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Positive Radial Solutions to Mean Curvature Equations with Singular Nonlinearity in Minkowski Space *

Chunmei Miao†
College of Science, Changchun University, Changchun 130022, PR China

Abstract In this paper, we consider the mean curvature equations with singular nonlinearity in Minkowski spaces. we get the existence of positive radial solutions to the mean curvature equations both in the unite ball and in the annular domain by Leray-Schauder degree arguments and truncation technique.

Keywords. Mean curvature equation, Minkowski space, radial solution, singular

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1 Introduction

The aim of this paper is to present positive radial solutions to the mean curvature equations with singular nonlinearity in Minkowski space

$$\mathbb{L}^{N+1} := \{(x,t): \ x \in \mathbb{R}^N, \ t \in \mathbb{R}\}$$

with coordinates $(x_1, x_2, \cdots, x_n, t)$ and the metric

$$\sum_{j=1}^{N}(dx_j)^2 - (dt)^2.$$

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It is well known that the study of hypersurfaces in the Minkowski space $\mathbb{L}^{N+1}$ leads to Dirichlet problems [1]:

$$\begin{cases} \mathcal{M}(u) = H(x,u), \ x \in \Omega, \\ u = 0, \ x \in \partial\Omega, \end{cases}$$

where $\mathcal{M}(u) = \operatorname{div}(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}})$ is the mean curvature operator, Ω is a bounded domain in $\mathbb{R}^n$ and $H : \Omega \times \mathbb{R} \to \mathbb{R}$ is a nonlinear term who can describe different situation on the mean curvature of the hypersurface.

The mean curvature problem has been first considered ($H = 0$) by Calabi [7]. Then it was improved by Cheng and Yau [8]. Later the case $H = c$ (c is a constant) was studied by Treibergs [9]. Recently, the radial solutions to the mean curvature problem $\mathcal{M}(u) = H$ with a general nonlinearity H have been studied by many authors (see [2], [3], [4], [5], [6]).

However, there are seldom results on the radial solutions for mean curvature equation with singular nonlinearity. In [11], Li and Yin obtained the existence and uniqueness of positive radial solutions to the mean curvature equation with singular nonlinearity in Euclidean space. Their results rely on the following conditions: there exist constants $M_1,\ M_2,\ M_3 > 0$ such that

$$M_1 g(t,y) \le f(t,y,u) \le M_2 g(t,y), \tag{1.1}$$

where $g(t,y) \ge 0$ is continuous, nonincreasing with respect to y and $\int_0^1 ds \int_0^s (\frac{\tau}{s})^{n-1} g(\tau,c) d\tau < \infty, \lim_{c\to 0^+} \int_0^1 g(s,c)ds \ge M_3,\ \lim_{c\to\infty} \int_0^t (\frac{\tau}{s})^{n-1} g(s,c)ds < \frac{1}{M_2}, t \in [0,1], \forall c > 0$.

Obviously, $f(t,y,u) = y^{-\mu} (\mu > 0)$ is a typical case satisfying conditions (1.1). But the case that $f(t,y,u) = y^{\alpha} + y^{-\beta} (\alpha,\beta > 0)$ can not be solved by previous methods. Motivated by [5], we extend the results in [11] to a move general singular situation. The aim of this paper is to present the mean curvature equation with singular nonlinearity in Minkowski spaces

$$\begin{cases} -\operatorname{div}(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}) = f(|x|,u), \ x \in \Omega, \\ u = 0, \ x \in \partial\Omega, \end{cases} \tag{1.2}$$

where Ω is a unit ball $\mathcal{B} = \{x \in \mathbb{R}^n |\ |x| < 1\}$ or an annular domain $\mathcal{A} = \{x \in \mathbb{R}^n |\ 1 < |x| < 2\}$, $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^n (n \ge 2)$. $f:\ [0,1] \times (0,+\infty) \to (0,+\infty)$ is continuous, and $f(t,y)$ may be singular at $y = 0$ (i.e. $f(t,y)$ becomes unbounded when $y \to 0+$).

This paper is organized as follows. In Section 2, we consider the existence of positive radial solutions to the mean curvature equation in a unit ball $\mathcal{B}$. In Section 3, we study on the existence of positive radial solutions to the mean curvature equation in an annular domain $\mathcal{A}$. In Section 4, some examples are presented to illustrate our results.

2 Radial solutions in the unit ball

In this section, we tackle the radial solutions to (1.2) in the unit open ball $\mathcal{B} \subset \mathbb{R}^n$. If we set $t = |x|$, $y(t) = u(|x|)$, the problem (1.2) can be transform to the following Robin boundary value problem

$$\begin{cases} -(\frac{t^{n-1}y'}{\sqrt{1-y'^2}})' = t^{n-1}f(t,y), \ t \in (0,1), \\ y'(0) = 0, \ y(1) = 0. \end{cases} \tag{2.1}$$

$X := \mathbb{C}[0,1]$ is the Banach space with maximum norm $||y|| := \max_{t\in[0,1]} |y(t)|$.

Definition 2.1. *We say y is a* solution *of* (2.1) *provided that $y \in \mathbb{C}^1[0,1]$ with $||y'|| < 1$, $t^{n-1}y'/\sqrt{1-y'^2}$ is differentiable and* (2.1) *is satisfied.*

Lemma 2.1. *Let $\Phi(x) = \frac{x}{\sqrt{1-x^2}}, x \in (-1,1)$, then Φ has an inverse function, $\phi : \mathbb{R} \to (-1,1), \phi(y) = \frac{y}{\sqrt{1+y^2}}$. Furthermore, $\phi(-y) = -\phi(y)$, and ϕ is strictly increasing on $\mathbb{R}$.*

The following hypotheses are adopted throughout this section:

(H_1) For any given constant $L > 0$, there is a continuous function $\psi_L(t) > 0$ on (0,1) such that $f(t,y) \geq \psi_L(t)$, $(t,y) \in [0,1] \times (0,L]$.

(H_2) $f(t,y) \leq q(t)[f_1(y) + f_2(y)]$, $(t,y) \in [0,1] \times (0,+\infty)$, where $q(t) > 0$ is continuous; $f_1(y) > 0$ is continuous, nonincreasing on $(0,\infty)$; $f_2(y) > 0$ is continuous and $f_2(y)/f_1(y)$ is nondecreasing on $[0,\infty)$; for any constant $K > 0$, $\int_0^1 \tau^{n-1} f_1((1-\tau)K)d\tau < \infty$.

(H_3) There exists a positive constant $R > 0$ such that

$$\frac{R}{\phi[||q||(1 + (\frac{f_2}{f_1})(R)) \int_0^1 \tau^{n-1} f_1((1-\tau)R)d\tau]} > 1.$$

Lemma 2.2. *Suppose that $e \in X$, $e(t) > 0$, $t \in (0,1)$, $a \geq 0$ is a constant. Then the BVP*

$$\begin{cases} -(\dfrac{t^{n-1}y'}{\sqrt{1-y'^2}})' = t^{n-1}e(t), \ t \in (0,1), \\ y'(0) = 0, \ y(1) = a \end{cases} \tag{2.2}$$

has a unique positive solution. Moreover this solution can be represented by

$$y(t) = a + \int_t^1 \phi[\int_0^s (\frac{\tau}{s})^{n-1} e(\tau)d\tau]ds. \tag{2.3}$$

Proof. It is easy to verify that (2.3) is a solution of (2.2). On the other hand, if y is a solution of (2.2), then

$$-(\frac{t^{n-1}y'}{\sqrt{1-y'^2}})' = t^{n-1}e(t), \ t \in (0,1).$$

For any $t \in (0,1)$, integrating on the both sides of the above equation from 0 to t, and using the boundary condition $y'(0) = 0$, we get

$$-\frac{y'}{\sqrt{1-y'^2}} = \int_0^t (\frac{s}{t})^{n-1} e(s)ds.$$

By Lemma 2.1, we have

$$-y'(t) = \phi[\int_0^t (\frac{s}{t})^{n-1} e(s)ds]. \tag{2.4}$$

Integrating on the both sides of (2.4) from t to 1, and one obtains

$$y(t) = y(1) + \int_t^1 \phi[\int_0^s (\frac{\tau}{s})^{n-1} e(\tau)d\tau]ds.$$

Using the boundary condition $y(1) = a$, we obtain

$$y(t) = a + \int_t^1 \phi[\int_0^s (\frac{\tau}{s})^{n-1} e(\tau)d\tau]ds.$$

The proof is complete. □

In order to solve (2.1), we consider the following BVP

$$\begin{cases} -(\dfrac{t^{n-1}y'}{\sqrt{1-y'^2}})' = t^{n-1}F(t,y), \ t \in (0,1), \\ y'(0) = 0, \ y(1) = a, \end{cases} \tag{2.5}$$

where $F : [0,1] \times \mathbb{R} \to (0,\infty)$ is continuous, $a \geq 0$ is a constant.

Let $y \in X$. We define an operator $T : X \to X$ by

$$(Ty)(t) = a + \int_t^1 \phi[\int_0^s (\frac{\tau}{s})^{n-1} F(\tau, y(\tau))d\tau]ds. \tag{2.6}$$

Let $X_1 = \{y \in X | \ y'(0) = 0, \ y(1) = a\}$ is a subspace of X.

Lemma 2.3. *$T : X_1 \to X_1$ is well defined, completely continuous, and $y \in X_1$ is a solution of* (2.5) *if and only if $T(y) = y$.*

Proof. It is easy to prove that $T : X_1 \to X_1$ is well defined, and $y \in X_1$ is a solution of (2.5) if and only if $T(y) = y$.

By the continuity of F, we have T is continuous.

Next we shall show that T is compact. Suppose $D = \{y \in X_1 | \ ||y|| \le r\} \subset X_1$ is a bounded set. For any $y \in D$, which implies $||y|| \le r$, we have

$$\begin{aligned} |(Ty)(t)| &= |a + \int_t^1 \phi[\int_0^s (\frac{\tau}{s})^{n-1} F(\tau, y(\tau))d\tau]ds| \\ &\le a + |\int_t^1 \phi[\int_0^s (\frac{\tau}{s})^{n-1} F(\tau, y(\tau))d\tau]ds| \\ &< a + (1-t) \\ &\le a + 1. \end{aligned} \tag{2.7}$$

This implies that $T(D)$ is uniformly bounded.

In addition,

$$|(Ty)'(t)| = | - \phi[\int_0^t (\frac{\tau}{t})^{n-1} F(\tau, y(\tau))d\tau]| < 1.$$

For any given $t_1, t_2 \in [0,1]$, we obtain

$$\begin{aligned} |(Ty)(t_1) - (Ty)(t_2)| &= | \int_{t_1}^{t_2} (Ty)'(s)ds| \\ &< |t_1 - t_2| \to 0 \quad \text{as} \quad t_1 \to t_2. \end{aligned}$$

This implies that $T(D)$ is equi-continuous.

By the Arzelà-Ascoli theorem, $T(D)$ is relatively compact. Therefore, $T : X_1 \to X_1$ is completely continuous. □

Now we state an existence principle which plays an important role in our proof of main results.

Lemma 2.4. (Existence principle) *Assume that there exists a constant $l > a \geq 0$ independent of λ, such that for $\lambda \in (0,1)$, $\|y\| \neq l$, where $y(t)$ satisfies*

$$\begin{cases} -(\dfrac{t^{n-1}y'}{\sqrt{1-y'^2}})' = \lambda t^{n-1}F(t,y), \ t \in (0,1), \\ y'(0) = 0, \ y(1) = a. \end{cases} \tag{$2.8)_\lambda$}$$

Then $(2.8)_1$ has at least one solution $y(t)$ such that $\|y\| \leq l$.

Proof. For any $\lambda \in [0,1]$, $y \in X_1$. We define one operator

$$(T_\lambda y)(t) = a + \int_t^1 \phi[\int_0^s \lambda(\frac{\tau}{s})^{n-1}F(\tau, y(\tau))d\tau]ds.$$

By Lemma 2.3, $T_\lambda : X_1 \to X_1$ is completely continuous. It can be verified that a solution of BVP $(2.8)_\lambda$ is equivalent to a fixed point of T_λ in X_1. Let $\Omega = \{y \in X_1 | \ \|y\| < l\}$, then Ω is an open set in X_1.

If there exists $y \in \partial\Omega$ such that $T_1 y = y$, then $y(t)$ is a solution of $(2.8)_1$ with $\|y\| \leq l$. Thus the conclusion is true. Otherwise, for any $y \in \partial\Omega$, $T_1 y \neq y$. If $\lambda = 0$, for $y \in \partial\Omega$, $(I - T_0)y(t) = y(t) - (T_0 y)(t) = y(t) - a \not\equiv 0$ since $\|y\| = l > a$, so $T_0 y \neq y$ for any $y \in \partial\Omega$. For $\lambda \in (0,1)$, if there is a solution $y(t)$ to BVP $(2.8)_\lambda$, by the assumption, one gets $\|y\| \neq l$, which is a contradiction to $y \in \partial\Omega$.

In a word, for any $y \in \partial\Omega$ and $\lambda \in [0,1]$, $T_\lambda y \neq y$. Homotopy invariance of Leray-Schauder degree deduce that

$$\text{Deg}\{I - T_1, \Omega, 0\} = \text{Deg}\{I - T_0, \Omega, 0\} = 1.$$

Hence, T_1 has a fixed point y in Ω. That is, BVP $(2.8)_1$ has at least one solution $y(t)$ with $\|y\| \leq l$. The proof is completed. □

Lemma 2.5. *If y is a solution to BVP (2.5), then*

(i) *$y(t)$ is concave on $[0,1]$;*

(ii) *$-1 < y'(t) \leq 0, \ t \in [0,1]$;*

(iii) *$y(t) \geq a$ and $y(t) \geq (1-t)\|y\|, \ t \in [0,1]$.*

Proof. Suppose y is a solution to BVP (2.5), then

$$(\frac{t^{n-1}y'}{\sqrt{1-y'^2}})' = -\lambda t^{n-1}F(t,y) \leq 0, \ t \in (0,1).$$

In addition,

$$y(t) = a + \int_t^1 \phi[\int_0^s (\frac{\tau}{s})^{n-1} F(\tau, y(\tau)) d\tau] ds \ge a,$$

we have $y''(t) \le 0$, $y(t) \ge a$ and $-1 < y'(t) \le 0$ on $[0,1]$.

By $y(t)$ is concave on $[0,1]$, we have

$$\frac{y(t) - 0}{t - 1} \le \frac{y(0) - 0}{0 - 1} \Rightarrow y(t) \ge (1-t)y(0) = (1-t)||y||,\ t \in [0,1].$$

The proof is completed. □

Theorem 2.6. *Assume* (H_1)-(H_3) *hold, then BVP* (2.1) *has at least one positive solution* $y \in X_1$ *such that* $||y|| \le R$ *and* $||y'|| < 1$.

Proof. Step 1. From (H_3), we choose $\varepsilon > 0$ such that

$$\frac{R}{\varepsilon + \phi[||q||(1 + (\frac{f_2}{f_1})(R)) \int_0^1 \tau^{n-1} f_1((1-\tau)R) d\tau]} \ge 1. \tag{2.8}$$

Let $n_0 \in \mathbb{N}_+ := \{1, 2, \cdots\}$ satisfying that $\frac{1}{n_0} < \varepsilon$, and set $\mathbb{N}_0 := \{n_0, n_0 + 1, n_0 + 2, \cdots\}$.

In what follows, we show that the following BVP

$$\begin{cases} -(\frac{t^{n-1} y'}{\sqrt{1 - y'^2}})' = t^{n-1} f(t, y),\ t \in (0,1), \\ y'(0) = 0,\ y(1) = \frac{1}{m} \end{cases} \tag{2.9}$$

has a positive solution for each $m \in \mathbb{N}_0$.

To this end, we consider the following BVP

$$\begin{cases} -(\frac{t^{n-1} y'}{\sqrt{1 - y'^2}})' = t^{n-1} f^*(t, y),\ t \in (0,1), \\ y'(0) = 0,\ y(1) = \frac{1}{m}, \end{cases} \tag{2.10}$$

where

$$f^*(t, y) = \begin{cases} f(t, y), & y \ge \frac{1}{m}, \\ f(t, \frac{1}{m}), & y < \frac{1}{m}, \end{cases}$$

then $f^* : [0,1] \times \mathbb{R} \to (0, \infty)$ is continuous.

To obtain a solution of BVP (2.10) for each $m \in \mathbb{N}_0$, by applying Lemma 2.4, we consider the family of BVPs

$$\begin{cases} -\left(\dfrac{t^{n-1}y'}{\sqrt{1-y'^2}}\right)' = \lambda t^{n-1} f^*(t,y), \ t \in (0,1), \\ y'(0) = 0, \ y(1) = \dfrac{1}{m}, \end{cases} \tag{2.11}$$

where $\lambda \in [0,1]$.

For any $\lambda \in [0,1]$ and $m \in \mathbb{N}_0$, by (H_2), one has

$$\begin{aligned} y_{\lambda m}(0) &= \frac{1}{m} + \int_0^1 \phi[\lambda \int_0^s (\frac{\tau}{s})^{n-1} f^*(\tau, y_{\lambda m}(\tau))d\tau]ds \\ &= \frac{1}{m} + \int_0^1 \phi[\lambda \int_0^s (\frac{\tau}{s})^{n-1} f(\tau, y_{\lambda m}(\tau))d\tau]ds \\ &< \varepsilon + \phi[\int_0^1 \tau^{n-1} f(\tau, y_{\lambda m}(\tau))d\tau]ds \\ &\le \varepsilon + \phi[\|q\|(1 + (\frac{f_2}{f_1})(y_{\lambda m}(0))) \int_0^1 \tau^{n-1} f_1((1-\tau)y_{\lambda m}(0))d\tau]. \end{aligned}$$

Furthermore, we have

$$\frac{y_{\lambda m}(0)}{\varepsilon + \phi[\|q\|(1 + (\frac{f_2}{f_1})(y_{\lambda m}(0))) \int_0^1 \tau^{n-1} f_1((1-\tau)y_{\lambda m}(0))d\tau]} < 1,$$

which together with (2.8) implies

$$\|y_{\lambda m}\| = y_{\lambda m}(0) \ne R.$$

Lemma 2.4 implies that (2.10) has at least one positive solution $y_m(t)$ such that $\|y_m\| \le R$ (independent of m) for any fixed $m \in \mathbb{N}_0$. From Lemma 2.5, we note that $y_m(t) \ge \frac{1}{m}$, $t \in [0,1]$, which implies that

$$f^*(t, y_m(t)) = f(t, y_m(t)).$$

Therefore, $y_m(t)$ is the solution of BVP (2.9).

Step 2. Note that (H_1) guarantees the existence of a function $\psi_R(t)$ which is continuous on [0,1] and positive on (0,1) with

$$f(t,y) \ge \psi_R(t), \quad t \in [0,1] \times (0,R]. \tag{2.12}$$

Let

$$\omega(t) := \int_0^t (\frac{\tau}{t})^{n-1} f(\tau, y_m(\tau))d\tau, \ t \in [0,1].$$

We conclude $\omega(t)$ is increasing on $[0,1]$. In fact, by $y_m(t)$ is the solution of (2.9), we have

$$\begin{aligned} y_m(t) &= \frac{1}{m} + \int_t^1 \phi[\int_0^s (\frac{\tau}{s})^{n-1} f(\tau, y_m(\tau))d\tau]ds \\ &= \frac{1}{m} + \int_t^1 \phi[\omega(s)]ds. \end{aligned}$$

In addition,

$$y_m'(t) = -\phi[\omega(t)] \le 0, \ t \in [0,1],$$

which together with the convexity of $y_m(t)$ and monotonicity of ϕ, we have $\omega(t)$ is increasing on [0,1].

Let $y_m(t)$ is the solution of (2.9). From the monotonicity of ϕ and ω, we have

$$\begin{aligned} y_m(t) &= \frac{1}{m} + \int_t^1 \phi[\int_0^s (\frac{\tau}{s})^{n-1} f(\tau, y_m(\tau))d\tau]ds \\ &\ge \frac{1}{m} + \phi[\int_0^t (\frac{\tau}{t})^{n-1} f(\tau, y_m(\tau))d\tau](1-t) \\ &\ge \phi[\int_0^t (\frac{\tau}{t})^{n-1} \psi_R(\tau)d\tau](1-t) \\ &:= \bar{\omega}(t)(1-t), \quad t \in [0,1]. \end{aligned} \tag{2.13}$$

Step 3. It remains to show that $\{y_m(t)\}_{m\in\mathbb{N}_0}$ is uniformly bounded and equi-continuous on [0,1]. By $y_m(t)$ is the solution of (2.9) one has $\|y_m\| \le R$, which implies that $\{y_m(t)\}_{m\in\mathbb{N}_0}$ is uniformly bounded on [0,1].

Next it suffices to show that $\{y_m(t)\}_{m\in\mathbb{N}_0}$ is equi-continuous on [0,1]. Since $y_m(t)$ is a solution of (2.9), we have

$$|y_m'(1)| = |\phi[\int_0^t (\frac{\tau}{t})^{n-1} f(\tau, y_m(\tau))d\tau]| < 1.$$

For any $t, s \in [0,1]$,

$$|y_m(t) - y_m(s)| = |\int_s^t y_m'(\tau)d\tau| < |t-s| \to 0 \text{ as } t \to s.$$

Therefore, $\{y_m(t)\}_{m\in\mathbb{N}_0}$ is equi-continuous on [0,1].

The Arzelà-Ascoli theorem guarantees that there is a subsequence $\mathbb{N}^*$ of $\mathbb{N}_0$ (without loss of generality, we assume $\mathbb{N}^* = \mathbb{N}_0$) and function $y(t)$ with $y_m(t) \to y(t)$ uniformly on [0,1] as $m \to +\infty$ through $\mathbb{N}^*$.

By $y_m(t) (m \in \mathbb{N}^*)$ is the solution of (2.9), we have

$$y_m(t) = \frac{1}{m} + \int_t^1 \phi[\int_0^s (\frac{\tau}{s})^{n-1} f(\tau, y_m(\tau))d\tau]ds. \tag{2.14}$$

Let $m \to +\infty$ through $\mathbb{N}^*$ in (2.14), by the Lebesgue dominated convergence theorem, one has

$$y(t) = \int_t^1 \phi[\int_0^s (\frac{\tau}{s})^{n-1} f(\tau, y(\tau))d\tau]ds, \ t \in [0,1],$$

and furthermore, we have $-(\frac{t^{n-1}y'}{\sqrt{1-y'^2}})' = t^{n-1}f(t,y), \ t \in (0,1)$ and $y'(0) = 0, \ y(1) = 0$, i.e. $y(t)$ is positive solution of BVP (2.1), and $\|y\| \le R, \ \|y'\| < 1, \ y(t) \ge \bar{\omega}(t)(1-t), \ t \in [0,1]$. The proof of Theorem 2.6 is complete. □

Note that BVP (2.1) has the positive solution $y(t)$, then $u(|x|) = y(t)$ is a positive radial solution of mean curvature problem (1.2).

3 Radial solutions in an annular domain

In this section, $\mathcal{A}$ denotes the annular domain $\mathcal{A} = \{x \in \mathbb{R}^n | \ 1 < |x| < 2\}$. When dealing with the radial solutions for (1.2), we are led to consider the Dirichlet BVP

$$\begin{cases} -(\frac{t^{n-1}y'}{\sqrt{1-y'^2}})' = t^{n-1}f(t,y), \ t \in (1,2), \\ y(1) = y(2) = 0. \end{cases} \tag{3.1}$$

The following hypotheses are adopted throughout this section:

(H_1^*) For each given constant $L > 0$, there is a continuous function $\psi_L(t) > 0$ on (1,2) such that $f(t,y) \ge \psi_L(t), \ (t,y) \in [1,2] \times (0,L]$.

(H_2^*) $0 < f(t,y) \le q(t)[f_1(y) + f_2(y)]$ for all $(t,y) \in [1,2] \times (0,+\infty)$, where $q(t) > 0$ is continuous; $f_1(y) > 0$ is continuous, nonincreasing on $(0,\infty)$; $f_1(y) > 0$ is continuous and $f_2(y)/f_1(y)$ is nondecreasing on $[0,\infty)$; for any constant $K > 0$, $\int_0^1 \tau^{n-1} f_1((2-\tau)(1-\tau)K)d\tau < \infty$.

(H_3^*) There exists a positive constant $R > 0$ such that

$$\frac{R}{\phi[\|q\|(1 + (\frac{f_2}{f_1})(R)) \int_1^2 \tau^{n-1} f_1(R(\tau-1)(2-\tau))d\tau]} > 1.$$

Lemma 3.1. *Suppose that* $e \in X, \ e(t) > 0, \ t \in (1,2), \ a \ge 0$ *is a constant. Then the BVP*

$$\begin{cases} -(\frac{t^{n-1}y'}{\sqrt{1-y'^2}})' = t^{n-1}e(t), \ t \in (1,2), \\ y(1) = y(2) = a \end{cases} \tag{3.2}$$

has a unique positive solution. Moreover this solution is given by

$$y(t)=\begin{cases}\int_1^t \phi[\int_s^\sigma (\frac{\tau}{s})^{n-1}e(\tau)d\tau]ds+a, & 1\le t\le\sigma,\\ \int_t^2 \phi[\int_\sigma^s (\frac{\tau}{s})^{n-1}e(\tau)d\tau]ds+a, & \sigma\le t\le 2.\end{cases} \tag{3.3}$$

where σ satisfies

$$\int_1^\sigma \phi[\int_s^\sigma (\frac{\tau}{s})^{n-1}e(\tau)d\tau]ds=\int_\sigma^2 \phi[\int_\sigma^s (\frac{\tau}{s})^{n-1}e(\tau)d\tau]ds.$$

Proof. First, we show the equation

$$\int_1^t \phi[\int_s^t (\frac{\tau}{s})^{n-1}e(\tau)d\tau]ds=\int_t^2 \phi[\int_t^s (\frac{\tau}{s})^{n-1}e(\tau)d\tau]ds \tag{3.4}$$

has a unique solution. Set

$$v_1(t)=\int_1^t \phi[\int_s^t (\frac{\tau}{s})^{n-1}e(\tau)d\tau]ds,$$
$$v_2(t)=\int_t^2 \phi[\int_t^s (\frac{\tau}{s})^{n-1}e(\tau)d\tau]ds,$$
$$V(t)=v_1(t)-v_2(t).$$

Clearly, $V(t)$ is continuous and strictly increasing on $[1,2]$, and

$$V(1)V(2)=-[\int_1^2 \phi[\int_1^s (\frac{\tau}{s})^{n-1}e(\tau)d\tau]ds][\int_1^2 \phi[\int_s^2 (\frac{\tau}{s})^{n-1}e(\tau)d\tau]ds]<0,$$

hence there exists a unique $\sigma\in(1,2)$ such that $V(\sigma)=0$, i.e., the equation (3.4) has a unique solution.

It is easy to verify that (3.3) is a solution of (3.2). On the other hand, if $y(t)$ is a solution of (3.2), then

$$-(\frac{t^{n-1}y'(t)}{\sqrt{1-y'^2(t)}})'=t^{n-1}e(t),\ t\in(0,1), \tag{3.5}$$

thus $y(t)$ is concave on [1,2], which together with the boundary condition $y(1)=y(2)=a$, show that there exists a unique $\hat\sigma\in(1,2)$ such that $y(\hat\sigma)=\|y\|$ and $y'(\hat\sigma)=0$.

For any $t\in(1,\hat\sigma)$, integrate on the both sides of (3.5) from t to $\hat\sigma$, we arrive at

$$\frac{y'(t)}{\sqrt{1-y'^2(t)}}=\int_t^{\hat\sigma}(\frac{s}{t})^{n-1}e(s)ds,$$

and from Lemma 2.1, we have

$$y'(t)=\phi[\int_t^{\hat\sigma}(\frac{s}{t})^{n-1}e(s)ds]. \tag{3.6}$$

Integrating on the both sides of (3.6) from 1 to t one obtains

$$y(t) = y(1) + \int_1^t \phi[\int_s^{\hat{\sigma}} (\frac{\tau}{s})^{n-1} e(\tau)d\tau]ds,$$

which together with the boundary condition $y(1) = a$, we obtain

$$y(t) = \int_1^t \phi[\int_s^{\hat{\sigma}} (\frac{\tau}{s})^{n-1} e(\tau)d\tau]ds + a.$$

For any $t \in (\hat{\sigma}, 2)$, integrating on the both sides of (3.5) from $\hat{\sigma}$ to t, one gets

$$-\frac{y'(t)}{\sqrt{1-y'^2(t)}} = \int_{\hat{\sigma}}^t (\frac{s}{t})^{n-1} e(s)ds,$$

furthermore, we have

$$-y'(t) = \phi[\int_{\hat{\sigma}}^t (\frac{s}{t})^{n-1} e(s)ds]. \tag{3.7}$$

Integrating on the both sides of (3.7) from t to 2 one obtains

$$y(t) = y(2) + \int_t^2 \phi[\int_{\hat{\sigma}}^s (\frac{\tau}{s})^{n-1} e(\tau)d\tau],$$

which together with the boundary condition $y(2) = a$, we obtain

$$y(t) = \int_t^2 \phi[\int_{\hat{\sigma}}^s (\frac{\tau}{s})^{n-1} e(\tau)d\tau] + a.$$

Having in mind the definition of σ we can see that $\hat{\sigma} = \sigma$. Therefore the unique solution to (3.2) can be expressed by (3.3). The proof is complete. □

In order to solve (3.1), we shall consider the following BVP

$$\begin{cases} -(\frac{t^{n-1}y'}{\sqrt{1-y'^2}})' = t^{n-1}F(t,y), \ t \in (0,1), \\ y(1) = y(2) = a, \end{cases} \tag{3.8}$$

where $F : [0,1] \times \mathbb{R} \to (0,\infty)$ is continuous, $a \geq 0$ is a constant.

Let $X_2 = \{y \in X : \ y(1) = y(2) = a\}$ is a subspace of X .

Let $y \in X_2$. We define an operator $T : X_2 \to X_2$ by

$$(Ty)(t) = \begin{cases} \int_1^t \phi[\int_s^{\sigma} (\frac{\tau}{s})^{n-1} F(\tau, y(\tau))d\tau]ds + a, \ 1 \leq t \leq \sigma, \\ \int_t^2 \phi[\int_{\sigma}^s (\frac{\tau}{s})^{n-1} F(\tau, y(\tau))d\tau]ds + a, \ \sigma \leq t \leq 2. \end{cases} \tag{3.9}$$

Lemma 3.2. *$T : X_2 \to X_2$ is well defined, completely continuous, and $y \in X_2$ is a solution of* (3.8) *if and only if $T(y) = y$.*

Proof. It is easy to prove that $T : X_2 \to X_2$ is well defined, and $y \in X_2$ is a solution of (3.8) if and only if $T(y) = y$.

First, we show that T is continuous. Let $y_m \to y_0 (m \to \infty)$ in X_2. Similarly to Lemma 3.1, for any y_m, there exists a unique $\sigma_m \in (1,2)$ such that $v_{1m}(\sigma_m) = v_{2m}(\sigma_m)$, where

$$v_{1m}(t) = \int_1^t \phi[\int_s^{\sigma_m} (\frac{\tau}{s})^{n-1} e(\tau) d\tau] ds,$$
$$v_{2m}(t) = \int_t^2 \phi[\int_{\sigma_m}^s (\frac{\tau}{s})^{n-1} e(\tau) d\tau] ds.$$

Meanwhile, we can obtain $\sigma_m \to \sigma_0 (m \to \infty)$, $v_{in} \to v_{i0} (m \to \infty)$, $i = 1,2$. Let $\underline{\sigma}_m = \min\{\sigma_m, \sigma_0\}$, $\bar{\sigma}_m = \max\{\sigma_m, \sigma_0\}$, $m = 1,2,\cdots$. Obviously, when $t \in \Delta_m = [\underline{\sigma}_m, \bar{\sigma}_m]$, $t - \sigma_0 \to 0$ as $m \to \infty$. Noticing that

$$\begin{aligned}\max_{t \in \Delta_m} |v_{in}(t) - v_{j0}(t)| &\le \max_{t \in \Delta_m} |v_{in}(t) - v_{in}(\sigma_m)| + |v_{jn}(\sigma_m) - v_{j0}(\sigma_0)| + \max_{t \in \Delta_m} |v_{j0}(t) - v_{j0}(\sigma_0)| \\ &\to 0 \quad \text{as } m \to \infty, \ i,j = 1,2, \ i \neq j,\end{aligned}$$

we have

$$\begin{aligned}\|Ty_m - Ty_0\| &= \max\{\|v_{1,m} - v_{1,0}\|_{[1,\underline{\sigma}_m]}, \|v_{1,m} - v_{2,0}\|_{\Delta_m}, \|v_{2m} - v_{10}\|_{\Delta_m}, \|v_{2m} - v_{20}\|_{[\bar{\sigma}_m, 2]}\} \\ &\to 0 \quad \text{as } m \to \infty.\end{aligned}$$

Therefore, T is continuous.

It is easy to prove that $T(D)$ is bounded and equi-continuous, where $D \subset X_2$ is a bounded set. By the Arzelà-Ascoli theorem, $T(D)$ is relatively compact. Thus $T : X_2 \to X_2$ is completely continuous. □

Now we state a existence principle which plays an important role in our proof of main results.

Lemma 3.3. (Existence principle) *Assume that there exists a constant $l > a \ge 0$ independent of λ, such that for $\lambda \in (0,1)$, $\|y\| \neq l$, where $y(t)$ satisfies*

$$\begin{cases} -(\dfrac{t^{n-1} y'}{\sqrt{1 - y'^2}})' = \lambda t^{n-1} F(t,y), \ t \in (1,2), \\ y(1) = y(2) = a. \end{cases} \tag{3.8$_\lambda$}$$

Then $(3.8)_1$ *has at least one solution $y(t)$ such that $\|y\| \le l$.*

Proof. For any $\lambda \in [0,1]$, $y \in X_2$, define one operator

$$(T_\lambda y)(t) = \begin{cases} \int_1^t \phi[\lambda \int_s^\sigma (\frac{\tau}{s})^{n-1} F(\tau, y(\tau)) d\tau] ds + a, & 1 \le t \le \sigma, \\ \int_t^2 \phi[\lambda \int_\sigma^s (\frac{\tau}{s})^{n-1} F(\tau, y(\tau)) d\tau] ds + a, & \sigma \le t \le 2. \end{cases}$$

By Lemma 3.2, $T_\lambda : X_2 \to X_2$ is completely continuous. It can be verified that a solution of BVP $(3.8)_\lambda$ equivalent to a fixed point of T_λ in X_2. Let $\Omega = \{y \in X_2 : \|y\| < l\}$, then Ω is an open set in X_2.

If there exists $y \in \partial\Omega$ such that $T_1 y = y$, then $y(t)$ is a solution of $(3.8)_1$ with $\|y\| \le l$. Thus the conclusion is true. Otherwise, for any $y \in \partial\Omega$, $T_1 y \neq y$. If $\lambda = 0$, for $y \in \partial\Omega$, $(I - T_0)y(t) = y(t) - (T_0 y)(t) = y(t) - a \not\equiv 0$ since $\|y\| = l > a$, so $T_0 y \neq y$ for any $y \in \partial\Omega$. For $\lambda \in (0,1)$, if there is a solution $y(t)$ to BVP $(3.8)_\lambda$, by the assumption, one gets $\|y\| \neq l$, which is a contradiction to $y \in \partial\Omega$.

In a word, for any $y \in \partial\Omega$ and $\lambda \in [0,1]$, $T_\lambda y \neq y$. Homotopy invariance of Leray-Schauder degree deduce that

$$\mathrm{Deg}\{I - T_1, \Omega, 0\} = \mathrm{Deg}\{I - T_0, \Omega, 0\} = 1.$$

Hence, T_1 has a fixed point y in Ω. That is, BVP $(3.8)_1$ has at least one solution $y(t)$ with $\|y\| \le l$. The proof is completed. □

Lemma 3.4. *If y is a solution to BVP (3.8), then*

(i) *$y(t)$ is strictly concave on $[1,2]$;*

(ii) *there exists a unique $\sigma \in (1,2)$ such that $y'(\sigma) = 0$, and $y'(t) > 0, t \in [1,\sigma)$, $y'(t) < 0, t \in (\sigma, 2]$.*

(iii) *$y(t) \ge a$ and $y(t) \ge \|y\|(t-1)(2-t)$, $t \in [1,2]$.*

Theorem 3.5. *Assume (H_1^*)-(H_3^*) hold, then BVP (3.1) has at least one positive solution $y \in X_2$, $\|y\| \le R$ and $\|y'\| < 1$.*

Proof. Step 1. From (H_3^*), we choose $\varepsilon > 0$ such that

$$\frac{R}{\varepsilon + \phi[\|q\|(1 + (\frac{f_2}{f_1})(R)) \int_1^2 \tau^{n-1} f_1(R(\tau-1)(2-\tau)) d\tau]} \ge 1. \tag{3.10}$$

Let $n_0 \in \mathbb{N}_+ := \{1, 2, \cdots\}$ satisfying that $\dfrac{1}{n_0} < \varepsilon$, and set $\mathbb{N}_0 := \{n_0, n_0+1, n_0+2, \cdots\}$.

In what follows, we show that the following BVP

$$\begin{cases} -\left(\dfrac{t^{n-1}y'}{\sqrt{1-y'^2}}\right)' = t^{n-1}f(t,y), \ t\in(1,2), \\ y(1) = y(2) = \dfrac{1}{m} \end{cases} \tag{3.11}$$

has a positive solution for each $m \in \mathbb{N}_0$.

To this end, we consider the following BVP

$$\begin{cases} -\left(\dfrac{t^{n-1}y'}{\sqrt{1-y'^2}}\right)' = t^{n-1}f^*(t,y), \ t\in(1,2), \\ y(1) = y(2) = \dfrac{1}{m}, \end{cases} \tag{3.12}$$

where

$$f^*(t,y) = \begin{cases} f(t,y), & y \ge \dfrac{1}{m}, \\ f(t,\dfrac{1}{m}), & y < \dfrac{1}{m}, \end{cases}$$

then $f^* : [1,2]\times\mathbb{R} \to (0,\infty)$ is continuous.

To obtain a solution of BVP (3.12) for each $m \in \mathbb{N}_0$, by applying Lemma 3.3, we consider the family of BVPs

$$\begin{cases} -\left(\dfrac{t^{n-1}y'}{\sqrt{1-y'^2}}\right)' = \lambda t^{n-1}f^*(t,y), \ t\in(1,2), \\ y(1) = y(2) = \dfrac{1}{m}, \end{cases} \tag{3.13}$$

where $\lambda \in [0,1]$.

For any $m \in \mathbb{N}_0$ and $\lambda \in [0,1]$, by (H_2^*), we get

$$\begin{aligned} y_{\lambda m}(\sigma_m) &= \int_1^{\sigma_m} \phi[\lambda \int_s^{\sigma_m} (\frac{\tau}{s})^{n-1} f^*(\tau, y_{\lambda m}(\tau))d\tau]ds + \frac{1}{m} \\ &= \int_1^{\sigma_m} \phi[\lambda \int_s^{\sigma_m} (\frac{\tau}{s})^{n-1} f(\tau, y_{\lambda m}(\tau))d\tau]ds + \frac{1}{m} \\ &< \varepsilon + \phi[\int_1^2 \tau^{n-1} f(\tau, y_{\lambda m}(\tau))d\tau] \\ &\le \varepsilon + \phi[\|q\|(1 + (\frac{f_2}{f_1})(y_{\lambda m}(\sigma_m))) \int_1^2 \tau^{n-1} f_1(y_{\lambda m}(\sigma_m)(\tau-1)(2-\tau))d\tau]. \end{aligned} \tag{3.14}$$

On the other hand,

$$\begin{aligned} y_{\lambda m}(\sigma_m) &= \int_{\sigma_m}^{2} \phi[\lambda \int_{\sigma_m}^{s} (\frac{\tau}{s})^{n-1} f^*(\tau, y_{\lambda m}(\tau)) d\tau] ds + \frac{1}{m} \\ &= \int_{\sigma_m}^{2} \phi[\lambda \int_{\sigma_m}^{s} (\frac{\tau}{s})^{n-1} f(\tau, y_{\lambda m}(\tau)) d\tau] ds + \frac{1}{m} \\ &< \varepsilon + \phi[\frac{1}{2^{n-1}} \int_{1}^{2} \tau^{n-1} f(\tau, y_{\lambda m}(\tau)) d\tau] \\ &\leq \varepsilon + \phi[\frac{1}{2^{n-1}} \|q\| (1 + (\frac{f_2}{f_1})(y_{\lambda m}(\sigma_m))) \int_{1}^{2} \tau^{n-1} f_1(y_{\lambda m}(\sigma_m)(\tau - 1)(2 - \tau)) d\tau]. \end{aligned} \tag{3.15}$$

By the inequality (3.14), (3.15) and (3.10), we have

$$\|y_{\lambda m}\| = y_{\lambda m}(\sigma_m) \neq R.$$

Lemma 3.3 implies that (3.12) has at least one positive solution $y_m(t)$ with $\|y_m\| \leq R$ (independent of m) for any fixed $m \in \mathbb{N}_0$. From Lemma 3.4, we note that $y_m(t) \geq \frac{1}{m}$, $t \in [0,1]$, which implies that

$$f^*(t, y_m(t)) = f(t, y_m(t)).$$

Consequently, $y_m(t)$ is the solution of BVP (3.11).

Step 2. Note that (H_1^*) guarantees the existence of a function $\psi_R(t)$ which is continuous on [1,2] and positive on (1,2) with

$$f(t,y) \geq \psi_R(t), \quad t \in [1,2] \times (0,R]. \tag{3.16}$$

Let $y_m(t)$ be a solution of (3.11), because for each $m \in \mathbb{N}_0$, $y_m(t)$ is strictly concave, then for any $t_1, t_2 \in (1,2)$, $\theta \in (0,1)$, we have

$$y_m(\theta t_1 + (1-\theta) t_2) > \theta y_m(t_1) + (1-\theta) y_m(t_2). \tag{3.17}$$

We conclude that there exist a_0 and a_1 with $a_0 > 1$, $a_1 < 2$, $a_0 \leq a_1$ such that

$$a_0 = \inf\{\sigma_m : m \in \mathbb{N}_0\} \leq \sup\{\sigma_m : m \in \mathbb{N}_0\} = a_1. \tag{3.18}$$

Where σ_m (as before) is the unique point in $(1,2)$ with $y_m'(\sigma_m) = 0$. We now show $\inf\{\sigma_m : m \in \mathbb{N}_0\} > 1$. If this is not true then there is a subsequence S of $\mathbb{N}_0$ with $\sigma_m \to 1$ as $m \to \infty$ in S. By Lemma 3.2, we have

$$y_m(\sigma_m) = \int_{1}^{\sigma_m} \phi[\int_{s}^{\sigma_m} (\frac{\tau}{s})^{n-1} f(\tau, y_m(\tau)) d\tau] ds + \frac{1}{m},$$

which together with (3.14) and Lebesgue's dominated convergence theorem, it follows that

$$y_m(\sigma_m) \to 0 \ \text{ as } \ m \to \infty \text{ in } S.$$

However since the maximum of y_m on [1,2] occurs at σ_m we have $y_m \to 0$ in X_2 as $m \to \infty$ in S. This is contradiction with (3.17). Therefore, $\inf\{\sigma_m : m \in \mathbb{N}_0\} > 1$. A similar argument shows $\sup\{\sigma_m : m \in \mathbb{N}_0\} < 2$.

By Lemma 3.1, (3.16), (3.18) and the monotonicity of ϕ, we have

$$\begin{aligned} y_m'(1) &= \phi[\int_1^{\sigma_m} \tau^{n-1} f(\tau, y_m(\tau) d\tau] ds \\ &\geq \phi[\int_1^{a_0} \tau^{n-1} f(\tau, y_m(\tau) d\tau] \\ &\geq \phi[\int_1^{a_0} \tau^{n-1} \psi_R(\tau) d\tau] > 0. \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} -y_m'(2) &= \phi[\int_{\sigma_m}^2 (\frac{\tau}{2})^{n-1} f(\tau, y_m(\tau) d\tau] ds \\ &\geq \phi[\int_{b_0}^2 (\frac{\tau}{2})^{n-1} f(\tau, y_m(\tau) d\tau] \\ &\geq \phi[\int_{b_0}^2 (\frac{\tau}{2})^{n-1} \psi_R(\tau) d\tau] > 0. \end{aligned} \tag{3.20}$$

Furthermore, from (3.19), (3.20) and the convexity of $y_m(t)$ on $[1,2]$, for any $m \in \mathbb{N}_0$, there exists a constant $c > 0$ such that

$$y_m(t) > c(t-1)(2-t), \ \ t \in (1,2). \tag{3.21}$$

Step 3. It remains to show that $\{y_m(t)\}_{m \in \mathbb{N}_0}$ is uniformly bounded and equi-continuous on [0,1]. By $y_m(t)$ is the solution of (3.11) one has $\|y_m\| \leq R$, which implies that $\{y_m(t)\}_{m \in \mathbb{N}_0}$ is uniformly bounded on [0,1].

Next we need only to show that $\{y_m(t)\}_{m \in \mathbb{N}_0}$ is equi-continuous on [0,1]. Since $y_m(t)$ is the solution of (3.11), then

$$\begin{aligned} y_m'(1) &= \phi[\int_1^{\sigma_m} \tau^{n-1} f(\tau, y_m(\tau) d\tau] ds < 1, \\ -y_m'(2) &= \phi[\int_{\sigma_m}^2 (\frac{\tau}{2})^{n-1} f(\tau, y_m(\tau) d\tau] ds < 1. \end{aligned}$$

Hence, for any $t \in [1,2]$, $|y_m'(t)| < 1$. Furthermore, for any $t, s \in [0,1]$, we have

$$|y_m(t) - y_m(s)| = |\int_s^t y_m'(\tau) d\tau| < |t - s| \to 0 \text{ as } t \to s.$$

Therefore, $\{y_m(t)\}_{m\in\mathbb{N}_0}$ is equi-continuous on [0,1].

The Arzelà-Ascoli theorem guarantees that there is a subsequence $\mathbb{N}^*$ of $\mathbb{N}_0$ (without loss of generality, we assume $\mathbb{N}^* = \mathbb{N}_0$) and function $y(t)$ with $y_m(t) \to y(t)$ uniformly on [0,1] and $\sigma_m \to \sigma$ as $m \to +\infty$ through $\mathbb{N}^*$.

Since $y_m(t)(m \in \mathbb{N}^*)$ is the solution of (3.11), we have

$$y_m(t) = \begin{cases} \frac{1}{m} + \int_1^t \phi[\int_s^{\sigma_m} (\frac{\tau}{s})^{n-1} f(\tau, y_m(\tau))d\tau]ds, 1 \le t \le \sigma_m, \\ \frac{1}{m} + \int_t^2 \phi[\int_{\sigma_m}^s (\frac{\tau}{s})^{n-1} f(\tau, y_m(\tau))d\tau]ds, \sigma_m \le t \le 2. \end{cases} \tag{3.22}$$

Let $m \to +\infty$ through $\mathbb{N}^*$ in (3.22). By the continuity of f and Lebesgue's dominated convergence theorem, one has

$$y(t) = \begin{cases} \int_1^t \phi[\int_s^{\sigma} (\frac{\tau}{s})^{n-1} f(\tau, y(\tau))d\tau]ds, 1 \le t \le \sigma, \\ \int_t^2 \phi[\int_{\sigma}^s (\frac{\tau}{s})^{n-1} f(\tau, y(\tau))d\tau]ds, \sigma \le t \le 2, \end{cases} \tag{3.23}$$

and furthermore, we have $-(\frac{t^{n-1}y'}{\sqrt{1-y'^2}})' = t^{n-1}f(t,y)$, $t \in (1,2)$ and $y(1) = y(2) = 0$, i.e. $y(t)$ is positive solution of BVP (3.1), and $\|y\| \le R$, $\|y'\| < 1$, $y(t) \ge c(t-1)(2-t)$, $t \in [1,2]$. The proof of Theorem 3.5 is complete. □

Note that BVP (3.1) has the positive solution $y(t)$, then $u(|x|) = y(t)$ is a positive radial solution of mean curvature problem (1.2).

4 Examples

In this section, we give some explicit examples to illustrate our results.

Example 4.1. *Consider the following mean curvature equation*

$$\begin{cases} -div(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}) = u^2 + u^{-\frac{1}{2}}, \ x \in B, \\ u = 0, \ x \in \partial B, \end{cases} \tag{4.1}$$

where $B = \{x \in \mathbb{R}^2 | \ |x| < 1\}$.

Conclusion. BVP (4.1) *has at least one positive radial solution.*

Proof. We consider the radial solutions on B. Set $t = |x|$, $y(t) = u(|x|)$, the mean curvature problem (4.1) reduces to the following boundary value problem

$$\begin{cases} -\left(\dfrac{ty'}{\sqrt{1-y'^2}}\right)' = t(y^2 + y^{-\frac{1}{2}}), \ t \in (0,1), \\ y'(0) = 0, \ y(1) = 0. \end{cases} \tag{4.2}$$

Here $f(t,y) = y^2 + y^{-\frac{1}{2}}$. For any given constant $L > 0$, there exists $\psi_L(t) = L^{-\frac{1}{2}}$ such that

$$f(t,y) = y^2 + y^{-\frac{1}{2}} \geq L^{-\frac{1}{2}}, \ (t,y) \in [0,1] \times (0,L].$$

Hence (H_1) holds. Let $q(t) = 2, f_1(y) = y^{-\frac{1}{2}}, f_2(y) = y^2$. For any constant $K > 0$, by a direct calculation, we have

$$K^{-1/2} \int_0^1 \frac{\tau}{(1-\tau)^{\frac{1}{2}}} d\tau < \infty.$$

Hence (H_2) holds. Let $R = 1$, it is easy to verify

$$\begin{aligned} &\phi\left[\frac{\|q\|}{n}(1 + R^{\frac{5}{2}}) \int_0^1 \tau(1-\tau)^{-\frac{1}{2}} R^{-\frac{1}{2}} d\tau\right] \\ &= \phi\left(\frac{8}{3}\right) \\ &\doteq 0.9363291776 < 1 = R. \end{aligned}$$

Hence (H_3) holds. Therefore, by Theorem 2.6, we obtain that (4.2) has at least one positive solution, furthermore, the mean curvature problem (4.1) has at least one positive radial solution. □

Example 4.2. *Consider the following mean curvature equation*

$$\begin{cases} -\, div\left(\dfrac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right) = u^2 + u^{-\frac{1}{2}}, \ x \in A, \\ u = 0, \ x \in \partial A, \end{cases} \tag{4.3}$$

where $A = \{x \in \mathbb{R}^2 | \ 1 < |x| < 2\}$.

Conclusion. BVP (4.2) *has at least one positive radial solution.*

Proof. We consider the radial solutions on A. Set $t = |x|$, $y(t) = u(|x|)$, the mean curvature problem (4.3) reduces to the following boundary value problem

$$\begin{cases} -\left(\dfrac{ty'}{\sqrt{1-y'^2}}\right)' = t(y^2 + y^{-\frac{1}{2}}), \ t \in (1,2), \\ y(1) = 0, \ y(2) = 0. \end{cases} \tag{4.4}$$

Here $f(t,y) = y^2 + y^{-\frac{1}{2}}$. For any given constant $L > 0$, there exists $\psi_L(t) = L^{-\frac{1}{2}}$ such that

$$f(t,y) = y^2 + y^{-\frac{1}{2}} \geq L^{-\frac{1}{2}}, \ (t,y) \in [0,1] \times (0,L].$$

Hence (H_1^*) holds. Let $q(t) = 1, f_1(y) = y^{-\frac{1}{2}}, f_2(y) = y^2$. For any constant $K > 0$, by a direct calculation, we have

$$K^{-1/2} \int_1^2 \frac{\tau}{\sqrt{(2-\tau)(\tau-1)}} d\tau = K^{-1/2} \frac{3}{2} \pi < \infty.$$

Hence (H_2^*) holds. Let $R = 1$, it is easy to verify

$$\begin{aligned} &\phi[\|q\|(1 + R^{\frac{5}{2}}) \int_1^2 \frac{\tau}{\sqrt{(2-\tau)(\tau-1)}} R^{-\frac{1}{2}} d\tau] \\ &= \phi(3\pi) \\ &\doteq 0.9944181312 < 1 = R. \end{aligned}$$

Hence (H_3^*) holds. Therefore, by Theorem 3.5, we obtain that (4.4) has at least one positive solution, furthermore, the mean curvature problem (4.3) has at least one positive radial solution.

□

References

[1] R. Bartnik, L. Simon, Spacelike hypersurfaces with prescribed boundary values and mean curvature, Comm. Math. Phys. 87 (1982-1983) 131-152.

[2] C. Bereanu, P. Jebelean, J. Mawhin, Radial solutions for some nonlinear problems involving mean curvature operators in Euclidean and Minkowski spaces, Proc. Amer. Math. Soc. 137 (2009) 171-178.

[3] C. Bereanu, P. Jebelean, J. Mawhin, Radial solutions for Neumann problems involving mean curvature operators in Euclidean and Minkowski spaces, Math. Nachr. 283 (2010) 379-391.

[4] C. Bereanu, P. Jebelean, J. Mawhin, Radial solutions of Neumann problems involvingmean extrinsic curvature and periodic nonlinearities, Calc. Var. Partial Differential Equations 46 (2013) 113-122.

[5] C. Bereanu, P. Jebelean, P.J. Torres, Positive radial solutions for Dirichlet problems with mean curvature operators in Minkowski space, J. Funct. Anal. 264 (2013) 270-287.

[6] C. Bereanu, P. Jebelean, P.J. Torres, Multiple positive radial solutions for a Dirichlet problem involving the mean curvature operator in Minkowski space, J. Funct. Anal. 265 (2013) 644-659.

[7] E. Calabi, Examples of Bernstein problems for some nonlinear equations. Proc. Symp. Pure Appl. Math. 15 (1968) 223-230.

[8] S.-Y. Cheng, S.-T. Yau, Maximal spacelike hypersurfaces in the Lorentz-Minkowski spaces, Ann. of Math. 104 (1976) 407-419.

[9] A.E. Treibergs, Entire spacelike hypersurfaces of constant mean curvature in Minkowski space, Invent. Math. 66 (1982) 39-56.

[10] N.D. Brubaker, J.A. Pelesko, Analysis of a one-dimensional prescribed mean curvature equation with singular nonlinearity, Nonlinear Anal. 75 (2012) 5086-5102.

[11] Y. Li, J.X. Yin, Radial symmetric solution for the generalized mean curvature equation with the singularity, Chinese Annals of Mathematics (in China), 21 A: 4 (2000) 483-490.

[12] T. Kusano, A. Ogota, H. Usami, On the oscillation of solutions second order quasilinear differential equations, Hiroshima Math. J., 23 (1993) 654-667.

[13] D. O'Regan, Theory of Sigular Boundary Value Problems. Singapore: World Scientific, 1994.

College of Science

Changchun University

Changchun 130022

PR China

E-mail address:mathchunmei2012@aliyun.com