t^{\alpha+1} L(t), \quad t \to \infty;$$

(iii) *if* $\alpha = -1$,

$$l(t) = \int_a^t \frac{L(s)}{s} ds \in \mathrm{SV} \quad \text{and} \quad \lim_{t\to\infty} \frac{L(t)}{l(t)} = 0,,$$

and

$$m(t) = \int_t^\infty \frac{L(s)}{s} ds \in \mathrm{SV} \quad \text{and} \quad \lim_{t\to\infty} \frac{L(t)}{m(t)} = 0,$$

provided $L(t)/t$ *is integrable near the infinity in the latter case.*

The reader is referred to Bingham et al [1] for the most complete exposition of theory of regular variation and its applications and to Marić [8] for the comprehensive survey of results up to 2000 on the asymptotic analysis of second order linear and nonlinear ordinary differential equations in the framework of regular variation.

3 Perturbations of the diagonal system

In this section we establish a criterion for the existence of intermediate regularly varying solutions by regarding (A) as a small perturbation of the system

$$(\mathrm{A_d}) \qquad x'' + p_1(t)x^{\alpha_1} = 0, \quad y'' + q_2(t)y^{\beta_2} = 0,$$

where

$$(3.1) \qquad \alpha_1 < 1, \quad \beta_2 < 1,$$

and

$$(3.2) \qquad p_1(t) \in \mathrm{RV}(\lambda_1), \quad q_2(t) \in \mathrm{RV}(\mu_2).$$

Use is made of the following results which are obtained by combining necessary and sufficient conditions for the existence of three types of intermediate regularly varying solutions of the sublinear Emden-Fowler equation established in [6] (see also [3]).

Proposition 3.1. *Let conditions* (3.1) *and* (3.2) *be satisfied. Then, system* $(\mathrm{A_d})$ *has intermediate regularly varying solutions* $(x(t), y(t))$ *of index* (ρ, σ) *with* $\rho \in (0,1)$ *and* $\sigma \in (0,1)$ *if and only if*

$$(3.3) \qquad -2 < \lambda_1 < -\alpha_1 - 1,$$

and

$$-2 < \mu_2 < -\beta_2 - 1, \tag{3.4}$$

in which case ρ *and* σ *are defined by*

$$\rho = \frac{\lambda_1 + 2}{1 - \alpha_1} \tag{3.5}$$

$$\sigma = \frac{\mu_2 + 2}{1 - \beta_2}, \tag{3.6}$$

and the asymptotic behavior of any such solution $(x(t), y(t))$ *is governed by the formulas*

$$x(t) \sim X_1(t), \quad y(t) \sim Y_1(t), \quad t \to \infty, \tag{3.7}$$

where $X_1(t) \in \mathrm{RV}(\rho)$ *and* $Y_1(t) \in \mathrm{RV}(\sigma)$ *are given by*

$$X_1(t) = \left[\frac{t^2 p_1(t)}{\rho(1-\rho)}\right]^{\frac{1}{1-\alpha_1}}, \tag{3.8}$$

$$Y_1(t) = \left[\frac{t^2 q_2(t)}{\sigma(1-\sigma)}\right]^{\frac{1}{1-\beta_2}}. \tag{3.9}$$

Proposition 3.2. *Let* (3.1) *and* (3.2) *hold. System* $(\mathrm{A_d})$ *has a solution* $(x(t), y(t)) \in$ ntr-RV$(1)\times$ RV(σ) *with* $\sigma \in (0,1)$ *if and only if* (3.4),

$$\lambda_1 = -\alpha_1 - 1 \quad \textit{and} \quad \int_a^\infty t^{\alpha_1} p_1(t)dt < \infty \tag{3.10}$$

hold, in which case σ *is defined by* (3.6) *and the asymptotic behavior of any such solution* $(x(t), y(t))$ *is governed by the formulas*

$$x(t) \sim X_2(t), \quad y(t) \sim Y_1(t), \quad t \to \infty, \tag{3.11}$$

where the functions $Y_1 \in \mathrm{RV}(\sigma)$ *and* $X_2 \in$ ntr-RV(1) *are defined by* (3.9) *and*

$$X_2(t) = t\big[(1-\alpha_1)\int_t^\infty s^{\alpha_1} p_1(s)ds\big]^{\frac{1}{1-\alpha_1}}, \tag{3.12}$$

respectively.

Proposition 3.3. *Let* (3.1) *and* (3.2) *hold. System* $(\mathrm{A_d})$ *has a solution* $(x(t), y(t)) \in$ ntr-RV$(0)\times$ RV(σ) *with* $\sigma \in (0,1)$ *if and only if* (3.4),

$$\lambda_1 = -2 \quad \textit{and} \quad \int_a^\infty \int_s^\infty p_1(r)drds = \infty \tag{3.13}$$

hold, in which case σ is given by (3.6) *and the asymptotic behavior of any such solution* $(x(t), y(t))$ *is governed by the formulas*

$$x(t) \sim X_3(t), \quad y(t) \sim Y_1(t), \quad t \to \infty, \tag{3.14}$$

where the functions $Y_1 \in \mathrm{RV}(\sigma)$ *and* $X_3 \in$ ntr-RV(0) *are defined by* (3.9) *and*

$$X_3(t) = \Big[(1-\alpha_1)\int_a^t \int_s^\infty p_1(r)drds\Big]^{\frac{1}{1-\alpha_1}}, \tag{3.15}$$

respectively.

Proposition 3.4. *Let* (3.1) *and* (3.2) *hold. System* (A$_\mathrm{d}$) *has a solution* $(x(t), y(t)) \in$ ntr-RV(1)$\times$ ntr-RV(1) *if and only if* (3.10) *and*

$$\mu_2 = -\beta_2 - 1 \quad and \quad \int_a^\infty t^{\beta_2} q_2(t)dt < \infty \tag{3.16}$$

hold, and the asymptotic behavior of any such solution $(x(t), y(t))$ *is governed by the formulas*

$$x(t) \sim X_2(t), \quad y(t) \sim Y_2(t), \quad t \to \infty, \tag{3.17}$$

where the functions $X_2 \in$ ntr-RV(1) *and* $Y_2 \in$ ntr-RV(1) *are defined by* (3.12) *and*

$$Y_2(t) = t\Big[(1-\beta_2)\int_t^\infty s^{\beta_2} q_2(s)ds\Big]^{\frac{1}{1-\beta_2}}, \tag{3.18}$$

respectively.

Proposition 3.5. *Let* (3.1) *and* (3.2) *hold. System* (A$_\mathrm{d}$) *has a solution* $(x(t), y(t)) \in$ ntr-RV(1)$\times$ ntr-RV(0) *if and only if* (3.10) *and*

$$\mu_2 = -2 \quad and \quad \int_a^\infty \int_s^\infty q_2(r)drds = \infty \tag{3.19}$$

hold, and the asymptotic behavior of any such solution $(x(t), y(t))$ *is governed by the formulas*

$$x(t) \sim X_2(t), \quad y(t) \sim Y_3(t), \quad t \to \infty, \tag{3.20}$$

where the functions $X_2 \in$ ntr-RV(1) *and* $Y_3 \in$ ntr-RV(0) *are defined by* (3.12) *and*

$$Y_3(t) = \Big[(1-\beta_2)\int_a^t \int_s^\infty q_2(r)drds\Big]^{\frac{1}{1-\beta_2}}, \tag{3.21}$$

respectively.

Proposition 3.6. *Let* (3.1) *and* (3.2) *hold. System* (A$_\mathrm{d}$) *has a solution* $(x(t), y(t)) \in$ ntr-RV(0)$\times$ ntr-RV(0) *if and only if* (3.13) *and* (3.19) *hold and the asymptotic behavior of any such solution* $(x(t), y(t))$ *is governed by the formulas*

$$x(t) \sim X_3(t), \quad y(t) \sim Y_3(t), \quad t \to \infty, \tag{3.22}$$

where the functions $X_3 \in$ ntr-RV(0) *and* $Y_3 \in$ ntr-RV(0) *are defined by* (3.15) *and* (3.21), *respectively.*

Theorem 3.1. *Assume that* (3.1)-(3.4) *hold. Let the constants* ρ *and* σ *be given by* (3.5) *and* (3.6), *and consider the functions* $X_1(t)$ *and* $Y_1(t)$ *defined by* (3.8) *and* (3.9). *Suppose that*

$$\lim_{t\to\infty} \frac{q_1(t)Y_1(t)^{\beta_1}}{p_1(t)X_1(t)^{\alpha_1}} = 0, \quad \lim_{t\to\infty} \frac{p_2(t)X_1(t)^{\alpha_2}}{q_2(t)Y_1(t)^{\beta_2}} = 0. \tag{3.23}$$

Then, system (A) *possesses intermediate regularly varying solutions* $(x(t), y(t))$ *of index* (ρ, σ) *whose asymptotic behavior is governed by the unique formula* (3.7).

Theorem 3.2. *Assume that* (3.1)-(3.2), (3.4) *and* (3.10) *hold. Let the constant* σ *be given by* (3.6) *and consider the functions* $Y_1(t)$ *and* $X_2(t)$ *defined by* (3.9) *and* (3.12). *Suppose that*

$$\lim_{t\to\infty} \frac{q_1(t)Y_1(t)^{\beta_1}}{p_1(t)X_2(t)^{\alpha_1}} = 0, \quad \lim_{t\to\infty} \frac{p_2(t)X_2(t)^{\alpha_2}}{q_2(t)Y_1(t)^{\beta_2}} = 0. \tag{3.24}$$

Then, system (A) *possesses solutions* $(x(t), y(t)) \in$ ntr-RV(1)$\times$ RV(σ) *whose asymptotic behavior is governed by the unique formula* (3.11).

Theorem 3.3. *Assume that* (3.1)-(3.2), (3.4) *and* (3.13) *hold. Let the constant* σ *be given by* (3.6) *and consider the functions* $Y_1(t)$ *and* $X_3(t)$ *defined by* (3.9) *and* (3.15). *Suppose that*

$$\lim_{t\to\infty} \frac{q_1(t)Y_1(t)^{\beta_1}}{p_1(t)X_3(t)^{\alpha_1}} = 0, \quad \lim_{t\to\infty} \frac{p_2(t)X_3(t)^{\alpha_2}}{q_2(t)Y_1(t)^{\beta_2}} = 0. \tag{3.25}$$

Then, system (A) *possesses solutions* $(x(t), y(t)) \in$ ntr-RV(0)$\times$ RV(σ) *whose asymptotic behavior is governed by the unique formula* (3.14).

Theorem 3.4. *Assume that* (3.1)-(3.2), (3.10) *and* (3.16) *hold. Consider the functions* $X_2(t)$ *and* $Y_2(t)$ *defined by* (3.12) *and* (3.18). *Suppose that*

$$\lim_{t\to\infty} \frac{q_1(t)Y_2(t)^{\beta_1}}{p_1(t)X_2(t)^{\alpha_1}} = 0, \quad \lim_{t\to\infty} \frac{p_2(t)X_2(t)^{\alpha_2}}{q_2(t)Y_2(t)^{\beta_2}} = 0. \tag{3.26}$$

Then, system (A) *possesses solutions* $(x(t), y(t)) \in$ ntr-RV(1)$\times$ ntr-RV(1) *whose asymptotic behavior is governed by the unique formula* (3.17).

Theorem 3.5. *Assume that* (3.1)-(3.2), (3.10) *and* (3.19) *hold. Consider the functions* $X_2(t)$ *and* $Y_3(t)$ *defined by* (3.12) *and* (3.21). *Suppose that*

$$\lim_{t\to\infty} \frac{q_1(t)Y_3(t)^{\beta_1}}{p_1(t)X_2(t)^{\alpha_1}} = 0, \quad \lim_{t\to\infty} \frac{p_2(t)X_2(t)^{\alpha_2}}{q_2(t)Y_3(t)^{\beta_2}} = 0. \tag{3.27}$$

Then, system (A) *possesses solutions* $(x(t), y(t)) \in$ ntr-RV(1)$\times$ ntr-RV(0) *whose asymptotic behavior is governed by the unique formula* (3.20).

Theorem 3.6. *Assume that* (3.1)-(3.2), (3.13) *and* (3.19) *hold. Consider the functions* $X_3(t)$ *and* $Y_3(t)$ *defined by* (3.15) *and* (3.21). *Suppose that*

$$\lim_{t\to\infty} \frac{q_1(t)Y_3(t)^{\beta_1}}{p_1(t)X_3(t)^{\alpha_1}} = 0, \quad \lim_{t\to\infty} \frac{p_2(t)X_3(t)^{\alpha_2}}{q_2(t)Y_3(t)^{\beta_2}} = 0. \tag{3.28}$$

Then, system (A) *possesses solutions* $(x(t), y(t)) \in$ ntr-RV(0)$\times$ ntr-RV(0) *whose asymptotic behavior is governed by the unique formula* (3.22).

PROOF. We will give a simultaneous proof of Theorems 3.1-3.6. Let $(X(t), Y(t))$ denote any of the six functions $(X_1(t), Y_1(t))$, $(X_2(t), Y_1(t))$, $(X_3(t), Y_1(t))$, $(X_2(t), Y_2(t))$, $(X_2(t), Y_3(t))$ and $(X_3(t), Y_3(t))$. It is known that $(X(t), Y(t))$ satisfies

$$\int_b^t \int_s^\infty p_1(r)X(r)^{\alpha_1} drds \sim X(t), \quad \int_b^t \int_s^\infty q_2(r)Y(r)^{\beta_2} drds \sim Y(t), \quad t\to\infty, \tag{3.29}$$

for any $b \geq a$. There exists $T_0 > a$ such that

$$\int_{T_0}^t \int_s^\infty p_1(r)X(r)^{\alpha_1} drds \leq 2X(t), \quad \int_{T_0}^t \int_s^\infty q_2(r)Y(r)^{\beta_2} drds \leq 2Y(t), t \geq T_0. \tag{3.30}$$

We may assume that T_0 is large enough so that $X(t)$ and $Y(t)$ are increasing for $t \geq T_0$. Since (3.29) holds for $b = T_0$, one finds $T_1 > T_0$ such that

$$\int_{T_0}^t \int_s^\infty p_1(r)X(r)^{\alpha_1} drds \geq \frac{1}{2}X(t), \quad \int_{T_0}^t \int_s^\infty q_2(r)Y(r)^{\beta_2} drds \geq \frac{1}{2}Y(t), \quad t \geq T_1. \tag{3.31}$$

Choose positive constants h, H, k and K so that $h < H, k < K$ and the following inequalities hold:

$$h \leq 2^{-\frac{1}{1-\alpha_1}}, \quad H \geq 8^{\frac{1}{1-\alpha_1}}, \quad k \leq 2^{-\frac{1}{1-\beta_2}}, \quad K \geq 8^{\frac{1}{1-\beta_2}}, \tag{3.32}$$

and

$$2hX(T_1) \leq HX(T_0), \quad 2kY(T_1) \leq KY(T_0). \tag{3.33}$$

We can choose $T_0 > a$ large enough so that in addition to (3.30)-(3.33) the following inequalities hold

$$\frac{q_1(t)Y(t)^{\beta_1}}{p_1(t)X(t)^{\alpha_1}} \leq \frac{h^{\alpha_1}}{K^{\beta_1}}, \quad \frac{p_2(t)X(t)^{\alpha_1}}{q_2(t)Y(t)^{\beta_1}} \leq \frac{k^{\beta_1}}{H^{\alpha_1}}, \tag{3.34}$$

which is possible because of (3.23)-(3.28). Define the set $\mathcal{X}$ by

$$\mathcal{X} = \{(x, y) \in C[T_0, \infty)^2 : hX(t) \leq x(t) \leq HX(t), \ kY(t) \leq y(t) \leq KY(t), \ t \geq T_0\}$$

and consider the mapping $\Phi : \mathcal{X} \to C[T_0, \infty)$ defined by

$$\Phi(x,y)(t) = \big(\mathcal{F}(x,y)(t), \mathcal{G}(x,y)(t)\big), \quad t \geq T_0, \tag{3.35}$$

where

$$\begin{aligned} \mathcal{F}(x,y)(t) &= x_0 + \int_{T_0}^{t}\int_{s}^{\infty} \big[p_1(r)x(r)^{\alpha_1} + q_1(r)y(r)^{\beta_1}\big]drds, \\ \mathcal{G}(x,y)(t) &= y_0 + \int_{T_0}^{t}\int_{s}^{\infty} \big[p_2(r)x(r)^{\alpha_2} + q_2(r)y(r)^{\beta_2}\big]drds \end{aligned} \tag{3.36}$$

with constants x_0 and y_0 satisfying

$$hX(T_1) \leq x_0 \leq \frac{1}{2}HX(T_0), \quad kY(T_1) \leq y_0 \leq \frac{1}{2}KY(T_0). \tag{3.37}$$

(i) $\Phi(\mathcal{X}) \subset \mathcal{X}$. Let $(x(t), y(t)) \in \mathcal{X}$. Then, using (3.34) we see that

$$\begin{aligned} p_1(t)x(t)^{\alpha_1} + q_1(t)y(t)^{\beta_1} &= p_1(t)x(t)^{\alpha_1}\left(1 + \frac{q_1(t)y(t)^{\beta_1}}{p_1(t)x(t)^{\alpha_1}}\right) \leq 2p_1(t)x(t)^{\alpha_1}, \\ p_2(t)x(y)^{\alpha_2} + q_2(t)y(t)^{\beta_2} &= q_2(t)y(t)^{\beta_2}\left(1 + \frac{p_2(t)x(t)^{\alpha_2}}{q_2(t)y(t)^{\beta_2}}\right) \leq 2q_2(t)y(t)^{\beta_2}. \end{aligned} \tag{3.38}$$

Thus, we obtain for $t \geq T_0$

$$\begin{aligned} \mathcal{F}(x,y)(t) &\leq \frac{1}{2}HX(T_0) + 2\int_{T_0}^{t}\int_{s}^{\infty} q_1(r)\big(HX(r)\big)^{\alpha_1}drds \leq \frac{1}{2}HX(T_0) + 4H^{\alpha_1}X(t) \\ &\leq \frac{1}{2}HX(t) + \frac{1}{2}HX(t) = HX(t), \quad t \geq T_0, \end{aligned}$$

$$\mathcal{F}(x,y)(t) \geq x_0 \geq hX(T_1) \geq hX(t) \quad \text{for } T_0 \leq t \leq T_1,$$

and

$$\mathcal{F}(x,y)(t) \geq \int_{T_0}^{t}\int_{s}^{\infty} p_1(r)\big(hX(r)\big)^{\alpha_1}drds \geq \frac{1}{2}h^{\alpha_1}X(t) \geq hX(t) \quad \text{for } t \geq T_1.$$

Likewise we prove that $kY(t) \leq \mathcal{G}(x,y)(t) \leq KY(t)$ for $t \geq T_0$. This shows in view of (3.35)-(3.36) that Φ is a self-map of $\mathcal{X}$.

(ii) $\Phi(\mathcal{X})$ is relative compact. The inclusion $\Phi(\mathcal{X}) \subset \mathcal{X}$ implies that $\Phi(\mathcal{X})$ is locally uniformly bounded on $[T_0, \infty)$. The inequalities

$$0 \leq (\mathcal{F}(x,y))'(t) \leq \int_{t}^{\infty} \big[H^{\alpha_1}p_1(s)X(s)^{\alpha_1} + K^{\beta_1}q_1(s)Y(s)^{\beta_1}\big]ds,$$

$$0 \leq (\mathcal{G}(x,y))'(t) \leq \int_{t}^{\infty} \big[H^{\alpha_2}p_2(s)X(s)^{\alpha_2} + K^{\beta_2}q_2(s)Y(s)^{\beta_2}\big]ds,$$

holding for $t \geq T_0$ and for all $(x,y) \in \mathcal{X}$ ensure that $\Phi(\mathcal{X})$ is locally equicontinuous on $[T_0, \infty)$. Then, the relative compactness of $\mathcal{F}(\mathcal{X})$ follows from the Arzela-Ascoli lemma.

(iii) Φ is continuous. Let $\{(x_n(t), y_n(t))\}$ be a sequence in $\mathcal{X}$ converging to $(x(t), y(t))$ as $t \to \infty$ uniformly on any compact subinterval of $[T_0, \infty)$. Noting that

$$|\mathcal{F}(x_n,y_n)(t) - \mathcal{F}(x,y)(t)| \leq t\int_{t}^{\infty} \big[p_1(s)|x_n(s)^{\alpha_1} - x(s)^{\alpha_1}| + q_1(s)|y_n(s)^{\beta_1} - y(s)^{\beta_1}|\big]ds,$$

$$|\mathcal{G}(x_n,y_n)(t)-\mathcal{G}(x,y)(t)|\le t\int_t^\infty \big[p_2(s)|x_n(s)^{\alpha_2}-x(s)^{\alpha_2}|+q_2(s)|y_n(s)^{\beta_2}-y(s)^{\beta_2}|\big]ds,$$

and applying the Lebesgue dominated convergence theorem to the right-hand sides of the above inequalities, it follows that

$$\mathcal{F}(x_n,y_n)(t)\to\mathcal{F}(x,y)(t),\quad \mathcal{G}(x_n,y_n)(t)\to\mathcal{G}(x,y)(t)$$

as $n\to\infty$ uniformly on compact subintervals of $[T_0,\infty)$. This implies the continuity of Φ.

Therefore, the Schauder-Tychonoff fixed theorem guarantees the existence of an element $(x(t),y(t))\in\mathcal{X}$ such that $(x(t),y(t))=\Phi(x(t),y(t)), t\ge T_0$, that is,

$$x(t)=\mathcal{F}(x,y)(t)=x_0+\int_{T_0}^t\int_s^\infty \big[p_1(r)x(r)^{\alpha_1}+q_1(r)y(r)^{\beta_1}\big]drds,$$

$$y(t)=\mathcal{G}(x,y)(t)=y_0+\int_{T_0}^t\int_s^\infty \big[p_2(r)x(r)^{\alpha_2}+q_2(r)y(r)^{\beta_2}\big]drds,$$

for $t\ge T_0$.

To complete the proof of Theorems 3.1-3.6, we have to verify that the intermediate solutions of (A) constructed above are actually regularly varying functions. For this purpose we use the generalized L'Hospital rule contained in the following lemma. (For the proof see Haupt and Aumann [2].)

Lemma 3.1. *Let $f(t),g(t)\in C^1[T,\infty)$ and suppose that*

$$\lim_{t\to\infty}f(t)=\lim_{t\to\infty}g(t)=\infty \quad \textit{and} \quad g'(t)>0 \quad \textit{for all large } t,$$

or

$$\lim_{t\to\infty}f(t)=\lim_{t\to\infty}g(t)=0 \quad \textit{and} \quad g'(t)<0 \quad \textit{for all large } t.$$

Then,

$$\liminf_{t\to\infty}\frac{f'(t)}{g'(t)}\le\liminf_{t\to\infty}\frac{f(t)}{g(t)},\quad \limsup_{t\to\infty}\frac{f(t)}{g(t)}\le\limsup_{t\to\infty}\frac{f'(t)}{g'(t)}.$$

Now, we define the functions $u(t)$ and $v(t)$ on $[a,\infty)$ by

$$u(t)=\int_a^t\int_s^\infty \big[p_1(r)X(r)^{\alpha_1}+q_1(r)Y(r)^{\beta_1}\big]drds,$$

$$v(t)=\int_a^t\int_s^\infty \big[p_2(r)X(r)^{\alpha_2}+q_2(r)Y(r)^{\beta_2}\big]drds.$$

Since (3.23)-(3.28) imply that

$$(3.39)\quad p_1(t)X(t)^{\alpha_1}+q_1(t)Y(t)^{\beta_1}\sim p_1(t)X(t)^{\alpha_1},\quad p_2(t)X(t)^{\alpha_2}+q_2(t)Y(t)^{\beta_2}\sim q_2(t)Y(t)^{\beta_2}$$

as $t\to\infty$, from the asymptotic relations (3.29) we obtain

$$(3.40)\qquad u(t)\sim X(t),\quad v(t)\sim Y(t),\quad t\to\infty.$$

We also use the relations

$$p_1(t)x(t)^{\alpha_1} + q_1(t)y(t)^{\beta_1} \sim p_1(t)x(t)^{\alpha_1}, \quad p_2(t)x(t)^{\alpha_2} + q_2(t)y(t)^{\beta_2} \sim q_2(t)y(t)^{\beta_2} \tag{3.41}$$

as $t \to \infty$, which follows from (3.39). Put

$$l = \liminf_{t\to\infty} \frac{x(t)}{u(t)}, \quad L = \limsup_{t\to\infty} \frac{x(t)}{u(t)}, \quad m = \liminf_{t\to\infty} \frac{y(t)}{v(t)}, \quad M = \limsup_{t\to\infty} \frac{y(t)}{v(t)}. \tag{3.42}$$

It is clear that $0 < l \le L < \infty$ and $0 < m \le M < \infty$. Applying Lemma 3.1 to l and m and taking (3.39)-(3.41) into account, we get

$$\begin{aligned} l &\ge \liminf_{t\to\infty} \frac{\int_t^\infty [p_1(s)x(s)^{\alpha_1} + q_1(s)y(s)^{\beta_1}]\,ds}{\int_t^\infty [p_1(s)X(s)^{\alpha_1} + q_1(s)Y(s)^{\beta_1}]\,ds} \\ &\ge \liminf_{t\to\infty} \frac{p_1(t)x(t)^{\alpha_1} + q_1(t)y(t)^{\beta_1}}{p_1(t)X(t)^{\alpha_1} + q_1(t)Y(t)^{\beta_1}} = \liminf_{t\to\infty} \frac{p_1(t)x(t)^{\alpha_1}}{p_1(t)X(t)^{\alpha_1}} \\ &= \left(\liminf_{t\to\infty} \frac{x(t)}{X(t)}\right)^{\alpha_1} = \left(\liminf_{t\to\infty} \frac{x(t)}{u(t)}\right)^{\alpha_1} = l^{\alpha_1}, \end{aligned}$$

and

$$\begin{aligned} m &\ge \liminf_{t\to\infty} \frac{\int_t^\infty [p_2(s)x(s)^{\alpha_2} + q_2(s)y(s)^{\beta_2}]\,ds}{\int_t^\infty [p_2(s)X(s)^{\alpha_2} + q_2(s)Y(s)^{\beta_2}]\,ds} \\ &\ge \liminf_{t\to\infty} \frac{p_2(t)x(t)^{\alpha_2} + q_2(t)y(t)^{\beta_2}}{p_2(t)X(t)^{\alpha_2} + q_2(t)Y(t)^{\beta_2}} = \liminf_{t\to\infty} \frac{q_2(t)y(t)^{\beta_2}}{q_2(t)Y(t)^{\beta_2}} \\ &= \left(\liminf_{t\to\infty} \frac{y(t)}{Y(t)}\right)^{\beta_2} = \left(\liminf_{t\to\infty} \frac{y(t)}{v(t)}\right)^{\beta_2} = m^{\beta_2}. \end{aligned}$$

Thus, we have

$$l \ge l^{\alpha_1} \quad \text{and} \quad m \ge m^{\beta_2}.$$

Since $\alpha_1 < 1$ and $\beta_2 < 1$, it follows that

$$l \ge 1 \quad \text{and} \quad m \ge 1. \tag{3.43}$$

Likewise, application of Lemma 3.1 to L and M yields

$$L \le L^{\alpha_1} \quad \text{and} \quad M \le M^{\beta_2},$$

which leads to

$$L \le 1 \quad \text{and} \quad M \le 1. \tag{3.44}$$

From (3.43) and (3.44) it follows that $l = L$ and $m = M$, that is,

$$\lim_{t\to\infty} \frac{x(t)}{u(t)} = 1, \quad \lim_{t\to\infty} \frac{y(t)}{v(t)} = 1.$$

Therefore we conclude from (3.40) that

$$x(t) \sim u(t) \sim X(t), \quad y(t) \sim v(t) \sim Y(t), \quad t \to \infty,$$

confirming that x and y are regularly varying functions of the desired indices. This completes the proof of Theorems 3.1-3.6.

Remark 3.1. In addition to (3.1) and (3.2) assume that $p_2(t) \in \mathrm{RV}(\lambda_2)$, $q_1(t) \in \mathrm{RV}(\mu_1)$ are expressed as

$$p_2(t) = t^{\lambda_2} l_2(t), \quad q_1(t) = t^{\mu_1} m_1(t), \quad l_2, m_1 \in \mathrm{SV}. \tag{3.45}$$

Using (3.23)-(3.28) we see that

$$\frac{q_1(t)Y_i(t)^{\beta_1}}{p_1(t)X_j(t)^{\alpha_1}} = t^{\mu_1+\beta_1\sigma-\lambda_1-\alpha_1\rho} L_{ij}(t), \quad \frac{p_2(t)X_j(t)^{\alpha_2}}{q_2(t)Y_i(t)^{\beta_2}} = t^{\lambda_2-\alpha_2\rho-\mu_2-\beta_2\sigma} M_{ij}(t),$$

for $i, j = 1, 2, 3$, and some $L_{ij}, M_{ij} \in \mathrm{SV}$. Thus, (3.23)-(3.28) are satisfied regardless of L_{ij} and M_{ij} if

$$\mu_1 + \beta_1\sigma < \lambda_1 + \alpha_1\rho \quad \text{and} \quad \lambda_2 + \alpha_2\rho < \mu_2 + \beta_2\sigma. \tag{3.46}$$

This can be used to get useful practical criteria for the existence of intermediate regularly varying solutions of the types (i)-(vi) for system (A).

Corollary 3.1. *Assume that* (3.1)-(3.3) *and* (3.4) *hold. Let* $\rho \in (0,1)$ *and* $\sigma \in (0,1)$ *be given by* (3.5) *and* (3.6). *If* (3.46) *holds, then system* (A) *possesses intermediate regularly varying solutions* $(x(t), y(t))$ *of index* (ρ, σ) *whose asymptotic behavior is governed by the unique formula* (3.7).

Corollary 3.2. *Assume that* (3.1)-(3.2), (3.4) *and* (3.10) *hold. Let* $\sigma \in (0,1)$ *be given by* (3.6). *If*

$$\mu_1 + \beta_1\sigma < -1 \quad \textit{and} \quad \lambda_2 + \alpha_2 < \mu_2 + \beta_2\sigma, \tag{3.47}$$

then system (A) *possesses intermediate solutions* $(x(t), y(t)) \in$ ntr-RV(1)$\times$ RV(σ) *whose asymptotic behavior is governed by the unique formula* (3.11).

Corollary 3.3. *Assume that* (3.1)-(3.2), (3.4) *and* (3.13) *hold. Let* $\sigma \in (0,1)$ *be given by* (3.6). *If*

$$\mu_1 + \beta_1\sigma < -2 \quad \textit{and} \quad \lambda_2 < \mu_2 + \beta_2\sigma, \tag{3.48}$$

then system (A) *possesses intermediate solutions* $(x(t), y(t)) \in$ ntr-RV(0)$\times$ RV(σ) *whose asymptotic behavior is governed by the unique formula* (3.14).

Corollary 3.4. *Assume that* (3.1)-(3.2), (3.10) *and* (3.16) *hold. If*

$$\mu_1 + \beta_1\sigma < -1 \quad \textit{and} \quad \lambda_2 + \alpha_2 < -1, \tag{3.49}$$

then system (A) *possesses intermediate solutions* $(x(t), y(t)) \in$ ntr-RV(1)$\times$ ntr-RV(1) *whose asymptotic behavior is governed by the unique formula* (3.17).

Corollary 3.5. *Assume that* (3.1)-(3.2), (3.10) *and* (3.19) *hold. If*

$$\mu_1 < -1 \quad \textit{and} \quad \lambda_2 + \alpha_2 < -2, \tag{3.50}$$

then system (A) *possesses intermediate solutions* $(x(t), y(t)) \in$ ntr-RV(1)$\times$ ntr-RV(0) *whose asymptotic behavior is governed by the unique formula* (3.20).

Corollary 3.6. *Assume that* (3.1)-(3.2), (3.13) *and* (3.19) *hold. If*

$$\mu_1 < -2 \quad \text{and} \quad \lambda_2 < -2, \tag{3.51}$$

then system (A) *possesses intermediate solutions* $(x(t), y(t)) \in$ ntr-RV(0)$\times$ ntr-RV(0) *whose asymptotic behavior is governed by the unique formula* (3.22).

4 Perturbations of the cyclic system

In this section we regard (A) as a small perturbation of the cyclic system

$$x'' + q_1(t)y^{\beta_1} = 0, \quad y'' + p_2(t)x^{\alpha_2} = 0, \tag{A$_c$}$$

where

$$\alpha_2\beta_1 < 1 \tag{4.1}$$

and $q_1(t)$ and $p_2(t)$ are continuous regularly varying functions of indices μ_1 and λ_2, respectively, expressed as

$$q_1(t) = t^{\mu_1}m_1(t), \quad p_2(t) = t^{\lambda_2}l_2(t), \quad m_1, l_2 \in \mathrm{SV}. \tag{4.2}$$

An intermediate positive solution $(x(t), y(t))$ of (Ac) defined on $[t_0, \infty)$ satisfies the system of integral equations

$$x(t) = x_0 + \int_{t_0}^{t}\int_{s}^{\infty} q_1(r)y(r)^{\beta_1}drds, \quad y(t) = y_0 + \int_{t_0}^{t}\int_{s}^{\infty} p_2(r)x(r)^{\alpha_2}drds, \tag{IA$_c$}$$

for some positive constants x_0 and y_0, and hence the system of asymptotic integral relations

$$x(t) \sim \int_{t_0}^{t}\int_{s}^{\infty} q_1(r)y(r)^{\beta_1}drds, \quad y(t) \sim \int_{t_0}^{t}\int_{s}^{\infty} p_2(r)x(r)^{\alpha_2}drds, \tag{AR$_c$}$$

Lemma 4.1 *Let* (4.1) *and* (4.2) *hold. System* (AR$_c$) *has regularly varying solutions of index* (ρ, σ) *with* $\rho \in (0,1)$ *and* $\sigma \in (0,1)$ *if and only if* (λ_2, μ_1) *satisfies the system of inequalities*

$$0 < \mu_1 + 2 + \beta_1(\lambda_2 + 2) < 1 - \alpha_2\beta_1, \quad 0 < \alpha_2(\mu_1 + 2) + \lambda_2 + 2 < 1 - \alpha_2\beta_1, \tag{4.3}$$

in which case ρ *and* σ *are given by*

$$\rho = \frac{\mu_1 + 2 + \beta_1(\lambda_2 + 2)}{1 - \alpha_2\beta_1}, \quad \sigma = \frac{\alpha_2(\mu_1 + 2) + \lambda_2 + 2}{1 - \alpha_2\beta_1}, \tag{4.4}$$

and the asymptotic behavior of any such solution $(x(t), y(t))$ *is governed by the formulas*

$$x(t) \sim X_1(t), \quad y(t) \sim Y_1(t), \quad t \to \infty, \tag{4.5}$$

where the functions $X_1 \in \mathrm{RV}(\rho)$ *and* $Y_1 \in \mathrm{RV}(\sigma)$ *on* $[a, \infty)$ *are defined by*

$$(4.6)\qquad X_1(t) = \left[\frac{t^{2(\beta_1+1)}q_1(t)p_2(t)^{\beta_1}}{\Delta(\rho)\Delta(\sigma)^{\beta_1}}\right]^{\frac{1}{1-\alpha_2\beta_1}}, \quad Y_1(t) = \left[\frac{t^{2(\alpha_2+1)}q_1(t)^{\alpha_2}p_2(t)}{\Delta(\rho)^{\alpha_2}\Delta(\sigma)}\right]^{\frac{1}{1-\alpha_2\beta_1}},$$

where $\Delta(\tau) = \tau(1-\tau)$ *for* $\tau \in (0,1)$.

PROOF. (The "only if" part) Suppose that $(\mathrm{AR_c})$ has a regularly varying solution $(x(t), y(t)), t \geq t_0$, of index (ρ, σ) with $\rho \in (0,1)$ and $\sigma \in (0,1)$. From $(\mathrm{AR_c})$ rewritten as

$$x(t) \sim \int_{t_0}^{t}\int_{s}^{\infty} r^{\mu_1+\beta_1\sigma}m_1(r)\eta(r)^{\beta_1}drds, \quad y(t) \sim \int_{t_0}^{t}\int_{s}^{\infty} r^{\lambda_2+\alpha_2\rho}l_2(r)\xi(r)^{\alpha_2}drds,$$

we see via Karamata's integration theorem that $-2 < \mu_1+\beta_1\sigma < -1$, $-2 < \lambda_2+\alpha_2\rho < -1$, and

$$(4.7)\quad x(t) \sim \frac{t^{\mu_1+\beta_1\sigma+2}m_1(t)\eta(t)^{\beta_1}}{[-(\mu_1+\beta_1\sigma+1)](\mu_1+\beta_1\sigma+2)}, \quad y(t) \sim \frac{t^{\lambda_2+\alpha_2\rho+2}l_2(t)\xi(t)^{\alpha_2}}{[-(\lambda_2+\alpha_2\rho+1)](\lambda_2+\alpha_2\rho+2)},$$

as $t \to \infty$. This means that $\rho = \mu_1+\beta_1\sigma+2$ and $\sigma = \lambda_2+\alpha_2\rho+2$, which implies that ρ and σ are determined by (4.4). Requiring that $\rho \in (0,1)$ and $\sigma \in (0,1)$ in (4.4) immediately leads to (4.3). Noting that (4.7) can be expressed as

$$x(t) \sim \frac{t^2q_1(t)y(t)^{\beta_1}}{\Delta(\rho)}, \quad y(t) \sim \frac{t^2p_2(t)x(t)^{\alpha_2}}{\Delta(\sigma)}, \quad t \to \infty,$$

and combining these two relations, we easily conclude that the asymptotic formulas for $x(t)$ and $y(t)$ are given by (4.5) with $X_1(t)$ and $Y_1(t)$ defined by (4.6).

(The "if" part) Suppose that (λ_2, μ_1) satisfies (4.3) and define (ρ, σ) by (4.4). We define $(X_1(t), Y_1(t))$ by (4.6), which can be rewritten as

$$X_1(t) = t^{\rho}\left[\frac{m_1(t)l_2(t)^{\beta_1}}{\Delta(\rho)\Delta(\sigma)^{\beta_1}}\right]^{\frac{1}{1-\alpha_2\beta_1}}, \quad Y_1(t) = t^{\sigma}\left[\frac{m_1(t)^{\alpha_2}l_2(t)}{\Delta(\rho)^{\alpha_2}\Delta(\sigma)}\right]^{\frac{1}{1-\alpha_2\beta_1}}.$$

It suffices to prove that

$$(4.8)\quad \int_{t_0}^{t}\int_{s}^{\infty} q_1(r)Y_1(r)^{\beta_1}drds \sim X_1(t), \quad \int_{t_0}^{t}\int_{s}^{\infty} p_2(r)X_1(r)^{\alpha_2}drds \sim Y_1(t), \quad t \to \infty.$$

Using Karamata's integration theorem, we compute as follows:

$$\int_{t}^{\infty} q_1(s)Y_1(s)^{\beta_1}ds = \int_{t}^{\infty} s^{\mu_1+\beta_1\sigma}m_1(s)\left[\frac{m_1(s)^{\alpha_2}l_2(s)}{\Delta(\rho)^{\alpha_2}\Delta(\sigma)}\right]^{\frac{\beta_1}{1-\alpha_2\beta_1}}ds$$

$$= \int_{t}^{\infty} s^{\rho-2}l_2(s)\left[\frac{m_1(t)^{\alpha_2}l_2(t)}{\Delta(\rho)^{\alpha_2}\Delta(\sigma)}\right]^{\frac{\beta_1}{1-\alpha_2\beta_1}}ds \sim \frac{t^{\rho-1}m_1(t)}{1-\rho}\left[\frac{m_1(t)^{\alpha_2}l_2(t)}{\Delta(\rho)^{\alpha_2}\Delta(\sigma)}\right]^{\frac{\beta_1}{1-\alpha_2\beta_1}}, \quad t \to \infty,$$

and hence

$$\int_{t_0}^{t}\int_{s}^{\infty} q_1(r)Y_1(r)^{\beta_1}drds \sim \frac{t^{\rho}m_1(t)}{\rho(1-\rho)}\left[\frac{m_1(t)^{\alpha_2}l_2(t)}{\Delta(\rho)^{\alpha_2}\Delta(\sigma)}\right]^{\frac{\beta_1}{1-\alpha_2\beta_1}} = X_1(t), \quad t \to \infty.$$

Similarly we obtain

$$\int_{t_0}^{t}\int_{s}^{\infty} p_2(r)X_1(r)^{\alpha_2}drds \sim \frac{t^{\sigma}l_2(t)}{\sigma(1-\sigma)}\Big[\frac{m_1(t)l_2(t)^{\beta_1}}{\Delta(\rho)\Delta(\sigma)^{\beta_1}}\Big]^{\frac{\alpha_2}{1-\alpha_2\beta_1}} = Y_1(t), \quad t\to\infty.$$

This ensures the truth of (4.8). This completes the proof of Lemma 4.1.

Lemma 4.2. *Let* (4.1) *and* (4.2) *hold. System* ($\mathrm{AR_c}$) *has a solution such that* $(x(t), y(t)) \in$ ntr-RV(1)$\times$RV(σ) with $\sigma \in (0,1)$ *if and only if*

$$-\beta_1 - 1 < \mu_1 < -1, \quad \mu_1 + 1 + \beta_1(\alpha_2 + \lambda_2 + 2) = 0, \tag{4.9}$$

and

$$\int_a^{\infty} t^{\beta_1(\alpha_2+2)}q_1(t)p_2(t)^{\beta_1}dt < \infty, \tag{4.10}$$

in which case σ *is given by*

$$\sigma = -\frac{\mu_1+1}{\beta_1} \quad (= \alpha_2 + \lambda_2 + 2), \tag{4.11}$$

and the asymptotic behavior of $(x(t), y(t))$ *is governed by the formulas*

$$x(t) \sim X_2(t), \quad y(t) \sim Y_2(t), \quad t\to\infty, \tag{4.12}$$

where the functions $X_2 \in$ ntr-RV(1) *and* $Y_2 \in$ RV(σ) *on* $[a,\infty)$ *are defined by*

$$\begin{aligned} X_2(t) &= t\Big[\frac{1-\alpha_2\beta_1}{\Delta(\sigma)^{\beta_1}}\int_t^{\infty} s^{\beta_1(\alpha_2+2)}q_1(s)p_2(s)^{\beta_1}ds\Big]^{\frac{1}{1-\alpha_2\beta_1}}, \\ Y_2(t) &= \frac{t^{\alpha_2+2}p_2(t)}{\Delta(\sigma)}\Big[\frac{1-\alpha_2\beta_1}{\Delta(\sigma)^{\beta_1}}\int_t^{\infty} s^{\beta_1(\alpha_2+2)}q_1(s)p_2(s)^{\beta_1}ds\Big]^{\frac{\alpha_2}{1-\alpha_2\beta_1}} \end{aligned} \tag{4.13}$$

where $\Delta(\sigma) = \sigma(1-\sigma)$.

PROOF. (The "only if" part) Suppose that ($\mathrm{AR_c}$) has a regularly varying solution $(x(t), y(t))$ on $[t_0,\infty)$ of index $(1,\sigma)$ with $\sigma \in (0,1)$. From ($\mathrm{AR_c}$) rewritten as

$$x(t) \sim \int_{t_0}^{t}\int_{s}^{\infty} r^{\mu_1+\beta_1\sigma}m_1(r)\eta(r)^{\beta_1}drds, \quad y(t) \sim \int_{t_0}^{t}\int_{s}^{\infty} r^{\lambda_2+\alpha_2}l_2(r)\xi(r)^{\alpha_2}drds, \tag{4.14}$$

we see via Karamata's integration theorem that $\mu_1 + \beta_1\sigma = -1$ and $-2 < \lambda_2 + \alpha_2 < -1$. Note that $\sigma = -(\mu_1+1)/\beta_1$, so that the requirement $\sigma \in (0,1)$ implies $\mu_1 \in (-\beta_1 - 1, -1)$. Using Karamata's integration theorem we transform (4.14) into

$$x(t) \sim t\int_t^{\infty} s^{-1}m_1(s)\eta(s)^{\beta_1}ds, \quad y(t) \sim \frac{t^{\lambda_2+\alpha_2+2}l_2(t)\xi(t)^{\alpha_2}}{[-(\lambda_2+\alpha_2+1)](\lambda_2+\alpha_2+2)}, \quad t\to\infty. \tag{4.15}$$

This shows that $\sigma = \alpha_2 + \lambda_2 + 2$, so that $\mu_1 + 1 + \beta_1(\alpha_2 + \lambda_2 + 2) = 0$. Rewrite the second relation in (4.15) as

$$\eta(t) \sim \frac{l_2(t)\xi(t)^{\alpha_2}}{\Delta(\sigma)}, \quad t\to\infty, \tag{4.16}$$

and combine it with the first relation in (4.15). We then obtain

$$\xi(t) \sim \frac{1}{\Delta(\sigma)^{\beta_1}} \int_t^\infty s^{\beta_1(\alpha_2+2)} q_1(s) p_2(s)^{\beta_1} \xi(s)^{\alpha_2\beta_1} ds, \quad t \to \infty. \tag{4.17}$$

Let $\widetilde{\xi}(t)$ denote the right-hand side of (4.17). Then, (4.17) can be transformed into the differential asymptotic relation for $\widetilde{\xi}(t)$

$$-\widetilde{\xi}(t)^{-\alpha_2\beta_1} \widetilde{\xi}'(t) \sim \frac{t^{\beta_1(\alpha_2+2)} q_1(t) p_2(t)^{\beta_1}}{\Delta(\sigma)^{\beta_1}}, \quad t \to \infty. \tag{4.18}$$

Since the left-hand side of (4.18) is integrable on $[t_0, \infty)$ because $\widetilde{\xi}(t) \to 0$ as $t \to \infty$, so is the right-hand side which ensures that (4.10) holds true, and integrating (4.18) from t to ∞, we obtain

$$\xi(t) \sim \widetilde{\xi}(t) \sim \left[\frac{1-\alpha_2\beta_1}{\Delta(\sigma)^{\beta_1}} \int_t^\infty s^{\beta_1(\alpha_2+2)} q_1(s) p_2(s)^{\beta_1} ds \right]^{\frac{1}{1-\alpha_2\beta_1}}, \quad t \to \infty,$$

which, combined with (4.16), establishes the asymptotic formula (4.12) for $(x(t), y(t))$.

(The "if" part) Let (λ_2, μ_1) satisfy (4.9) and σ be given by (4.11). Consider the vector function $(X_2(t), Y_2(t))$ defined on $[t_0, \infty)$ by (4.13). Using Karamata's integration theorem, we can show that $(X_2(t), Y_2(t))$ satisfies (AR$_c$), i.e.,

$$\int_{t_0}^t \int_s^\infty q_1(r) Y_2(r)^{\beta_1} dr ds \sim X_2(t), \quad \int_{t_0}^t \int_s^\infty p_2(r) X_2(r)^{\alpha_2} dr ds \sim Y_2(t), \quad t \to \infty. \tag{4.19}$$

This completes the proof of Lemma 4.2.

Lemma 4.3. *Let* (4.1) *and* (4.2) *hold. System* (AR$_c$) *has a solution such that* $(x(t), y(t)) \in$ ntr-RV(0)×RV(σ) *with* $\sigma \in (0,1)$ *if and only if*

$$-\beta_1 - 2 < \mu_1 < -2, \quad \mu_1 + 2 + \beta_1(\lambda_2 + 2) = 0, \tag{4.20}$$

and

$$\int_a^\infty t^{2\beta_1+1} q_1(t) p_2(t)^{\beta_1} dt = \infty, \tag{4.21}$$

in which case σ *is given by*

$$\sigma = -\frac{\mu_1 + 2}{\beta_1} \quad (= \lambda_2 + 2), \tag{4.22}$$

and the asymptotic behavior of $(x(t), y(t))$ *is governed by the formulas*

$$x(t) \sim X_3(t), \quad y(t) \sim Y_3(t), \quad t \to \infty, \tag{4.23}$$

where the functions $X_3 \in$ ntr-RV(0) *and* $Y_3 \in$ RV(σ) *on* $[a, \infty)$ *are defined by*

$$\begin{aligned} X_3(t) &= \left[\frac{1-\alpha_2\beta_1}{\Delta(\sigma)^{\beta_1}} \int_a^t s^{2\beta_1+1} q_1(s) p_2(s)^{\beta_1} ds \right]^{\frac{1}{1-\alpha_2\beta_1}}, \\ Y_3(t) &= \frac{t^2 p_2(t)}{\Delta(\sigma)} \left[\frac{1-\alpha_2\beta_1}{\Delta(\sigma)^{\beta_1}} \int_a^t s^{2\beta_1+1} q_1(s) p_2(s)^{\beta_1} ds \right]^{\frac{\alpha_2}{1-\alpha_2\beta_1}}, \end{aligned} \tag{4.24}$$

where $\Delta(\sigma) = \sigma(1-\sigma)$.

PROOF. (The "only if" part) Suppose that (AR_c) has a regularly varying solution $(x(t), y(t))$ on $[t_0, \infty)$ of index $(0, \sigma)$ with $\sigma \in (0,1)$. From (AR_c) rewritten as

$$x(t) \sim \int_{t_0}^{t}\int_{s}^{\infty} r^{\mu_1+\beta_1\sigma} m_1(r)\eta(r)^{\beta_1} drds, \quad y(t) \sim \int_{t_0}^{t}\int_{s}^{\infty} r^{\lambda_2} l_2(r)\xi(r)^{\alpha_2} drds, \tag{4.25}$$

it follows that $\mu_1 + \beta_1\sigma = -2$ and $-2 < \lambda_2 < -1$. Thus, $\sigma = -(\mu_1+2)/\beta_1$ and this together with $\sigma \in (0,1)$ implies $\mu_1 \in (-\beta_1 - 2, -2)$. Karamata's integration theorem applied to (4.25) yields

$$x(t) \sim \int_{t_0}^{t} s^{-1} m_1(s)\eta(s)^{\beta_1} ds, \quad y(t) \sim \frac{t^{\lambda_2+2} l_2(t)\xi(t)^{\alpha_2}}{[-(\lambda_2+1)](\lambda_2+2)}, \quad t \to \infty. \tag{4.26}$$

This shows that $\sigma = \lambda_2 + 2$, and hence $\mu_1 + 2 + \beta_1(\lambda_2+2) = 0$. The second relation in (4.26) is rewritten as

$$\eta(t) \sim \frac{l_2(t)\xi(t)^{\alpha_2}}{\Delta(\sigma)}, \quad t \to \infty, \tag{4.27}$$

which, combined with the first relation in (4.26), gives

$$\xi(t) \sim \frac{1}{\Delta(\sigma)^{\beta_1}} \int_{t_0}^{t} s^{2\beta_1+1} q_1(s) p_2(s)^{\beta_1} \xi(s)^{\alpha_2\beta_1} ds, \quad t \to \infty. \tag{4.28}$$

We denote the right-hand side of (4.28) by $\widetilde{\xi}(t)$ and transform (4.28) into the following differential asymptotic relation for $\widetilde{\xi}(t)$:

$$\widetilde{\xi}(t)^{-\alpha_2\beta_1}\widetilde{\xi}'(t) \sim \frac{t^{2\beta_1+1} q_1(t) p_2(t)^{\beta_1}}{\Delta(\sigma)^{\beta_1}}, \quad t \to \infty. \tag{4.29}$$

The left-hand side of (4.29) is not integrable on $[t_0, \infty)$, nor is the right-hand side, that is, (4.21) must hold. Integrating (4.29) on $[t_0, t]$ shows that

$$\xi(t) \sim \widetilde{\xi}(t) \sim \left[\frac{1-\alpha_2\beta_1}{\Delta(\sigma)^{\beta_1}} \int_{t_0}^{t} s^{2\beta_1+1} q_1(s) p_2(s)^{\beta_1} ds\right]^{\frac{1}{1-\alpha_2\beta_1}} \sim \left[\frac{1-\alpha_2\beta_1}{\Delta(\sigma)^{\beta_1}} \int_{a}^{t} s^{2\beta_1+1} q_1(s) p_2(s)^{\beta_1} ds\right]^{\frac{1}{1-\alpha_2\beta_1}},$$

as $t \to \infty$, from which the asymptotic formulas (4.23) for $x(t)$ and $y(t)$ follow immediately.

(The "if" part) Consider the functions $X_3(t)$ and $Y_3(t)$ defined on $[a, \infty)$ by (4.24). Then, $(X_3(t), Y_3(t))$ satisfies (AR_c), i.e.,

$$\int_{t_0}^{t}\int_{s}^{\infty} q_1(r) Y_3(r)^{\beta_1} drds \sim X_3(t), \quad \int_{t_0}^{t}\int_{s}^{\infty} p_2(r) X_3(r)^{\alpha_2} drds \sim Y_3(t), \quad t \to \infty. \tag{4.30}$$

Theorem 4.1. *Let* (4.1), (4.2) *and* (4.3) *hold. Define the constants* ρ *and* σ *by* (4.4) *and consider the functions* $X_1(t)$ *and* $Y_1(t)$ *given by* (4.6). *Suppose that*

$$\lim_{t\to\infty} \frac{p_1(t) X_1(t)^{\alpha_1}}{q_1(t) Y_1(t)^{\beta_1}} = 0, \quad \lim_{t\to\infty} \frac{q_2(t) Y_1(t)^{\beta_2}}{p_2(t) X_1(t)^{\alpha_2}} = 0. \tag{4.31}$$

Then, system (A) *possesses solutions* $(x(t), y(t)) \in \mathrm{RV}(\rho)\times \mathrm{RV}(\sigma)$ *such that* (4.5) *holds.*

Theorem 4.2. *Let* (4.1), (4.2), (4.9) *and* (4.10) *hold. Define* σ *by* (4.11) *and consider functions* $X_2(t)$ *and* $Y_2(t)$ *given by* (4.13). *Suppose that*

$$\lim_{t\to\infty} \frac{p_1(t)X_2(t)^{\alpha_1}}{q_1(t)Y_2(t)^{\beta_1}} = 0, \quad \lim_{t\to\infty} \frac{q_2(t)Y_2(t)^{\beta_2}}{p_2(t)X_2(t)^{\alpha_2}} = 0. \tag{4.32}$$

Then, system (A) *possesses solutions* $(x(t), y(t)) \in$ ntr-$\mathrm{RV}(1)\times \mathrm{RV}(\sigma)$ *such that* (4.12) *holds.*

Theorem 4.3. *Let* (4.1), (4.2), (4.20) *and* (4.21) *hold. Define* σ *by* (4.22) *and consider functions* $X_3(t)$ *and* $Y_3(t)$ *given by* (4.24). *Suppose that*

$$\lim_{t\to\infty} \frac{p_1(t)X_3(t)^{\alpha_1}}{q_1(t)Y_3(t)^{\beta_1}} = 0, \quad \lim_{t\to\infty} \frac{q_2(t)Y_3(t)^{\beta_2}}{p_2(t)X_3(t)^{\alpha_2}} = 0. \tag{4.33}$$

Then, system (A) *possesses solutions* $(x(t), y(t)) \in \mathrm{RV}(0)\times \mathrm{RV}(\sigma)$ *such that* (4.23) *holds.*

PROOF. A simultaneous proof of the above theorems will be given. Let $(X(t), Y(t))$ denote any of the three functions $(X_i(t), Y_i(t)), i = 1, 2, 3$, defined, respectively, by (4.6), (4.13) and (4.24). (Naturally $(X_i(t), Y_i(t))$ should be used in proving Theorem 4.i, i = 1, 2, 3.) It is known that $(X(t), Y(t))$ satisfies

$$\int_b^t \int_s^\infty q_1(r)Y(r)^{\beta_1} dr ds \sim X(t), \quad \int_b^t \int_s^\infty p_2(r)X(r)^{\alpha_2} dr ds \sim Y(t), \quad t \to \infty, \tag{4.34}$$

for any $b \geq a$. There exists $T_0 > a$ such that

$$\int_{T_0}^t \int_s^\infty q_1(r)Y(r)^{\beta_1} dr ds \leq 2X(t), \quad \int_{T_0}^t \int_s^\infty p_2(r)X(r)^{\alpha_2} dr ds \leq 2Y(t), \quad t \geq T_0. \tag{4.35}$$

We may assume that T_0 is large enough so that $X(t)$ and $Y(t)$ are increasing for $t \geq T_0$. Since (4.34) holds for $b = T_0$, one finds $T_1 > T_0$ such that

$$\int_{T_0}^t \int_s^\infty q_1(r)Y(r)^{\beta_1} dr ds \geq \frac{1}{2}X(t), \quad \int_{T_0}^t \int_s^\infty p_2(r)X(r)^{\alpha_2} dr ds \geq \frac{1}{2}Y(t), \quad t \geq T_1. \tag{4.36}$$

Choose positive constants h, H, k and K so that $h < H,\ k < K$ and the following inequalities hold

$$2h \leq k^{\beta_1}, \quad 2k \leq h^{\alpha_2}, \quad 8K^{\beta_1} \leq H, \quad 8H^{\alpha_2} \leq K, \tag{4.37}$$

and

$$2hX(T_1) \leq HX(T_0), \quad 2kY(T_1) \leq KY(T_0). \tag{4.38}$$

Because of (4.31)-(4.33) one can choose $T_0 > a$ large enough so that in addition to (4.35)-(4.36) and (4.38) the following inequalities hold for $t \geq T_0$:

$$\frac{p_1(t)X(t)^{\alpha_1}}{q_1(t)Y(t)^{\beta_1}} \leq \frac{k^{\beta_1}}{H^{\alpha_1}}, \quad \frac{q_2(t)Y(t)^{\beta_2}}{p_2(t)X(t)^{\alpha_2}} \leq \frac{h^{\alpha_2}}{K^{\beta_2}}. \tag{4.39}$$

With these constants we define the set $\mathcal{X}$ comprised of continuous vector functions $(x(t), y(t))$ on $[T_0, \infty)$ such that

$$hX(t) \le x(t) \le HX(t), \quad kY(t) \le y(t) \le KY(t), \quad t \ge T_0.$$

It is clear that $\mathcal{X}$ is closed and convex in $C[T_0,\infty) \times C[T_0,\infty)$. Finally consider the mapping $\Phi : \mathcal{X} \to C[T_0,\infty) \times C(T_0,\infty)$ defined by

$$\Phi(x(t), y(t)) = \big(\mathcal{F}(x,y)(t), \mathcal{G}(x,y)(t)\big), \quad t \ge T_0, \tag{4.40}$$

where

$$\begin{aligned} \mathcal{F}(x,y)(t) &= x_0 + \int_{T_0}^{t}\int_{s}^{\infty} \big[p_1(r)x(r)^{\alpha_1} + q_1(r)y(r)^{\beta_1}\big]drds, \\ \mathcal{G}(x,y)(t) &= y_0 + \int_{T_0}^{t}\int_{s}^{\infty} \big[p_2(r)x(r)^{\alpha_2} + q_2(r)y(r)^{\beta_2}\big]drds. \end{aligned} \tag{4.41}$$

Here x_0 and y_0 are constants satisfying

$$hX(T_1) \le x_0 \le \frac{1}{2}HX(T_0), \quad kY(T_1) \le y_0 \le \frac{1}{2}KY(T_0). \tag{4.42}$$

(i) $\Phi(\mathcal{X}) \subset \mathcal{X}$. Let $(x(t), y(t)) \in \mathcal{X}$. Using (4.39) we see that

$$\begin{aligned} p_1(t)x(t)^{\alpha_1} + q_1(t)y(t)^{\beta_1} &= q_1(t)y(t)^{\beta_1}\left(1 + \frac{p_1(t)x(t)^{\alpha_1}}{q_1(t)y(t)^{\beta_1}}\right) \le 2q_1(t)y(t)^{\beta_1}, \\ p_2(t)x(t)^{\alpha_2} + q_2(t)y(t)^{\beta_2} &= p_2(t)x(t)^{\alpha_2}\left(1 + \frac{q_2(t)y(t)^{\beta_2}}{p_2(t)x(t)^{\alpha_2}}\right) \le 2p_2(t)x(t)^{\alpha_2}. \end{aligned} \tag{4.43}$$

Thus, we obtain for $t \ge T_0$

$$\begin{aligned} \mathcal{F}(x,y)(t) &\le \frac{1}{2}HX(T_0) + 2\int_{T_0}^{t}\int_{s}^{\infty} q_1(r)\big(KY(r)\big)^{\beta_1}drds \le \frac{1}{2}HX(T_0) + 4K^{\beta_1}X(t) \\ &\le \frac{1}{2}HX(t) + \frac{1}{2}HX(t) = HX(t), \quad t \ge T_0, \end{aligned}$$

$$\mathcal{F}(x,y)(t) \ge x_0 \ge hX(T_1) \ge hX(t) \quad \text{for } T_0 \le t \le T_1,$$

and

$$\mathcal{F}(x,y)(t) \ge \int_{T_0}^{t}\int_{s}^{\infty} q_1(r)\big(kY(r)\big)^{\beta_1}drds \ge \frac{1}{2}k^{\beta_1}X(t) \ge hX(t) \quad \text{for } t \ge T_1.$$

Likewise we prove that $kY(t) \le \mathcal{G}(x,y)(t) \le KY(t)$ for $t \ge T_0$. This shows in view of (4.40) that Φ is a self-map of $\mathcal{X}$.

(ii) $\Phi(\mathcal{X})$ is relative compact. The inclusion $\Phi(\mathcal{X}) \subset \mathcal{X}$ implies that $\Phi(\mathcal{X})$ is locally uniformly bounded on $[T_0, \infty)$. The inequalities

$$0 \le (\mathcal{F}(x,y))'(t) \le \int_{t}^{\infty} \big[H^{\alpha_1}p_1(s)X(s)^{\alpha_1} + K^{\beta_1}q_1(s)Y(s)^{\beta_1}\big]ds,$$

$$0 \le (\mathcal{G}(x,y))'(t) \le \int_{t}^{\infty} \big[H^{\alpha_2}p_2(s)X(s)^{\alpha_2} + K^{\beta_2}q_2(s)Y(s)^{\beta_2}\big]ds,$$

holding for $t \ge T_0$ and for all $(x,y) \in \mathcal{X}$ ensure that $\Phi(\mathcal{X})$ is locally equicontinuous on $[T_0, \infty)$. Then, the relative compactness of $\mathcal{F}(\mathcal{X})$ follows from the Arzela-Ascoli lemma.

(iii) Φ is continuous. Let $\{(x_n(t), y_n(t))\}$ be a sequence in $\mathcal{X}$ converging to $(x(t), y(t))$ as $t \to \infty$ uniformly on any compact subinterval of $[T_0, \infty)$. Noting that

$$|\mathcal{F}(x_n, y_n)(t) - \mathcal{F}(x, y)(t)| \leq t \int_t^\infty \left[p_1(s)|x_n(s)^{\alpha_1} - x(s)^{\alpha_1}| + q_1(s)|y_n(s)^{\beta_1} - y(s)^{\beta_1}|\right] ds,$$

$$|\mathcal{G}(x_n, y_n)(t) - \mathcal{G}(x, y)(t)| \leq t \int_t^\infty \left[p_2(s)|x_n(s)^{\alpha_2} - x(s)^{\alpha_2}| + q_2(s)|y_n(s)^{\beta_2} - y(s)^{\beta_2}|\right] ds,$$

and applying the Lebesgue dominated convergence theorem to the right-hand sides of the above inequalities, it follows that

$$\mathcal{F}(x_n, y_n)(t) \to \mathcal{F}(x, y)(t), \quad \mathcal{G}(x_n, y_n)(t) \to \mathcal{G}(x, y)(t)$$

as $n \to \infty$ uniformly on compact subintervals of $[T_0, \infty)$. This implies the continuity of Φ.

Therefore, the Schauder-Tychonoff fixed theorem guarantees the existence of an element $(x(t), y(t)) \in \mathcal{X}$ such that $(x(t), y(t)) = \Phi(x(t), y(t)), t \geq T_0$, that is,

$$x(t) = \mathcal{F}(x, y)(t) = x_0 + \int_{T_0}^t \int_s^\infty \left[p_1(r)x(r)^{\alpha_1} + q_1(r)y(r)^{\beta_1}\right] dr ds,$$

$$y(t) = \mathcal{G}(x, y)(t) = y_0 + \int_{T_0}^t \int_s^\infty \left[p_2(r)x(r)^{\alpha_2} + q_2(r)y(r)^{\beta_2}\right] dr ds,$$

for $t \geq T_0$.

To complete the proof of Theorems 4.1-4.3, we have to verify the intermediate solutions of (A) constructed above are actually regularly varying functions.

We define the functions $u(t)$ and $v(t)$ on $[a, \infty)$ by

$$u(t) = \int_a^t \int_s^\infty \left[p_1(r)X(r)^{\alpha_1} + q_1(r)Y(r)^{\beta_1}\right] dr ds,$$

$$v(t) = \int_a^t \int_s^\infty \left[p_2(r)X(r)^{\alpha_2} + q_2(r)Y(r)^{\beta_2}\right] dr ds.$$

Since (4.31)-(4.33) imply that

$$(4.44) \quad p_1(t)X(t)^{\alpha_1} + q_1(t)Y(t)^{\beta_1} \sim q_1(t)Y(t)^{\beta_1}, \quad p_2(t)X(t)^{\alpha_2} + q_2(t)Y(t)^{\beta_2} \sim p_2(t)X(t)^{\alpha_2}$$

as $t \to \infty$, from the asymptotic relations (4.34) we obtain

$$(4.45) \qquad u(t) \sim X(t), \quad v(t) \sim Y(t), \quad t \to \infty.$$

We also use the relations

$$(4.46) \qquad p_1(t)x(t)^{\alpha_1} + q_1(t)y(t)^{\beta_1} \sim q_1(t)y(t)^{\beta_1}, \quad p_2(t)x(t)^{\alpha_2} + q_2(t)y(t)^{\beta_2} \sim p_2(t)x(t)^{\alpha_2}$$

as $t \to \infty$, which follows from (4.45). Put

$$(4.47) \qquad l = \liminf_{t\to\infty} \frac{x(t)}{u(t)}, \quad L = \limsup_{t\to\infty} \frac{x(t)}{u(t)}, \quad m = \liminf_{t\to\infty} \frac{y(t)}{v(t)}, \quad M = \limsup_{t\to\infty} \frac{y(t)}{v(t)}.$$

It is clear that $0 < l \le L < \infty$ and $0 < m \le M < \infty$. Applying Lemma 3.1 to l and m and taking (4.44)-(4.46) into account, we get

$$\begin{aligned} l &\ge \liminf_{t\to\infty} \frac{\int_t^\infty \left[p_1(s)x(s)^{\alpha_1} + q_1(s)y(s)^{\beta_1}\right]ds}{\int_t^\infty \left[p_1(s)X(s)^{\alpha_1} + q_1(s)Y(s)^{\beta_1}\right]ds} \\ &\ge \liminf_{t\to\infty} \frac{p_1(t)x(t)^{\alpha_1} + q_1(t)y(t)^{\beta_1}}{p_1(t)X(t)^{\alpha_1} + q_1(t)Y(t)^{\beta_1}} = \liminf_{t\to\infty} \frac{q_1(t)y(t)^{\beta_1}}{q_1(t)Y(t)^{\beta_1}} \\ &= \left(\liminf_{t\to\infty} \frac{y(t)}{Y(t)}\right)^{\beta_1} = \left(\liminf_{t\to\infty} \frac{y(t)}{v(t)}\right)^{\beta_1} = m^{\beta_1}, \end{aligned}$$

and

$$\begin{aligned} m &\ge \liminf_{t\to\infty} \frac{\int_t^\infty \left[p_2(s)x(s)^{\alpha_2} + q_2(s)y(s)^{\beta_2}\right]ds}{\int_t^\infty \left[p_2(s)X(s)^{\alpha_2} + q_2(s)Y(s)^{\beta_2}\right]ds} \\ &\ge \liminf_{t\to\infty} \frac{p_2(t)x(t)^{\alpha_2} + q_2(t)y(t)^{\beta_2}}{p_2(t)X(t)^{\alpha_2} + q_2(t)Y(t)^{\beta_2}} = \liminf_{t\to\infty} \frac{p_2(t)x(t)^{\alpha_2}}{p_2(t)X(t)^{\alpha_2}} \\ &= \left(\liminf_{t\to\infty} \frac{x(t)}{X(t)}\right)^{\alpha_2} = \left(\liminf_{t\to\infty} \frac{x(t)}{u(t)}\right)^{\alpha_2} = l^{\alpha_2}. \end{aligned}$$

Thus, we have

$$l \ge m^{\beta_1} \quad \text{and} \quad m \ge l^{\alpha_2},$$

which implies that

$$l \ge l^{\alpha_2\beta_1} \quad \text{and} \quad m \ge m^{\alpha_2\beta_1} \quad \Longrightarrow \quad l \ge 1 \quad \text{and} \quad m \ge 1 \quad \text{because } \alpha_2\beta_1 < 1. \tag{4.48}$$

Likewise, application of Lemma 3.1 to L and M yields

$$L \le M^{\beta_1} \quad \text{and} \quad M \le L^{\alpha_2},$$

which leads to

$$L \le L^{\alpha_2\beta_1} \quad \text{and} \quad M \le M^{\alpha_2\beta_1} \quad \Longrightarrow \quad L \le 1 \quad \text{and} \quad M \le 1 \quad \text{because } \alpha_2\beta_1 < 1. \tag{4.49}$$

From (4.48) and (4.49) it follows that $l = L$ and $m = M$, that is,

$$\lim_{t\to\infty} \frac{x(t)}{u(t)} = 1, \quad \lim_{t\to\infty} \frac{y(t)}{v(t)} = 1.$$

Therefore we conclude from (4.45) that

$$x(t) \sim u(t) \sim X(t), \quad y(t) \sim v(t) \sim Y(t), \quad t \to \infty,$$

confirming that x and y are regularly varying functions of the desired indices. This completes the proof of Theorems 4.1-4.3.

Remark 4.1. Let $p_1(t) \in \mathrm{RV}(\lambda_1)$ and $q_2(t) \in \mathrm{RV}(\mu_2)$, i. e.,

$$p_1(t) = t^{\lambda_1} l_1(t), \quad q_2(t) = t^{\mu_2} m_2(t), \quad l_1, m_2 \in \mathrm{SV}. \tag{4.50}$$

From (4.31)-(4.33) we see that

$$\frac{p_1(t)X_i(t)^{\alpha_1}}{q_1(t)Y_i(t)^{\beta_1}} = t^{\lambda_1+\alpha_1\rho-\mu_1-\beta_1\sigma}L_i(t), \quad \frac{q_2(t)Y_i(t)^{\beta_2}}{p_2(t)X_i(t)^{\alpha_2}} = t^{\mu_2+\beta_2\sigma-\lambda_2-\alpha_2\rho}M_i(t),$$

for $i = 1, 2, 3$, and some L_i, $M_i \in$ SV. Thus, conditions (4.31)-(4.33) are satisfied (regardless of L_i and M_i), if

$$\lambda_1 + \alpha_1\rho < \mu_1 + \beta_1\sigma \quad \text{and} \quad \mu_2 + \beta_2\sigma < \lambda_2 + \alpha_2\rho. \tag{4.51}$$

Corollary 4.1. *Assume that* (4.1)-(4.3) *and* (4.50) *hold. Let* ρ *and* σ *be given by* (4.4). *If* (4.51) *holds, then system* (A) *possesses intermediate solutions* $(x(t), y(t)) \in \mathrm{RV}(\rho)\times \mathrm{RV}(\sigma)$ *such that* (4.5) *holds.*

Corollary 4.2. *Assume that* (4.1), (4.2), (4.9), (4.10) *and* (4.50) *hold. Let* σ *be given by* (4.11). *If*

$$\lambda_1 + \alpha_1 < \mu_1 + \beta_1\sigma, \quad \mu_2 + \beta_2\sigma < \lambda_2 + \alpha_2, \tag{4.52}$$

then system (A) *possesses intermediate solutions* $(x(t), y(t)) \in$ ntr-RV(1)$\times$ RV(σ) *such that* (4.12) *holds.*

Corollary 4.3. *Assume that* (4.1), (4.2), (4.20), (4.21) *and* (4.50) *hold. Let* σ *be given by* (4.22). *If*

$$\lambda_1 < \mu_1 + \beta_1\sigma, \quad \mu_2 + \beta_2\sigma < \lambda_2, \tag{4.53}$$

then system (A) *possesses intermediate solutions* $(x(t), y(t)) \in$ ntr-RV(0)$\times$ RV(σ) *such that* (4.23) *holds.*

(ADDITION)

Using similar arguments like in the necessity parts of the proofs of Lemmas 4.1-4.3 we can easily prove the following lemmas.

Lemma 4.4. *Suppose that system* (AR$_c$) *has a solution such that* $(x(t), y(t)) \in$ ntr-RV(1)$\times$ ntr-RV(1). *Then,*

$$\mu_1 = -\beta_1 - 1, \quad \lambda_2 = -\alpha_2 - 1 \tag{4.54}$$

and the slowly varying parts of $x(t)$ *and* $y(t)$ *satisfy the asymptotic relations*

$$\xi(t) \sim \int_t^\infty s^{\beta_1}q_1(s)\eta(s)^{\beta_1}ds, \quad \eta(t) \sim \int_t^\infty s^{\alpha_2}p_2(s)\xi(s)^{\alpha_2}ds, \quad t \to \infty. \tag{4.55}$$

Lemma 4.5. *Suppose that system* (AR$_c$) *has a solution such that* $(x(t), y(t)) \in$ ntr-RV(0)$\times$ ntr-RV(0). *Then*

$$\mu_1 = -2, \quad \lambda_2 = -2 \tag{4.56}$$

and the slowly varying parts of $x(t)$ and $y(t)$ satisfy the asymptotic relations

$$\xi(t) \sim \int_{t_0}^{t} s q_1(s)\eta(s)^{\beta_1} ds, \quad \eta(t) \sim \int_{t_0}^{t} s p_2(s)\xi(s)^{\alpha_2} ds, \quad t \to \infty. \tag{4.57}$$

Lemma 4.6. *Suppose that system* ($\mathrm{AR_c}$) *has a solution such that* $(x(t), y(t)) \in$ ntr-RV(1)× ntr-RV(0). *Then*

$$\mu_1 = -1, \quad \lambda_2 = -\alpha_2 - 2 \tag{4.58}$$

and the slowly varying parts of $x(t)$ and $y(t)$ satisfy the asymptotic relations

$$\xi(t) \sim \int_{t}^{\infty} q_1(s)\eta(s)^{\beta_1} ds, \quad \eta(t) \sim \int_{t_0}^{t} s^{\alpha_2+1} p_2(s)\xi(s)^{\alpha_2} ds, \quad t \to \infty. \tag{4.59}$$

Remark. Under the additional assumptions

$$t^{\beta_1} q_1(t) \sim t^{\alpha_2} p_2(t) \quad \text{as } t \to \infty \tag{4.60}$$

and

$$\int_a^{\infty} t^{\beta_1} q_1(t) dt < \infty \quad \Longleftrightarrow \quad \int_a^{\infty} t^{\alpha_2} p_2(t) dt < \infty, \tag{4.61}$$

resp.

$$q_1(t) \sim p_2(t) \quad \text{as } t \to \infty \tag{4.62}$$

and

$$\int_a^{\infty} t q_1(t) dt = \infty \quad \Longleftrightarrow \quad \int_a^{\infty} t p_2(t) dt = \infty, \tag{4.63}$$

and using the functions

$$\begin{aligned} X_4(t) &= t\left[\frac{1-\alpha_2\beta_1}{\beta_1+1}\left(\frac{\beta_1+1}{\alpha_2+1}\right)^{\frac{\beta_1}{\beta_1+1}} \int_t^{\infty} s^{\beta_1} q_1(s) ds\right]^{\frac{\beta_1+1}{1-\alpha_2\beta_1}}, \\ Y_4(t) &= t\left[\frac{1-\alpha_2\beta_1}{\alpha_2+1}\left(\frac{\alpha_2+1}{\beta_1+1}\right)^{\frac{\alpha_2}{\alpha_2+1}} \int_t^{\infty} s^{\alpha_2} p_2(s) ds\right]^{\frac{\alpha_2+1}{1-\alpha_2\beta_1}}, \end{aligned} \tag{4.64}$$

resp.

$$\begin{aligned} X_5(t) &= \left[\frac{1-\alpha_2\beta_1}{\beta_1+1}\left(\frac{\beta_1+1}{\alpha_2+1}\right)^{\frac{\beta_1}{\beta_1+1}} \int_a^{t} s q_1(s) ds\right]^{\frac{\beta_1+1}{1-\alpha_2\beta_1}}, \\ Y_5(t) &= \left[\frac{1-\alpha_2\beta_1}{\alpha_2+1}\left(\frac{\alpha_2+1}{\beta_1+1}\right)^{\frac{\alpha_2}{\alpha_2+1}} \int_a^{t} s p_2(s) ds\right]^{\frac{\alpha_2+1}{1-\alpha_2\beta_1}}, \end{aligned} \tag{4.65}$$

it can be shown easily that conditions (4.54) and (4.56) in Lemmas 4.4 and 4.5, respectively, are not only necessary, but also sufficient conditions for the existence of solutions $(x(t), y(t)) \in$ ntr-RV(1)×ntr-RV(1) (resp. $(x(t), y(t)) \in$ ntr-RV(0)×ntr-RV(0)).

Theorem 4.4. *Let* (4.1), (4.2), (4.54), (4.60) *and* (4.61) *hold. Consider functions* $X_4(t)$ *and* $Y_4(t)$ *given by* (4.64) *and suppose that*

$$\lim_{t\to\infty} \frac{p_1(t)X_4(t)^{\alpha_1}}{q_1(t)Y_4(t)^{\beta_1}} = 0, \quad \lim_{t\to\infty} \frac{q_2(t)Y_4(t)^{\beta_2}}{p_2(t)X_4(t)^{\alpha_2}} = 0. \tag{4.66}$$

Then, system (A) *possesses intermediate solutions* $(x(t), y(t)) \in$ ntr-RV(1)× ntr-RV(1), *all of which enjoy one and the same asymptotic behavior*

$$x(t) \sim X_4(t), \quad y(t) \sim Y_4(t), \quad t \to \infty. \tag{4.67}$$

Theorem 4.5. *Let* (4.1), (4.2), (4.56), (4.62) *and* (4.63) *hold. Consider functions* $X_5(t)$ *and* $Y_5(t)$ *given by* (4.65) *and suppose that*

$$\lim_{t\to\infty} \frac{p_1(t)X_5(t)^{\alpha_1}}{q_1(t)Y_5(t)^{\beta_1}} = 0, \quad \lim_{t\to\infty} \frac{q_2(t)Y_5(t)^{\beta_2}}{p_2(t)X_5(t)^{\alpha_2}} = 0. \tag{4.68}$$

Then, system (A) *possesses intermediate solutions* $(x(t), y(t)) \in$ ntr-RV(0)× ntr-RV(0), *all of which enjoy one and the same asymptotic behavior*

$$x(t) \sim X_5(t), \quad y(t) \sim Y_5(t), \quad t \to \infty. \tag{4.69}$$

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