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STRUCTURE OF THE POSITIVE RADIAL SOLUTIONS FOR A SUPERCRITICAL NEUMANN PROBLEM IN A BALL

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Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$, be a bounded domain with smooth boundary. We study the positive solution of the Neumann problem

$$\begin{cases} \varepsilon^2 \Delta u - u + u^p = 0 & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\varepsilon > 0$ is a positive parameter. This problem arises in stationary problems of the shadow system of the Gierer-Meinhardt model and the Keller-Segel model with logarithmic sensitivity function. When the domain is the entire space $\mathbb{R}^N$, the problem (1) also appears in the study of the standing wave of the nonlinear Schrödinger equation. The problem (1) has attracted much attention for more than two decades. Solutions of various shapes have been found in [4, 6, 11, 12]. However, many authors study the case $1 < p < p_S$. Here,

$$p_S := \begin{cases} \dfrac{N+2}{N-2} & \text{if } N \geq 3, \\ \infty & \text{if } N = 1, 2. \end{cases}$$

When $p > p_S$, the Sobolev embedding $H^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ does not hold and it is difficult to use variational methods. There are few results about the structure of the positive solutions in the case $p \geq p_S$.

We consider the positive radial solutions when $p > p_S$ and $\Omega = B$. Then (1) can be reduced to the ODE

$$\begin{cases} u_{rr} + \dfrac{N-1}{r} u_r + \lambda f(u) = 0 & (0 < r < 1), \\ u_r(1) = 0, & \\ u(r) > 0 & (0 \leq r \leq 1), \end{cases} \tag{2}$$

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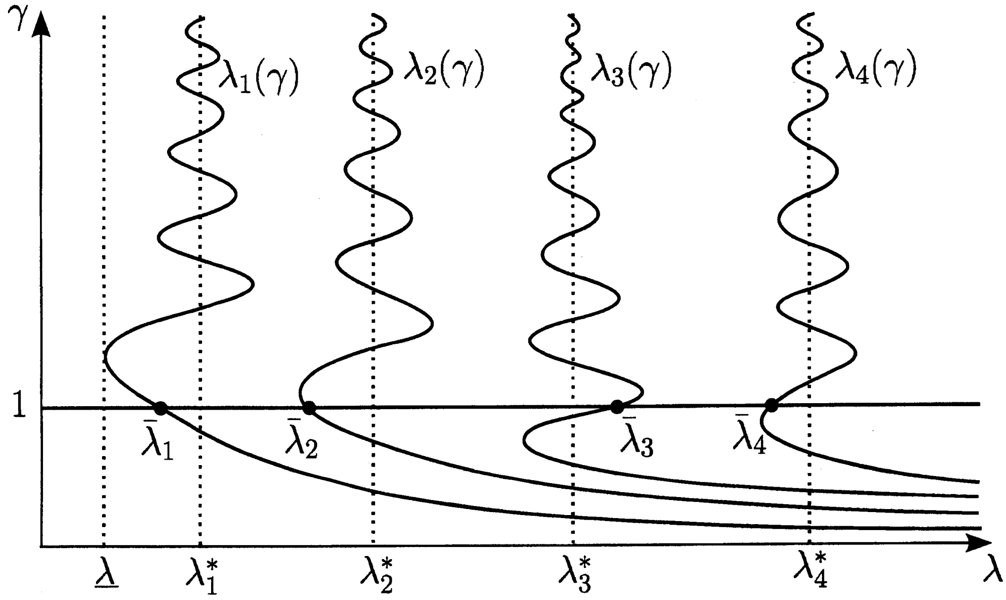

FIGURE 1. Schematic picture of the bifurcation diagram of (2) in the case $p_S < p < p_{JL}$.

where $f(u) = -u + u^p$ and $\lambda = 1/\varepsilon^2 > 0$. Adimurthi and Yadava [1, 2] studied the critical case $p = p_S$ when Ω is a unit ball B. They have shown that if $N \geq 7$, then (2) has a solution for all small $\lambda > 0$, while if $N \in \{4, 5, 6\}$, then (2) has no solution for small $\lambda > 0$. Del Pino et.al. [3] constructed a bubble tower solution when p is slightly greater than p^*.

We study the bifurcation diagram of the radial solutions of (1), using ODE techniques. In this study the existence of the singular solution of (2) plays an important role.

Theorem A. *Suppose that $p > p_S$. The problem (1) has infinitely many singular solutions $(\lambda_n^*, U_n^*(r)) \in \mathbb{R}_+ \times (C^2(0,1) \cap C^0(0,1] \cap H^1(B))$ ($n = 1, 2, \cdots$ and $\lambda_1^* < \lambda_2^* < \cdots \to \infty$) such that the following assertions hold:*
(i) $U_n^*(r)$ satisfies

$$U_n^*(r) = A(p,N)(\sqrt{\lambda_n^*}r)^{-\theta}(1 + o(1)) \quad \text{as} \quad (r \downarrow 0), \tag{3}$$

where

$$A(p,N) := \{\theta(N-2-\theta)\}^{\frac{1}{p-1}}. \tag{4}$$

(ii) $\mathcal{Z}_{(0,1]}[U_n^*(\cdot) - 1] = n$.
(iii) $U_n^*(r) > 0$ $(0 < r \leq 1)$.
Moreover, the singular solution (λ_n^*, U_n^*) is unique, i.e., if $(\tilde{\lambda}_n^*, \tilde{U}_n^*)$ is a singular solution such that (i) and (ii) hold, then $(\tilde{\lambda}_n^*, \tilde{U}_n^*) = (\lambda_n^*, U_n^*)$.

The main result is the following:

Theorem B. *Suppose that $p > p_S$. Let $\mathcal{S}$ be the set of the regular solutions. Then*

$$\mathcal{S} = \mathcal{C}_0 \cup \bigcup_{n=1}^{\infty} (\mathcal{C}_n^+ \cup \mathcal{C}_n^+),$$

where $\mathcal{C}_n^+$ (resp. $\mathcal{C}_n^-$) is the branch emanating from the trivial branch $\{(\lambda, 1)\}_{\lambda>0}$, which we denote by $\mathcal{C}_0$, such that $u(0) > 1$ (resp. $u(0) < 1$). $\mathcal{C}_n^{\pm}$ is a C^1-function of $\gamma := u(0)$, hence $\mathcal{C}_n^{\pm}$ can be described as $\{(\lambda_n(\gamma), u_n(r, \gamma))\}$. Moreover, the following hold:
(i) $\lambda_n(1) = \bar{\lambda}_n$,
(ii) $\lambda_n(\gamma) \to \lambda_n^*$ $(\gamma \to \infty)$,
(iii) if $p_s < p < p_{JL}$, then $\lambda_n(\gamma)$ oscillates infinitely many times around λ_n^*, where

$$p_{JL} := \begin{cases} 1 + \dfrac{4}{N - 4 - 2\sqrt{N-1}} & \text{if } N \geq 11, \\ \infty & \text{if } 2 \leq N \leq 10, \end{cases}$$

(iv) $\lambda_n(\gamma) \to \infty$ $(\gamma \downarrow 0)$,
(v) if $\gamma > 0$ is small, then $u_1(r, \gamma)$ is non-degenerate in the space of radial functions and it concentrates on the boundary,
(vi) $\lambda_1(\gamma) < \lambda_2(\gamma) < \cdots$.

Figure 1 is a schematic picture of the bifurcation diagram of (2) in the case $p_S < p < p_{JL}$. When $p_S < p < p_{JL}$, (2) has infinitely many regular solutions for $\lambda = \lambda_n^*$. Each branch blows up at λ_n^*, while it is unbounded in the positive direction of λ in the subcritical case $1 < p < p_S$.

The following corollary is an immediate consequence of Theorem B.

Corollary C. *Suppose that $p > p_S$. There exist $\underline{\lambda} > 0$ and $\bar{\lambda}(> \underline{\lambda})$ such that a radially decreasing solution of (2), which belongs to $\mathcal{C}_1^+$, does not exist for $\lambda \in (0, \underline{\lambda}) \cup (\bar{\lambda}, \infty)$.*

The main tool of the proof is an intersection number between the singular solution and a regular solution. Using a scaling argument, one can show that each branch has infinitely many turning points if $p_S < p < p_{JL}$. See [8] for details of the proof. In the case $p \geq p_{JL}$ we do not know the number of the turning points of each branch.

Let us explain the strategy of the proof. Let $u(s) := U(r)$ and $s := \sqrt{\lambda}r$. The equation (1) is transformed to the problem

$$(5) \qquad \begin{cases} u_{ss} + \frac{N-1}{s}u_s + f(u) = 0, & 0 < s < \sqrt{\lambda}, \\ u_s(\sqrt{\lambda}) = 0, & \\ u > 0, & 0 \le s \le \sqrt{\lambda}. \end{cases}$$

First we construct the singular solution $u^*(s)$ of the equation in (5) near $s = 0$ and show that $u^*(s) = As^{-\theta}(1+o(1))$ $(s \downarrow 0)$. Here $A := A(p, N)$ and $A(p, N)$ is defined by (4). Second we show that the domain of $u^*(s)$ can be extended to $0 < s < \infty$, that $u^*(s)$ satisfies the equation in (5), and that $u^*(s) > 0$ for $s > 0$. Third we show that $u^*(s)$ oscillates around 1 infinitely many times as $s \to \infty$ and that $u^*(s)$ has the set of the critical points $\{s_n^*\}_{n=1}^{\infty}$ of u^* such that $0 < s_1^* < s_2^* < \cdots \to \infty$ and

$$\begin{cases} s_n^* \text{ is a local minimum point of } u^* \text{ and } u^*(s_n^*) < 1 \text{ if } n \in \{1, 3, 5, \cdots\}, \\ s_n^* \text{ is a local maximum point of } u^* \text{ and } u^*(s_n^*) > 1 \text{ if } n \in \{2, 4, 6, \cdots\}. \end{cases}$$

We set $\lambda_n^* := (s_n^*)^2$ and $U_n^*(r) := u^*(s)$ $(s = \sqrt{\lambda_n^*}r)$. Then, (λ_n^*, U_n^*) is a singular solution stated in Theorem A.

Let $(\lambda_n(\gamma), u(s, \gamma))$ denote the solution of (5) such that $u(0, \gamma) = \gamma$ and $u_s(0, \gamma) = 0$. We show that $\lambda_n(\gamma) \to \lambda_n^*$ as $\gamma \to \infty$ and that $u(s, \gamma)$ converges to $u^*(s)$ in an appropriate sense. In [7] Merle and Peletier proved a similar convergence result for the Dirichlet problem

$$\begin{cases} U_{rr} + \frac{N-1}{r}U_r + \lambda U + U^p = 0, & 0 < r < 1, \\ U(1) = 0, & \\ U > 0, & 0 \le r < 1. \end{cases}$$

when $p > p_S$. We show that $u(s, \gamma) \to u^*(s)$, following arguments in the proof of [7, Theorem A].

We show that $\lambda_n(\gamma)$ oscillates around λ_n^* if $p_S < p < p_{JL}$. Let $\rho := \gamma^{\frac{p-1}{2}}s$. We define $\tilde{u}(\rho, \gamma) := u(s, \gamma)/\gamma$ and $\tilde{u}^*(\rho) := u^*(s)/\gamma$. We use the intersection number between $\tilde{u}$ and $\tilde{u}^*$. The function $\tilde{u}(\rho, \gamma)$ satisfies

$$(6) \qquad \begin{cases} \tilde{u}_{\rho\rho} + \frac{N-1}{\rho}\tilde{u}_\rho + \tilde{u}^p - \frac{1}{\gamma^{p-1}}\tilde{u} = 0, & 0 < \rho < \infty, \\ \tilde{u}(0) = 1, \ \tilde{u}_\rho(0) = 0. \end{cases}$$

Let $\bar{u}(\rho, \gamma)$ be the regular solution of

$$(7) \qquad \begin{cases} \bar{u}_{\rho\rho} + \frac{N-1}{\rho}\bar{u}_\rho + \bar{u}^p = 0, & 0 < \rho < \infty, \\ \bar{u}(0) = \gamma, \ \bar{u}_\rho(0) = 0. \end{cases}$$

We show that as $\gamma \to \infty$,

$$\tilde{u}(\rho, \gamma) \to \bar{u}(\rho, 1) \quad \text{in} \quad C^2_{loc}(0, \infty) \cap C^0_{loc}[0, \infty)$$

and

$$\tilde{u}^*(\rho) \to \bar{u}^*(\rho) \quad \text{in} \quad C^0_{loc}(0, \infty),$$

where $\bar{u}^*(\rho)$ a singular solution of the equation in (7). We recall the fact that $\mathcal{Z}_{(0,\infty)}[\bar{u}^*(\,\cdot\,) - \bar{u}(\,\cdot\,, 1)] = \infty$. Hence, for each $\delta > 0$,

$$\mathcal{Z}_{(0,\delta)}[u^*(\,\cdot\,) - u(\,\cdot\,, \gamma)] \to \infty \quad (\gamma \to \infty), \tag{8}$$

since $s \in (0, \delta)$ is corresponding to $\rho \in (0, \delta\gamma^{\frac{N-1}{2}})$ and $\delta\gamma^{\frac{N-1}{2}} \to \infty$ $(\gamma \to \infty)$. Since each zero of $u^*(\,\cdot\,) - u(\,\cdot\,, \gamma)$ is simple, each zero depends continuously on γ. The divergence (8) tells us that a zero which is simple enters the interval $(0, \sqrt{\lambda_n^*}]$ from $s = \sqrt{\lambda_n^*}$ infinitely many times. Therefore, there exists a sequence of large numbers $\{\gamma_j\}_{j=1}^\infty$ $(\gamma_1 < \gamma_2 < \cdots \to \infty)$ such that $u^*(\sqrt{\lambda_n^*}) = u(\sqrt{\lambda_n^*}, \gamma_j)$ and the following holds: $u_s(\sqrt{\lambda_n^*}, \gamma_j) < 0$ for $j \in \{1, 3, 5, \cdots\}$ and $u_s(\sqrt{\lambda_n^*}, \gamma_j) > 0$ for $j \in \{2, 4, 6, \cdots\}$. Using the convergence $u(s, \gamma) \to u^*(s)$, we show that if $n \in \{1, 3, 5, \cdots\}$ (resp. $n \in \{2, 4, 6, \cdots\}$)

$$\lambda_n(\gamma_j) \begin{cases} > \lambda_n^*, & (j \in \{1, 3, 5, \cdots\}), \\ < \lambda_n^*, & (j \in \{2, 4, 6, \cdots\}), \end{cases} \tag{9}$$

$$\left(\text{resp. } \lambda_n(\gamma_j) \begin{cases} < \lambda_n^*, & (j \in \{1, 3, 5, \cdots\}), \\ > \lambda_n^*, & (j \in \{2, 4, 6, \cdots\}). \end{cases} \right)$$

which implies that $\lambda_n(\gamma)$ oscillates around λ_n^* infinitely many times as $\gamma \to \infty$.

This method can be applied to Dirichlet problems. In [9] the author studies

$$\begin{cases} \Delta u + \lambda f(u) = 0 & \text{in } B, \\ u > 0 & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{10}$$

where $f(u) = u^p + g(u)$, $p > p^*$, and $|g(u)| < Cu^{p-\varepsilon}$. We assume that $f \in C^1$ and $f(u) > 0$ for $u \geq 0$. If $3 \leq N \leq 10$, then the branch of the solutions of (10) has infinitely many turning points. It is shown that there is a nonlinear term $f(u)$ such that the branch has finitely many turning points.

In [10] he studies (10), where $f(u) = e^u + g(u)$ and $|g(u)| < Ce^{(1-\varepsilon)u}$. In [10] it is shown that if $3 \leq N \leq 9$, then the branch of the solutions of (10) has infinitely many turning points around some $\lambda^* > 0$ and that if

$$\text{(f1')}\quad N \geq 10,\ -e^u < g'(u) \leq \frac{N-10}{8}e^u \text{ in } (0,\infty), \\ \text{and } g''(u) > -e^u \text{ in } (0,\infty),$$

then the branch does not have a turning point and blows up at λ^*. In particular, the branch consists only of the minimal solutions. Thus, when (f1') is satisfied, then the bifurcation diagram is qualitatively the same as the case $f(u) = e^u$.

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