On multidimensional inverse scattering in time-dependent electric fields†

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1 Introduction

Throughout this paper, we assume the spatial dimension $d \geq 2$. We report one of the inverse scattering problems for quantum systems in a time-dependent electric field $E(t) \in \mathbb{R}^d$, which was obtained in Adachi-Fujiwara-I [1]. By Enss-Weder time-dependent method [5], we can show that the high speed limit of the scattering operator determines uniquely the potential $V$ belonging to the wider class than the classes given by the previous work in Adachi-Maehara [3], Adachi-Kamada-Kazuno-Toratani [2], Nicoleau [10] and Fujiwara [6].

The free and full Hamiltonians under the consideration are given by

$$H_0(t) = p^2/2 - E(t) \cdot x, \quad H(t) = H_0(t) + V$$

acting as the self-adjoint operators on $L^2(\mathbb{R}^d)$, where $p = -i\nabla_x$ is the momentum, $E(t)$ is the time-dependent electric field and the interaction potential $V$ is real-valued multiplicative operator. $E(t)$ and $V = V^{v_0} + V^s + V^l \in \mathcal{V}_{v_0}^{\mu,\alpha_\mu} + \mathcal{V}_{s,\gamma_\mu}^{l}$ satisfy following assumptions.

Assumption 1.1. The time-dependent electric field $E(t) \in \mathbb{R}^d$ is represented as

$$E(t) = E_0(1 + |t|)^{-\mu} + E_1(t),$$

where $0 \leq \mu < 1$, $E_0 \in \mathbb{R}^d \setminus \{0\}$ and $E_1(t) \in C(\mathbb{R}, \mathbb{R}^d)$ such that

$$\left| \int_0^t \int_0^s E_1(\tau) d\tau ds \right| \leq C \max\{|t|, |t|^2-\mu_1\}$$

with $\mu < \mu_1 \leq 1$.

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Roughly speaking about the perturbation part $E_1(t)$, we assume that $|E_1(t)| \leq C(1 + |t|)^{-\mu_2}$ for some $\mu_2 > \mu$ and take $\mu_1$ as follows:

$$
\begin{cases}
\mu_1 = \mu_2 & \mu < \mu_2 < 1 \\
\mu < \mu_1 < \mu_2 & \mu_2 = 1 \\
\mu_1 = 1 & \mu_2 > 1.
\end{cases}
$$

(S.4)

Such $E(t)$ was first dealt with in Adachi-Kamada-Kazuno-Toratani [2]. For brevity’s sake, we suppose that $E_0 = e_1 = (1,0,\ldots,0) \in \mathbb{R}^d$.

**Assumption 1.2.** $\mathcal{V}^{vs}$ is the class of real-valued multiplicative operators $V^{vs}$ is satisfying that $V^{vs}$ is decomposed into the sum of a singular part $V_1^{vs}$ and a regular part $V_2^{vs}$. $V_1^{vs}$ is compactly supported, belongs to $L^q(\mathbb{R}^d)$ and satisfies $|\nabla V_1^{vs}| \in L^q(\mathbb{R}^d)$. $V_2^{vs} \in C^1(\mathbb{R}^d)$ satisfies that $V_2^{vs}$ and its first derivatives are all bounded in $\mathbb{R}^d$ and that

$$
\int_0^\infty \| F(|x| \geq R) V_2^{vs}(x) \|_{\mathfrak{R}(L^2)} dR < \infty.
$$

Here $q_1$ satisfies that $q_1 > d/2$ and $q_1 \geq 2$, $q_2$ satisfies

$$
\begin{cases}
1/q_2 = 1/(2q_1) + 2/d & d \geq 5 \\
1/q_2 = 1/(2q_1) + 1/2 & d = 4 \\
1/q_2 = 1/(2q_1) + 1/2 & d \leq 3,
\end{cases}
$$

and $F(|x| \geq R)$ is the characteristic function of $\{x \in \mathbb{R}^d \mid |x| \geq R\}$.

$\mathcal{V}_{\mu,\alpha_\mu}^{s}$ with some $\alpha_\mu > 0$ is the class of real-valued multiplicative operators $V^{s}$ is satisfying that $V^{s}$ belongs to $C^1(\mathbb{R}^d)$ and satisfies

$$
|V^{s}(x)| \leq C\langle x \rangle^{-\gamma}, \quad |\partial_x^\beta V^{s}(x)| \leq C_\beta \langle x \rangle^{-1-\alpha}, \quad |\beta| = 1
$$

(S.7)

with some $\gamma$ and $\alpha$ such that $1/(2-\mu) < \gamma \leq 1$ and $\alpha_\mu < \alpha \leq \gamma$.

Finally, $\mathcal{V}_{\mu,\gamma_\mu}^{1}$ with some $\gamma_\mu > 1/(2-\mu)$ is the class of real-valued multiplicative operators $V^{1}$ is satisfying that $V^{1}$ belongs to $C^2(\mathbb{R}^d)$ and satisfies

$$
|\partial_x^\beta V^{1}(x)| \leq C\langle x \rangle^{-\gamma_D-|\beta|/(2-\mu)}, \quad |\beta| \leq 2
$$

(S.8)

with some $\gamma_D$ such that $\gamma_\mu < \gamma_D \leq 1/(2-\mu)$.

We note that one can obtain

$$
\int_0^\infty \| F(|x| \geq R) V^{vs}(x)(p)^{-2} \|_{\mathfrak{R}(L^2)} dR < \infty
$$

(S.9)

by this assumption and it is equivalent to

$$
\int_0^\infty \| V^{vs}(x)(p)^{-2} F(|x| \geq R) \|_{\mathfrak{R}(L^2)} dR < \infty
$$

(S.10)
because $V^{vs}$ is a multiplicative operator (see e.g. Reed-Simon [11]). As for the class $\mathcal{V}_{\mu,\alpha_{\mu}}^{s}$, we also note that by virtue of $\alpha \leq \gamma$, we can treat an oscillation part. For example, the following function belongs to $\mathcal{V}_{\mu,\alpha_{\mu}}^{s}$:

$$V^{s}(x) = \langle x \rangle^{-\gamma} \cos \langle x \rangle^{\gamma-\alpha}. \quad (1.11)$$

In fact, we can verify easily that $|\nabla_{x}V^{s}(x)| \leq C(\langle x \rangle^{-1-\gamma} + \langle x \rangle^{-1-\alpha}) \leq C\langle x \rangle^{-1-\alpha}$ holds with some $C > 0$.

2 Results

We first state the case where $V^{1} = 0$. Then we can see the wave operators

$$W^{\pm} = \lim_{t \to \pm\infty} U(t, 0)^{*}U_{0}(t, 0) \quad (2.1)$$

exist as this fact was shown in Adachi-Kamada-Kazuno-Toratani [2], where we denote the propagators generated by $H_{0}(t)$ and $H(t)$ as $U_{0}(t, 0)$ and $U(t, 0)$. The existence and uniqueness of these propagators are guaranteed by virtue of Yajima [14]. The scattering operator $S = S(V)$ is defined by

$$S = (W^{+})^{*}W^{-} . \quad (2.2)$$

The following obtained in [1] is one of those which we would like to report in this paper.

**Theorem 2.1. (Adachi-Fujiwara-I [1])** Put

$$\tilde{\alpha}_{\mu} = \left\{ \begin{array}{ll}
7 - 3\mu - \sqrt{(1 - \mu)(17 - 9\mu)} & 0 \leq \mu \leq 1/2 \\
1 + \mu & 1/2 < \mu < 1.
\end{array} \right. \quad (2.3)$$

Let $V_{1}, V_{2} \in \mathcal{V}^{vs} + \mathcal{V}_{\mu_{0},\alpha_{\mu}}^{s}$. If $S(V_{1}) = S(V_{2})$, then $V_{1} = V_{2}$.

In the case where $E(t) \equiv E_{0}$, that is, the case of the Stark effect, this theorem was first proved by Weder [12] under the condition $V^{s} \in \mathcal{V}_{0,0}^{s}$ and the additional assumption $\gamma > 3/4$. However, as it is well-known, the short-range condition on $V$ under the Stark effect is $\gamma > 1/2$. Later Nicoleau [9] proved this theorem for real-valued $V \in C^{\infty}(\mathbb{R}^{d})$ satisfying $|\partial_{x}^{\beta}V(x)| \leq C_{\beta}\langle x \rangle^{-\gamma-|\beta|}$ with $\gamma > 1/2$, under the spatial dimension $d \geq 3$. After that, this theorem was obtained by Adachi-Maehara [3] for $V^{s} \in \mathcal{V}_{0,1/2}^{s}$. In our case where $\mu = 0$, substitute $\mu = 0$ in $\tilde{\alpha}_{\mu}$. We have

$$\tilde{\alpha}_{0} = \left. \frac{7 - 3\mu - \sqrt{(1 - \mu)(17 - 9\mu)}}{4(2 - \mu)} \right|_{\mu=0} = \frac{7 - \sqrt{17}}{8} < \frac{1}{2}. \quad (2.4)$$

If $a < b$, then $\mathcal{V}_{\mu,b}^{s} \subsetneq \mathcal{V}_{\mu,a}^{s}$. Therefore this implies

$$\mathcal{V}_{0,1/2}^{s} \subsetneq \mathcal{V}_{0,\tilde{\alpha}_{0}}^{s} . \quad (2.5)$$
In the time-dependent case where $0 < \mu < 1$ and $E_1(t) \not\equiv 0$, the result corresponding to Theorem 2.1 was also obtained by Adachi-Kamada-Kazuno-Toratani [2] under the assumption that $V \in \mathcal{V}^s + \mathcal{V}_{\mu,1/(2-\mu)}$. We can verify that $\tilde{\alpha}_\mu < 1/(2 - \mu)$ and this implies that above result is finer than the previous one:

$$\mathcal{V}_{\mu,1/(2-\mu)} \subsetneq \mathcal{V}_{\mu,\tilde{\alpha}_\mu}. \quad (2.6)$$

We next state the case where $V^1 \not\equiv 0$. If $V^1 \in \mathcal{V}_{\mu,\tilde{\gamma}_\mu}$, the Dollard-type modified wave operators due to White [13] (see also Adachi-Tamura [4] and Jensen-Yajima [8])

$$W^\pm_D = \lim_{t \to \pm \infty} U(t, 0)^* U_0(t, 0) M_D(t), \quad M_D(t) = e^{-i \int_0^t V^1(\sigma+c(\tau))d\tau} \quad (2.7)$$

can exist by virtue of the condition $\gamma_D > 1/(2(2 - \mu))$ (see [2]), where we put

$$c(t) = \int_0^t b(\tau)d\tau, \quad b(t) = \int_0^t E(\tau)d\tau. \quad (2.8)$$

Then the Dollard-type modified scattering operator $S_D = S_D(V^1, V^{\nu} + V^s)$ is defined by

$$S_D = (W^+_D)^* W^-_D. \quad (2.9)$$

Then we also report the following result.

**Theorem 2.2. (Adachi-Fujiiwara-I [1])** Suppose that a given $V^1$ satisfies $V^1 \in \mathcal{V}_{\mu,\tilde{\gamma}_\mu}$ with

$$\tilde{\gamma}_\mu = \frac{1}{2(2 - \mu)} + \frac{1 - \mu}{4(2 - \mu)}. \quad (2.10)$$

Put

$$\tilde{\alpha}_{\mu,D} = \begin{cases} \frac{13 - 5\mu - \sqrt{(1 - \mu)(41 - 25\mu)}}{8(2 - \mu)} & 0 \leq \mu \leq 5/7 \\ \frac{1 + \mu}{2(2 - \mu)} & 5/7 < \mu < 1. \end{cases} \quad (2.11)$$

Let $V_1, V_2 \in \mathcal{V}^s + \mathcal{V}_{\mu,\tilde{\alpha}_{\mu,D}}$. If $S_D(V^1, V_1) = S(V^1, V_2)$, then $V_1 = V_2$. Moreover, any one of the Dollard-type modified scattering operators $S_D$ determines uniquely the total potential $V$.

In the case where $0 < \mu < 1$ and $E_1(t) \not\equiv 0$, Adachi-Kamada-Kazuno-Toratani [2] proved this theorem under the condition that

$$V^1 \in \mathcal{V}_{\mu,\hat{\gamma}_\mu}, \quad V_1, V_2 \in \mathcal{V}^s + \mathcal{V}_{\mu,1/(2-\mu)} \quad (2.12)$$

with $\hat{\mu} = (7 - \sqrt{3} - \sqrt{60 - 22\sqrt{3}})/4$ and

$$\tilde{\gamma}_\mu = \begin{cases} -\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2(1-\mu)^2}{1+\mu}} & 0 < \mu \leq \hat{\mu} \\ -\frac{3 - \mu}{8} + \sqrt{\frac{(3 - \mu)^2}{64} + \frac{2\mu^2 - 7\mu + 7}{4(2 - \mu)^2}} & \hat{\mu} < \mu < 1. \end{cases} \quad (2.13)$$

We can verify that $\tilde{\alpha}_\mu < 1/(2 - \mu)$ and this implies that above result is finer than the previous one:
Computing straightforwardly, we can see $\tilde{\alpha}_{\mu,D} < 1/(2-\mu)$ and $\tilde{\gamma}_{\mu} < \hat{\gamma}_{\mu}$. We thus obtain

$$\mathcal{V}_{\mu,1/(2-\mu)}^{S} \subsetneq \mathcal{V}_{\mu,\tilde{\alpha}_{\mu,D}}^{S}, \mathcal{V}_{\mu,\hat{\gamma}_{\mu}}^{1} \subsetneq \mathcal{V}_{\mu,\overline{\gamma}_{\mu}}^{1}.$$  \tag{2.14}

In particular, there was no result for the case where $\mu = 0$. Here we emphasize that if $5/7 \leq \mu < 1$, then $\tilde{\alpha}_{\mu,D} = \tilde{\alpha}_{\mu}$ holds, although if $0 \leq \mu < 5/7$, then $\tilde{\alpha}_{\mu,D} > \tilde{\alpha}_{\mu}$ holds.

**Remark 2.3.** We assume that $E(t) \in C(\mathbb{R}, \mathbb{R}^{d})$ is $T$-periodic in time with non-zero mean $E_{0}$, that is,

$$E_{0} = \int_{0}^{T} E(\tau)d\tau/T \neq 0,$$  \tag{2.15}

which was treated by Nicoleau [10] and Fujiwara [6]. In this case, the method in the proofs of Theorems 2.1 and 2.2 does work well also, because we have

$$|b(t) - tE_{0}| \leq \int_{0}^{T} |E(\tau) - E_{0}|d\tau,$$  \tag{2.16}

$$|c(t) - t^{2}E_{0}/2| \leq \int_{0}^{|t|} |b(\tau) - \tau E_{0}|d\tau \leq C|t|,$$  \tag{2.17}

with $C = \int_{0}^{T} |E(\tau) - E_{0}|d\tau$ by the periodicity of $E(t)$. (2.17) implies $\mu = 0$ in (1.2) and $\mu_{1} = 1$ in (1.3).

By virtue of this fact, we can obtain an improvement of the results of [10] and [6].

**Theorem 2.4.** Suppose that $E(t) \in C(\mathbb{R}, \mathbb{R}^{d})$ is $T$-periodic in time with non-zero mean $E_{0}$. Then the followings hold.

1. Let $V_{1}, V_{2} \in \mathcal{V}^{vs} + \mathcal{V}^{s}_{0,\tilde{\alpha}_{0}}$. If $S(V_{1}) = S(V_{2})$, then $V_{1} = V_{2}$.

2. Suppose that a given $V^{1}$ satisfies $V^{1} \in \mathcal{V}^{1}_{0,\tilde{\alpha}_{0}}$. Let $V_{1}, V_{2} \in \mathcal{V}^{vs} + \mathcal{V}^{s}_{0,\tilde{\alpha}_{0,D}}$. If $S_{D}(V^{1}, V_{1}) = S_{D}(V^{1}, V_{2})$, then $V_{1} = V_{2}$. Moreover, any one of the Dollard-type modified scattering operators $S_{D}$ determines uniquely the total potential $V$.

Nicoleau [10] proved the uniqueness assuming that $|\partial_{x}^{\beta}V(x)| \leq C_{\beta}(x)^{-\gamma-|\beta|}$ with $\gamma > 1/2$ for $V \in C^{\infty}(\mathbb{R})^{d}$ and the additional condition $d \geq 3$. Fujiwara [6] assumed that $V \in \mathcal{V}^{vs} + \mathcal{V}^{s}_{0,1/2}$. These two results did not treat the long-range potentials.

### 3 Short-range Case

By virtue of Theorem 3.1 below and the Plancherel formula associated with the Radon transform (see Helgason [7]), Theorem 2.1 can be shown in the quite same way as in the proof of Theorem 1.2 in [12] (see also Enss-Weder [5]).
Theorem 3.1. (Reconstruction Formula [1]) Let $\hat{v} \in \mathbb{R}^d$ be given such that $|\hat{v} \cdot e_1| < 1$. Put $v = |v|\hat{v}$. Let $\eta > 0$ be given, and $\Phi_0, \Psi_0 \in \mathcal{S}(\mathbb{R}^d)$ be such that $\mathcal{F}\Phi_0, \mathcal{F}\Psi_0 \in C^\infty_0(\mathbb{R}^d)$ with $\text{supp} \mathcal{F}\Phi_0, \text{supp} \mathcal{F}\Psi_0 \subset \{\xi \in \mathbb{R}^d \mid |\xi| \leq \eta\}$. Put $\Phi_v = e^{iv \cdot x} \Phi_0, \Psi_v = e^{iv \cdot x} \Psi_0$. Let $V^{\nu} \in \mathcal{V}^{\nu}$ and $V^s \in \mathcal{V}^{s}_{\mu, \tilde{\alpha}_\mu}$, where $\tilde{\alpha}_\mu$ is the same as in Theorem 2.1 and $\mathcal{F}$ is the Fourier transformation. Then

$$|v|(i[S, p_j] \Phi_v, \Psi_v) = \int_{-\infty}^{\infty} \left( (V^{\nu}(x + \hat{v}t)p_j \Phi_0, \Psi_0) - (V^{\nu}(x + \hat{v}t)\Phi_0, p_j \Psi_0) ight. $$
$$\left. + (i(\partial_{x_j} V^s)(x + \hat{v}t) \Phi_0, \Psi_0) \right) dt + o(1) \tag{3.1}$$

holds as $|v| \to \infty$ for $1 \leq j \leq d$.

To prove Theorem 3.1, the following estimate is the key.

Proposition 3.2. Let $v$ and $\Phi_v$ be as in Theorem 3.1 and $\epsilon > 0$. Put

$$\Theta(\alpha) = \left\{ \begin{array}{ll}
\alpha + \frac{(\alpha - \mu)(1 - \alpha)}{(1 - \mu)(2 - \alpha)} & \alpha > \mu \\
\alpha - \frac{\mu - \alpha}{1 - \mu} & \mu/(2 - \mu) < \alpha \leq \mu.
\end{array} \right. \tag{3.2}$$

Then

$$\int_{-\infty}^{\infty} \|(V^s(x) - V^s(x + c(t)))U_0(t, 0)\Phi_v\| dt = O(|v|^{-\Theta(\alpha) + \epsilon}) \tag{3.3}$$

holds as $|v| \to \infty$ for $V^s \in \mathcal{V}^{s}_{\mu, \mu/(2-\mu)}$.

In Adachi-Maehara [3], the corresponding estimate to this proposition was

$$\int_{-\infty}^{\infty} \|(V^s(x) - V^s(x + c(t)))U_0(t, 0)\Phi_v\| dt = O(|v|^{-\alpha}) \tag{3.4}$$

(see Lemma 2.2 in [3]). When we denote the error term of (3.1) by $R(v)$, $\lim_{|v| \to \infty} R(v) = 0$ is equivalent to $2(-\alpha) + 1 < 0$. Therefore $\alpha > 1/2$ was required. On the other hand, in Adachi-Kamada-Kazuno-Toratani [2], the corresponding one was

$$\int_{-\infty}^{\infty} \|(V^s(x) - V^s(x + c(t)))U_0(t, 0)\Phi_v\| dt = O(|v|^{-\rho}) \tag{3.5}$$

where

$$\rho = \frac{(2 - \mu)\alpha - \mu}{2(1 - \mu)} \tag{3.6}$$

(see Lemma 3.4 in [2]) and $\lim_{|v| \to \infty} R(v) = 0$ is equivalent to $2(-\rho) + 1 < 0$. Solving this inequality for $\alpha$, we see that $\alpha > 1/(2-\mu)$ was required. In our estimate, $\alpha > \tilde{\alpha}_\mu$ comes from the inequality $2(-\Theta(\alpha)) + 1 < 0$ which is equivalent to $\lim_{|v| \to \infty} R(v) = 0$. 

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4 Long-range Case

In the case where $V^1 \neq 0$, the reconstruction formula is represented as follows, which also yields the proof of Theorem 2.2.

**Theorem 4.1. (Reconstruction Formula [1])** Let $\hat{v} \in \mathbb{R}^d$ be given such that $|\hat{v} \cdot e_i| < 1$. Put $v = |v|\hat{v}$. Let $\eta > 0$ be given, and $\Phi_0, \Psi_0 \in \mathcal{S}(\mathbb{R}^d)$ be such that $\mathcal{F}\Phi_0, \mathcal{F}\Psi_0 \in C_0^\infty(\mathbb{R}^d)$ with $\text{supp}\mathcal{F}\Phi_0, \text{supp}\mathcal{F}\Psi_0 \subset \{ \xi \in \mathbb{R}^d \mid |\xi| \leq \eta \}$. Put $\Phi_v = e^{ivx}\Phi_0, \Psi_v = e^{ivx}\Psi_0$. Let $V^{vs} \in \mathcal{V}^{vs}, V^s \in \mathcal{V}_{\mu,\overline{\alpha}_{\mu,D}}^s$ and $V^1 \in \mathcal{V}_{\mu,\overline{\gamma}_{\mu}}^1$, where $\overline{\alpha}_{\mu,D}$ and $\overline{\gamma}_{\mu}$ are the same as in Theorem 2.2. Then

\[
|v|(i[S_D, p_j]\Phi_v, \Psi_v) = \int_{-\infty}^{\infty}((V^{vs}(x+\hat{v}t)p_j\Phi_0, \Psi_0) - (V^{vs}(x+\hat{v}t)\Phi_0, p_j\Psi_0)
+ (i(\partial_{x_j}V^s)(x+\hat{v}t)\Phi_0, \Psi_0) + (i(\partial_{x_j}V^1)(x+\hat{v}t)\Phi_0, \Psi_0))dt + o(1)
\] (4.1)

holds as $|v| \to \infty$ for $1 \leq j \leq d$.

To prove Theorem 4.1, the following estimate is the key.

**Proposition 4.2.** Let $v$ and $\Phi_v$ be as in Theorem 4.1, $\epsilon > 0$ and $V^1 \in \mathcal{V}_{\mu,1/(2(2-\mu))}^1$. Put

\[
\Theta_D(\gamma_D) = \begin{cases} 1 & \gamma_D > 1/2 \\ 2\gamma_D(2-\mu) - 1 & 1 - \mu \leq \gamma_D \leq 1/2. \end{cases}
\] (4.2)

Then

\[
\int_{-\infty}^{\infty}||(V^1(x) - V^1(t(p-b(t))) + c(t))U_D(t)\Phi_v||dt = O(|v|^{-\Theta_D(\gamma_D)+\epsilon})
\] (4.3)

holds as $|v| \to \infty$, where $U_D(t) = U_0(t, 0)M_D(t)$ and $M_D(t)$ is the same as in (2.7).

In Adachi-Kamada-Kazuno-Toratani [2], the corresponding estimate to this proposition was

\[
\int_{-\infty}^{\infty}||(V^1(x) - V^1(t(p-b(t))) + c(t))U_D(t)\Phi_v||dt = O(|v|^{-\rho_1}),
\] (4.4)

where

\[
\rho_1 = \frac{(1 - \sigma_k)(\gamma_D + 2 - \mu)}{(1 - \mu)(\sigma_k(\gamma_D + 2 - \mu) - 1 + \gamma_D)}
\] (4.5)

with $\gamma_D = (2 - \mu)\gamma_D$ and $\sigma_k = 1 - \kappa(1 - \mu)/(2 - \mu)$ for $0 < \kappa < 1$ (see Lemma 4.5 in [2]). When we denote the error term of (4.1) by $R_D(v)$, $\lim_{|v| \to \infty} R_D(v) = 0$ is equivalent to $2(-\rho_1) + 1 < 0$. Therefore $\gamma_D > \gamma_D$ was required. In our case, $\lim_{|v| \to \infty} R_D(v) = 0$ is equivalent to $2(-\Theta_D(\gamma_D)) + 1 < 0$ and $\gamma_D > \gamma_D$ comes from this inequality.
References


