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Efficient multi-particle localization bounds in continuous random media

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Abstract

We study multi-particle interactive quantum disordered systems in the Euclidean space of arbitrary dimension and in the so-called quantum (or metric) graphs, subject to a Gaussian random potential field with continuous argument. We prove exponential and strong dynamical localization in presence of a nontrivial interaction between the particles. The obtained bounds on the decay of the eigenfunction correlators are uniform in finite or infinite volumes, so the Anderson localization phenomenon is effectively proved in finite volumes of arbitrarily large size. In earlier works, the multi-particle localization bounds were established only in the entire (infinitely extended) physical configuration space.

1 Introduction. The model and results

Until recently, the rigorous Anderson localization theory focused on single-particle models. (In the physical community, notable papers on multi-particle systems with interaction appeared as early as in 2005–2006; see [3, 17].) Initial rigorous results on multi-particle lattice localization were presented in [8–10] and [2]; continuous models have been considered in [4], [6]. Further progress was shown in [18, 19], [21].

In the present paper we consider an interactive multi-particle Anderson models in configuration spaces of two types:

(1) a Euclidean space $\mathbb{R}^d$ of arbitrary dimension $d \geq 1$;

(2) a quantum graph associated with a combinatorial graph with polynomial growth of balls; formally, we focus on the graphs over $\mathbb{Z}^d \hookrightarrow \mathbb{R}^d$, $d \geq 2$, but our technique easily extends to more general underlying graphs.

The main method used is a new modification of the multi-particle multi-scale analysis (MPMSA). More precisely, in contrast to earlier works [6, 10, 21], we use the fixed-energy multi-particle variant of the multi-scale analysis, and then infer from the fixed-energy estimates their energy-interval counterparts required for the proof of spectral and dynamical localization. This results in a simpler and more straightforward scale induction than in earlier works.

The results are summarized as follows.

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- For an N-particle system we prove uniform localization bounds valid for finite or infinite regions of the physical, single-particle configuration space. All previously published results (except for [9] where two-particle systems were considered) provided much less efficient bounds in finite-size systems.
- In our models, the external random potential is generated by a random field with continuous argument (in the Euclidean space or, respectively, on a metric (a.k.a. quantum) graph. Such model, in the single-particle setting, was studied in the works [14,15] where spectral (but not dynamical) localization was established. Generally speaking, this class of random potentials has been studied much less than the so-called alloy-type potentials with random scatterer amplitudes.
- Working with lower-unbounded random potentials, we are able to relax the positivity assumption on the interaction energy, assumed in [6,19,21].
- Compared to the works [6,19,21], we relax the assumption of Independence At Distance (IAD) and allow for strongly mixing random potentials with correlations of infinite range.

1.1 The N-particle Hamiltonian in $\mathbb{R}^d$

We consider a system of $N > 1$ distinguishable quantum particles in the Euclidean space $\mathbb{R}^d$, $d \geq 1$, with positions $x_j = (x_j^{(1)}, \ldots, x_j^{(d)}) \in \mathbb{R}^d$, so the configuration space is $(\mathbb{R}^d)^N \cong \mathbb{R}^{Nd}$. The configurations will be denoted by boldface lowercase lettres, e.g., $\mathbf{x} = (x_1, \ldots, x_N) \in (\mathbb{R}^d)^N$. It is convenient to use both in $\mathbb{R}^d$ and in $(\mathbb{R}^d)^N$ the max-norms, viz.

$$|x| = \max_{1 \leq i \leq d} |x^{(i)}|, \quad |\mathbf{x}| = \max_{1 \leq j \leq N} |x_j|.$$

Sometimes we also make use of the Euclidean distance $|\mathbf{x}|_2$.

We introduce the mappings $\Pi_{\mathcal{J}} : \mathbf{x} = (x_1, \ldots, x_N) \mapsto (x_j, j \in \mathcal{J})$, for the index subsets $\mathcal{J} \subset \{1, \ldots, N\}$, called the partial projections.

Each particle is subject to the (common) external random potential, given by a sample of a Gaussian random field (r.f.) $V : \mathbb{R}^d \times \Omega \to \mathbb{R}$, relative to some probability space $(\Omega, \mathfrak{F}, \mathbb{P})$, with a.s. continuous samples. (See Sect. 1.3 and 1.4 where the assumptions on the r.f. V are listed.)

The Hamiltonian of the N-particle system in $\mathbb{R}^d$ is assumed to have the form $\mathbf{H}^{(N)}(\omega) = -\boldsymbol{\Delta} + \mathbf{V}(\mathbf{x}; \omega) + \mathbf{U}(\mathbf{x})$, namely

$$\mathbf{H}^{(N)}(\omega) = -\boldsymbol{\Delta} + \sum_{1 \leq j \leq N} V(x_j; \omega) + \sum_{1 \leq j < k \leq N} U^{(2)}(|x_j - x_k|), \tag{1.1}$$

where $\boldsymbol{\Delta}$ is the standard Laplacian in $(\mathbb{R}^d)^N$,

$$\boldsymbol{\Delta} = \sum_{1 \leq j \leq N1} \Delta_j,$$

and the two-body interaction potential $U^{(2)}$ generating the interaction energy operator $\mathbf{U}(\mathbf{x})$ is measurable, bounded and compactly supported .

With probability 1, the random Hamiltonian $\mathbf{H}(\omega)$ is well-defined as a self-adjoint operator in the Hilbert space $\mathcal{H}^{(N)} \cong L^2(\mathbb{R}^{Nd})$, e.g., with the domain

formed by compactly supported smooth functions. This follows from the standard results on Schrödinger operators with the potential of tempered growth. This is well-known to be true[1], if for $\mathbb{P}$-a.e. $\omega \in \Omega$, the total potential energy $\mathbf{W}(\mathbf{x};\omega)$ satisfies $|\mathbf{W}(\mathbf{x};\omega)| \leq C(|\mathbf{x}|_2^2 + 1)^{1/2}$, and this (indeed, much stronger) growth bound follows from our assumptions on the Gaussian r.f. V.

Apart from the max-norm distance, it will be convenient to use also the symmetrized max-distance $\boldsymbol{\rho}_{\mathrm{S}}$, defined as follows:

$$\boldsymbol{\rho}_{\mathrm{S}}(\mathbf{x},\mathbf{y}) = \min_{\pi \in \mathfrak{S}_N} |\pi(\mathbf{x}) - \mathbf{y}|. \tag{1.2}$$

Here the symmetric group $\mathfrak{S}_N$ acts on $\mathbb{R}^N$ by permutations of the particle positions:

$$\pi(x_1, \ldots, x_N) = (x_{\pi(1)}, , \ldots, x_{\pi(N)}).$$

The reason for using the symmetrized distance is that the quantum particles are considered as distinguishable, while the Hamiltonian is permutation-symmetric, thus leaves invariant the subspace of symmetric functions. As a result, one cannot achieve efficient decay bounds on the (localized) eigenfunctions in the original, max-norm distance in $(\mathbb{R}^d)^N$. In fact, even if the interaction potential were not permutation-symmetric, the symmetry of the external potential would make it difficult to rule out "resonances" between arbitrarily distant points of the orbits of the symmetric group $\mathfrak{S}_N$ acting on $(x_1, \ldots, x_N)$ by permutations of the positions x_j.

We will work with the balls (in the max-norm), $\mathbf{B}_L^{(N)}(\mathbf{x}) \subset (\mathbb{R}^d)^N$, of radius $L > 0$, centered at $\mathbf{x} = (x_1, \ldots, x_N)$:

$$\mathbf{B}_L^{(N)}(\mathbf{x}) := \{\mathbf{y} : |\mathbf{x} - \mathbf{y}| \leq L\} = \bigtimes_{j=1}^{N} \mathrm{B}_L(x_j), \tag{1.3}$$

where $\mathrm{B}_L(x) = \{y \in \mathbb{R}^d : |y - x| < L\}$.

Our techniques apply also to a large class of quantum graphs of tempered growth, but for the sake of notational brevity, we consider in the present paper the N-particle systems in the quantum graph embedded in the Euclidean space. Specifically, we consider the usual (combinatorial) oriented countable graph with the vertex set $\Gamma^{(1)} = \mathbb{Z}^d \hookrightarrow \mathbb{R}^d$ and the edge set $\mathcal{E}^{(1)}$ formed by the pairs of nearest neighbors in $\mathbb{Z}^d$:

$$\mathbb{Z}^d \times \mathbb{Z}^d \ni e = \langle x, y\rangle \in \mathcal{E}^{(1)} \Leftrightarrow |x - y|_2 = 1$$

(recall: the graph is oriented, so the edge $\langle x, y\rangle$ is an ordered pair). Further, we associate with each edge $e = \langle x, y\rangle$:

- the initial point $\iota_e = x$,
- the terminal point $\tau_e = y$,
- the edge interval $I_e \cong (0, 1)$.

Formally speaking, I_e is merely a replica of the open interval $(0, 1)$, but the embedding $\mathbb{Z}^d \hookrightarrow \mathbb{R}^d$ allows one to identify it with the affine interval in $\mathbb{R}^d$ of the form $\{t \cdot x + (1 - t) \cdot y,\ t \in (0, 1)\}$.

[1] See e.g., [7].

A combinatorial graph equipped with the association $e \leftrightarrow I_e$ is called metric graph; the term "quantum graph" emerged in the wake of numerous researches, in theoretical and mathematical physics, of real quantum systems with the configuration space that can be efficiently reduced to a collection of interconnected thin tubular fragments. Naturally, the edge intervals need not be of unit length; for exemple, the metric graphs with random edge lengths have also been considered in the literature (cf., e.g., [20]). A complete specification of a quantum graph should include, therefore, the mapping $\mathcal{I} : e \mapsto I_e$, $e \in \mathcal{E}^{(1)}$, but since we consider only the model with the identical, unit lengths, the underlying combinatorial graph structure $(\Gamma^{(1)}, \mathcal{E}^{(1)})$ is sufficient, and we keep this shorter (albeit slightly abusive) notation referring to a quantum graph, and omit $\mathcal{I}$.

The Hilbert space of single-particle quantum states in the quantum graph $(\Gamma^{(1)}, \mathcal{E}^{(1)})$ is the direct sum

$$\mathcal{H}^{(1)} = \bigoplus_{e \in \mathcal{E}^{(1)}} L^2(I_e) \cong \bigoplus_{e \in \mathcal{E}^{(1)}} L^2(0,1).$$

It is convenient to define the Laplace operator Δ on the quantum graph at hand via the bilinear form

$$(\phi, \psi) \mapsto - \sum_{e \in \mathcal{E}^{(1)}} (\phi', \psi')_{L^2(I_e)} = - \sum_{e \in \mathcal{E}^{(1)}} \int_{I_e} \overline{\phi'(x)}\, \psi'(x)\, dx,$$

well-defined, e.g., for the smooth, compactly supported functions ϕ, ψ. Given a measurable, locally bounded function $V : \sqcup_{e \in \mathcal{E}^{(1)}} I_e \to \mathbb{R}$, referred to as the external potential, the Schrödinger operator $H = -\Delta + V$ on the quantum graph, with the potential V, is also defined via its bilinear form:

$$\begin{aligned} \mathfrak{h}[\phi, \psi] = (\phi, \psi) &\mapsto - \sum_{e \in \mathcal{E}^{(1)}} \left((\phi', \psi')_{L^2(I_e)} + (\phi, V\psi)_{L^2(I_e)} \right) \\ &= - \sum_{e \in \mathcal{E}^{(1)}} \int_{I_e} \left(\overline{\phi'(x)}\, \psi'(x) + V(x) \overline{\phi(x)}\, \psi(x) \right) dx, \end{aligned}$$

for the smooth, compactly supported functions ϕ, ψ. To obtain a self-adjoint extension of H, one usually considers the Kirchhoff "boundary" (or rather, contact) conditions. First, it is convenient to compactify the edge intervals I_e and consider the closed intervals $\bar{I}_e \cong [0,1]$. Next, given a test function ϕ, assume that it extends to a continuous function on the union of $\bar{I}_e$. The embedding $\mathbb{Z}^d \hookrightarrow \mathbb{R}^d$ makes these abstract construction more straightforward: the collection of the unit edges between the vertices of $\mathbb{Z}^d$ is endowed with the metric induced by the Euclidean distance in $\mathbb{R}^d$, so one can simply identify $\cup_{e \in \mathcal{E}^{(1)}} \bar{I}_e$ with this subset of $\mathbb{R}^d$, which has the structure of a (singular) Riemannian manifold of dimension 1; the singularity comes of course from the vertex points which do not have a neighborhood homeomorphic to an open interval. Making use of the identification $\bar{I}_e \leftrightarrow [0,1]$, we will write $\phi(0_e)$ and $\phi(1_e)$ meaning the limiting value at the left and, respectively, right endpoint of the interval $[0,1]$.

The Kirchhoff conditions require that for each vertex point v of the quantum graph, the test function ϕ be continuous at v, and that

$$\sum_{e:\, \iota_e = v} \phi'(0_e) - \sum_{e:\, \tau_e = v} \phi'(1_e) = 0.$$

It can be shown that H can be defined as a self-adjoint Schrödinger operator with Kirchhoff conditions, for slowly growing potentials V. We work with Gaussian random fields on the quantum graphs embedded in the Euclidean space, and it follows from general results going back to the pioneering work by Fernique [13] that $\mathbb{P}$-a.e. sample $V(x;\omega)$ of a continuous Gaussian field $V(\cdot;\omega)$ on $\Gamma^{(1)}$ grows not faster than $c_1(\omega)\ln^{c_2}(|x|+1)$.

The notion of quantum graph and the key constructions used in the localization theory on quantum graphs have been developed by Sabri in his recent PhD project. Following [21], we will the N-particle analog of the quantum graph the N-graph.

Specifically, the configuration space of an N-particle system on a conventional (single-particle) quantum graph $(\Gamma^{(1)},\mathcal{E}^{(1)})$, with identical, unit lengths of the edges, is the collection of open N-dimensional unit cubes $\boldsymbol{\sigma}_{\mathbf{e}} = \boldsymbol{\sigma}_{\mathbf{e}}^{(N)} \cong (0,1)^N$, labeled by the N-tuples $\mathbf{e} = (e_1,\dots,e_n) \in \left(\mathcal{E}^{(1)}\right)^N$. The kinetic energy operator of the N-particle system is the Laplace operator on the union of the N-edges, and the potential energy operator (including the external random potential and the interaction potential) is the operator of multiplication by the potential energy function. It is shown in [21] that the N-particle Hamiltonian is well-defined for interactions of finite range and bounded external potentials. This result can be easily extended to the external random potentials of slow growth, including the Gaussian random fields growing at logarithmic rate. For further details, we refer the reader to the paper [21].

1.2 Some basic facts on Gaussian fields

For brevity, we consider in this subsection only the random fields in a Euclidean space. Working with Gaussian r.f. on a metric graph requires a more general framework of random fields on metric spaces (X,ρ) endowed with the structure of a measure space $(X,\mathfrak{B},\mathrm{mes})$, where $\mathfrak{B}$ is generated by the Borel sets in X (cf., e.g., [1]), but such a generalization is quite straightforward.

A convenient way to define a Gaussian r.f. with regular samples in $\mathbb{R}^d$ is as follows. Fix a function $\mathrm{C} : \mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}$, which, in general, has to be measurable and positive definite: for any measurable function $\phi : \mathbb{R}^d \to \mathbb{C}$

$$\langle\phi,\phi\rangle_{\mathrm{C}} := \int_{\mathbb{R}^d\times\mathbb{R}^d} \mathrm{C}(x,y)\,\overline{\phi(x)}\phi(y)\,dx\,dy \geq 0,$$

and $\langle\phi,\phi\rangle_{\mathrm{C}} > 0$ unless $\phi = 0$ a.e.

We assume that C is continuous (for we need the samples of the r.f. to be continuous).

Then there exists a Gaussian r.f. V in $\mathbb{R}^d$ with zero mean and the covariance function $\mathbb{E}\,[\,V(x;\omega)V(y;\omega)\,] = \mathrm{C}(x,y)$. Its local regularity properties, e.g., a.s. local boundedness and a.s. continuity, can be studied with the help of its restrictions to bounded Borel sets (e.g., cubes) $Q \subset \mathbb{R}^d$.

For the Gaussian r.f. in a cube $Q \subset \mathbb{R}^d$, the a.s. local boundedness and a.s. continuity can be inferred from the continuity of the covariance function; in fact, this result can be extended[2] to a large class of Gaussian r.f. in metric space Q satisfying some entropy condition (cf. [1, Condition (iii) in Theorem

[2]See, e.g., the original papers by Fernique [13], Talagrand [23] and the discussion in the monograph by Adler [1], where an extensive bibliography can be found.

4.16]). For brevity, we refer to it as the entropy-regularity, but do not introduce it formally, for it would take one on a rather long technical detour.

In particular, for stationary Gaussian r.f. on a compact group, and for a restriction of a stationary G.r.f. to a compact subset of a non-compact group Q, both the a.s. boundedness and the a.s. continuity of the samples are equivalent to the entropy-regularity condition on Q (cf. [1, Theorem 4.16]).

Next, given a bounded Borel set $Q \subset \mathbb{R}^d$, introduce the Hilbert space $\mathcal{L}^2_{\mathrm{C}}(Q)$ formed by measurable functions ϕ with support in Q and finite value of $\langle \phi, \phi \rangle_{\mathrm{C}}$, with the scalar product

$$\langle \phi, \psi \rangle = \int_{Q \times Q} \mathrm{C}(x, y)\, \overline{\phi(x)} \psi(y)\, dx\, dy$$

and the norm denoted by $\| \cdot \|_{\mathrm{C}}$.

Given an orthonormal basis $\{\eta_i, i = 1, 2, \ldots\}$ in $\mathcal{L}^2_{\mathrm{C}}(Q)$, one can expand V in Q in a norm-convergent series

$$V(x; \omega) = \sum_{i \geq 1} c_i(\omega) \eta_i(x), \quad x \in Q, \tag{1.4}$$

with uncorrelated (hence, independent) Gaussian r.v. $c_i = c_i^{(Q)}$. Our crucial eigenvalue concentration (EVC) bound, resulting in the new, efficient decay bounds on the EF correlators, is based on the decomposition

$$V(x; \omega) = \xi_Q(\omega) \mathbf{1}_Q(x) + \widetilde{V}(x; \omega), \tag{1.5}$$

which is obtained as a particular case of the expansion (1.4) with $\eta_1(x) = \mathrm{Const}$ (note that a proper normalization is required for η_1).

1.3 Assumptions for the Hamiltonian in $\mathbb{R}^d$

(V1) The centered Gaussian r.f. $V : \mathbb{R}^d \times \Omega \to \mathbb{R}$ generating the external potential is translation invariant, has variance $\sigma^2 > 0$ and a.s. continuous samples. It satisfies the strong mixing condition with the Rosenblatt coefficient $\alpha(r) \leq C\mathrm{e}^{-r^\delta}$, $\delta > 0$ (this includes the case where V has finite-range correlations). There exists $C' > 0$ such that

$$\mathbb{E}\left[\left(\int_{\{|x| \leq L\}} V(x; \omega)\, dx \right)^2 \right] \geq C' L^d, \quad L \geq 1. \tag{1.6}$$

Remark 1.1. *The role of the condition* (1.6) *is to ensure the growth of the variance of the integral of V over large balls, resulting in a controllable, satisfactory growth of the normalization constant in the r.v. ξ_Q figuring in the key decomposition* (1.5).

(U1) The interaction energy,

$$\mathbf{U}(\mathbf{x}) = \mathbf{U}(x_1, \ldots, x_N) = \sum_{1 \leq i < j \leq N} U^{(2)}(|x_i - x_j|),$$

is generated by a two-body interaction potential $U^{(2)} : \mathbb{R}_+ \to \mathbb{R}$ which is bounded and compactly supported:

$$\exists\, r_0 \in (0, +\infty) \ \ \forall r \geq r_0 \quad U^{(2)}(r) = 0.$$

1.4 Assumptions for the Hamiltonian on a quantum graph

(V2) The centered Gaussian r.f. $V : \Gamma^{(N)} \times \Omega \to \mathbb{R}$ is $\mathbb{Z}^d$-translation invariant, has uniformly bounded variance and a.s. continuous samples. It satisfies the strong mixing condition with the Rosenblatt coefficient $\alpha(r) \leq Ce^{-r^\delta}$, $\delta > 0$. There exists $C' > 0$ such that

$$\mathbb{E}\left[\left(\int_{\{|x|\leq L\}} V(x;\omega)\, d\mu(x)\right)^2\right] \geq C' L^d, \quad L \geq 1. \tag{1.7}$$

Here $d\mu(x)$ is the measure on the underlying quantum graph $\Gamma^{(1)}$, induced by the Lebesgue measures on the edge intervals I_e, $e \in \mathcal{E}^{(1)}$.

(U2) The interaction energy is generated by a two-body interaction potential $U^{(2)} : \mathbb{R}_+ \to \mathbb{R}$ which is bounded and compactly supported:

$$\exists r_0 \in (0, +\infty) \ \forall r \geq r_0 \quad U^{(2)}(r) = 0.$$

2 Main results

2.1 Localization in a Euclidean space

Theorem 1. *Let be given integers $d \geq 1$, $N^* \geq 2$, a stationary Gaussian random field with continuous argument, $V : \mathbb{R}^d \times \Omega \to \mathbb{R}$, satisfying the assumptions (V1), and an interaction potential $U^{(2)}$ satisfying the assumptions (U1). Consider the N-particle random Anderson Hamiltonian $\mathbf{H}^{(N)}(\omega) = -\boldsymbol{\Delta} + \mathbf{V}(\mathbf{x};\omega) + \mathbf{U}(\mathbf{x})$ of the form (1.1).*

Then for all $1 \leq N \leq N^*$, for any $m > 0$ and $\nu > 0$ there exists $E^* > -\infty$ such that, for $\mathbb{P}$-a.e. ω,

(A) *$\mathbf{H}^{(N)}(\omega)$ has pure point spectrum in $(-\infty, E^*]$;*

(B) *all eigenfunctions $\psi_j(\omega)$ of $\mathbf{H}^{(N)}(\omega)$ with eigenvalues in $(-\infty, E^*]$ decay exponentially:*

$$|\psi_j(\mathbf{x},\omega)| \leq C_j(\omega) e^{-m|\mathbf{x}|},$$

for some $C_j(\omega) \in (0, \infty)$;

(C) *$\mathbf{H}^{(N)}(\omega)$ features strong dynamical localization in $(-\infty, E^*]$: there is a non-random number $\kappa > 0$ such that*

$$\forall \mathbf{x}, \mathbf{y} \in (\mathbb{R}^d)^N \quad \Upsilon_{\mathbf{x},\mathbf{y}} := \mathbb{E}\left[\sup_{t\in\mathbb{R}} \langle \mathbf{1}_\mathbf{x} \,|\, P_{(-\infty,E^*]}(\mathbf{H}^{(N)})\, e^{-it\mathbf{H}^{(N)}} \,|\, \mathbf{1}_\mathbf{y}\rangle\right] \leq e^{-\nu\left(\rho_S(\mathbf{x},\mathbf{y})\right)^\kappa}.$$

Here $P_{(-\infty,E^*]}(\mathbf{H}^{(N)})$ is the spectral projection for the operator $\mathbf{H}^{(N)}$ on the interval $(-\infty, E^*]$.

In contrast to the works [6,19], we do not assume the non-negativity of the interaction potential. The latter condition has been used (cf. [6,18,19]) in the proofs of multi-particle Anderson localization at "extreme" energies, viz. in the Lifshitz tails zone. In fact, in absence of the strong disorder condition, the

positivity of the interaction is a convenient (and so far, the only) tool allowing one to establish the initial-scale MSA bounds for multi-particle systems, reducing them to their single-particle counterparts. The specificity of the Gaussian (and, more generally, lower-unbounded) potentials is that the localization mechanism at sufficiently low, negative energies is quite similar to that responsible for the onset of localization under strong disorder, regardless of how small is the (nonzero) amplitude of the random potential.

But a more important improvement comes from the fact that the decay of the EF correlators is now proved with respect to the (symmetrized) norm-distance, and not in the Hausdorff distance, used explicitly in [2] (in the lattice systems) and in [18, 19] (in lattice and continuous systems). Implicitly, the decay estimates related to the Hausdorff distance appeared also in our earlier papers [6, 10]. While the probabilistic decay estimates for the eigenfunctions and their correlators ultimately give rise to spectral and dynamical localization in an infinitely extended configuration space, they do not imply directly reasonable localization bounds in arbitrarily large, but bounded regions of the (single-particle) configuration space. This makes them much less suitable for the applications to the physical models, e.g., in the solid state physics, where a disordered sample has always a finite size.

Remark 2.1. Theorem 1 establishes strong dynamical localization for $N \leq N^*$, including the case where $N = 1$. Earlier, Fischer et al. [15] proved spectral localization for the 1-particle Anderson Hamiltonian in $\mathbb{R}^d$ with Gaussian random potential (of continuous argument, as in our model), under a slightly more restrictive hypothesis on the covariance function C. They also used the fixed-energy MSA induction. The derivation of spectral localization was obtained, as in the paper by von Dreifus and Klein [12] (devoted to the 1-particle Anderson lattice model with Gaussian potential), with the help of the Simon–Wolf criterion [22]. Unfortunately, the latter deep, remarkable result

- does not lead directly to the energy-interval localization bounds, sufficient for the proof of the dynamical localization, and
- has not been extended so far to the multi-particle systems.

Note also that our technique allows to infer some energy-interval estimates from the fixed-energy analysis performed in [15], thus proving strong dynamical localization under the hypotheses of [15], but the resulting decay rate of the eigenfunction correlators would be significantly slower than in Theorem 1.

2.2 Localization in a quantum graph

Theorem 2. *Let be given integers $d \geq 1$, $N^* \geq 2$. Consider the N-graph $\mathbf{\Gamma}^{(N)}$ over the single-particle quantum graph $\Gamma^{(1)}$, a $\mathbb{Z}^d$-stationary Gaussian random field $V : \Gamma^{(1)} \times \Omega \to \mathbb{R}$, satisfying the assumptions (V2), an interaction potential $U^{(2)}$ satisfying the assumptions (U2), and the N-particle random Anderson Hamiltonian $\mathbf{H}^{(N)}(\omega) = -\mathbf{\Delta} + \mathbf{V}(\mathbf{x};\omega) + \mathbf{U}(\mathbf{x})$.*

Then for all $1 \leq N \leq N^*$, for any $m > 0$ and $\nu > 0$, there exists $E^* > -\infty$ such that, for $\mathbb{P}$-a.e. ω,

(A) $\mathbf{H}^{(N)}(\omega)$ *has pure point spectrum in* $(-\infty, E^*]$;

(B) *all eigenfunctions* $\psi_j(\omega)$ *of* $\mathbf{H}^{(N)}(\omega)$ *with eigenvalues in* $(-\infty, E^*]$ *decay exponentially:*

$$|\psi_j(\mathbf{x},\omega)| \le C_j(\omega)\mathrm{e}^{-m|\mathbf{x}|},$$

for some $C_j(\omega) \in (0,\infty)$*;*

(C) $\mathbf{H}^{(N)}(\omega)$ *features strong dynamical localization in* $(-\infty, E^*]$*: there is a non-random number* $\kappa > 0$ *such that*

$$\forall \mathbf{x},\mathbf{y} \in \mathbf{\Gamma}^{(N)} \quad \Upsilon_{\mathbf{x},\mathbf{y}} := \mathbb{E}\left[\sup_{t\in\mathbb{R}} \langle \mathbf{1}_{\mathbf{x}} \,|\, P_{(-\infty,E^*]}\big(\mathbf{H}^{(N)}\big)\, \mathrm{e}^{-\mathrm{i}t\mathbf{H}^{(N)}} \,|\, \mathbf{1}_{\mathbf{y}}\rangle\right] \le \mathrm{e}^{-\nu\left(\rho_{\mathrm{S}}(\mathbf{x},\mathbf{y})\right)^{\kappa}}.$$

Here, comparing our results with those obtained in the work by Sabri [21], we make similar remarks.

First of all, our decay bounds are established with respect to the (symmetrized) norm distance. Secondly, the external potential does not have the discrete, alloy-type structure, but is generated by a random field with continuous argument. Finally, we do not assume the non-negativity of the interaction potential.

3 The main strategy of the proofs

Although the geometric nature of the configuration space is quite different for the multi-particle systems in a Euclidean space and in a quantum graph, the main distinction of the techniques used in our work resides in a fairly general probabilistic argument, for which the Gaussian random potentials represent the simplest case.

3.1 Efficient multi-particle EVC bounds

Definition 3.1. *An* N*-particle ball* $\mathbf{B}_L^{(N)}(\mathbf{x})$ *in the max-norm distance is called weakly separated from* $\mathbf{B}_L^{(N)}(\mathbf{y})$ *iff there exists a single-particle ball* Q *of diameter* $\le 2NL$ *and index subsets* $\mathcal{J}_1, \mathcal{J}_2 \subset \{1,\dots,N\}$ *such that* $|\mathcal{J}_1| > |\mathcal{J}_2|$ *(possibly* $\mathcal{J}_2 = \varnothing$*) and*

$$\begin{aligned} \Pi_{\mathcal{J}_1}\mathbf{B}_L^{(N)}(\mathbf{x}) \cup \Pi_{\mathcal{J}_2}\mathbf{B}_L^{(N)}(\mathbf{y}) &\subset Q, \\ \Pi_{\mathcal{J}_1^{\mathrm{c}}}\mathbf{B}_L^{(N)}(\mathbf{x}) \cup \Pi_{\mathcal{J}_2^{\mathrm{c}}}\mathbf{B}_L^{(N)}(\mathbf{y}) &\cap Q = \varnothing. \end{aligned} \tag{3.1}$$

A pair of balls $\mathbf{B}_L^{(N)}(\mathbf{x})$, $\mathbf{B}_L^{(N)}(\mathbf{y})$ *is called weakly separated iff at least one of these balls is weakly separated from the other.*

Lemma 3 (Cf. [5, Lemma 3.7])**.** *Any pair of balls* $\mathbf{B}_L^{(N)}(\mathbf{x})$, $\mathbf{B}_L^{(N)}(\mathbf{y})$ *with* $\rho_{\mathrm{S}}(\mathbf{x},\mathbf{y}) > 3NL$ *is weakly separated.*

We will call $3NL$-distant the balls $\mathbf{B}_L^{(N)}(\mathbf{x})$, $\mathbf{B}_L^{(N)}(\mathbf{y})$ with $\rho_{\mathrm{S}}(\mathbf{x},\mathbf{y}) > 3NL$.

For brevity, we formulate below one statement, which applies to the model in the Euclidean space and in the quantum graph. In both cases, the main conditions ((U1) & (V1) or, respectively, (V2) & (U2)) upon the potentials V and $U^{(2)}$ are implicitly assumed.

Theorem 4. *Let* $\mathbf{B}_L^{(N)}(\mathbf{x})$ *and* $\mathbf{B}_L^{(N)}(\mathbf{y})$ *be two weakly separated balls. Denote by* $\Sigma_{\mathbf{x},L}(\omega)$ *and* $\Sigma_{\mathbf{y},L}(\omega)$ *the spectra of the respective Hamiltonians* $\mathbf{H}_{\mathbf{B}_L(\mathbf{x})}(\omega)$, $\mathbf{H}_{\mathbf{B}_L(\mathbf{y})}(\omega)$. *Fix* $\beta \in (0,1)$.

Then for all $L > 0$ *large enough, the following bound holds true:*

$$\mathbb{P}\left\{ \operatorname{dist}\left(\Sigma_{\mathbf{x},L}(\omega), \Sigma_{\mathbf{y},L}(\omega)\right) \le \mathrm{e}^{-L^\beta} \right\} \le \mathrm{e}^{-L^{\beta/2}}. \tag{3.2}$$

The estimate (3.3) is a fairly straightforward adaptation to Gaussian r.f. with continuous argument of an EVC bound given in [5, Lemma 3.8] in the case of lattice Hamiltonians.

On account of Lemma 3, we infer from Lemma 4 the key EVC estimate (cf. [5, Theorem 2.1]).

Theorem 5. *Let* $\mathbf{B}_L^{(N)}(\mathbf{x})$ *and* $\mathbf{B}_L^{(N)}(\mathbf{y})$ *be two* $3NL$*-distant cubes, and* $\Sigma_{\mathbf{x},L}(\omega)$, $\Sigma_{\mathbf{y},L}(\omega)$ *the spectra of the respective Hamiltonians* $\mathbf{H}_{\mathbf{B}_L(\mathbf{x})}(\omega)$, $\mathbf{H}_{\mathbf{B}_L(\mathbf{y})}(\omega)$.

Then for all $L > 0$ *large enough, the following bounds holds true:*

$$\mathbb{P}\left\{ \operatorname{dist}\left(\Sigma_{\mathbf{x},L}(\omega), \Sigma_{\mathbf{y},L}(\omega)\right) \le \mathrm{e}^{-L^\beta} \right\} \le \mathrm{e}^{-L^{\beta/2}}. \tag{3.3}$$

The main tool for the proof of the above Wegner-type estimate is the decomposition of a Gaussian r.f. in a subset Q of the underlying configuration space of finite measure:

$$V(x;\omega)\mathbf{1}_Q(x) = \xi_Q(\omega)\mathbf{1}_Q(x) + \widetilde{V}_Q(x;\omega), \tag{3.4}$$

where the random variable ξ_Q is independent of the "fluctuation" field $\widetilde{V}(\cdot;\omega)$.

Such a decomposition can be obtained by expanding the restriction $V_Q := V \restriction Q$ in an orthogonal series $V_Q = \sum_{n\ge 0} c_n(\omega)\eta_n(x)$ over an orthonormal basis $\{\eta_n(\cdot)\}$ in the Hilbert space $\mathcal{L}^2_{\mathbb{C}}(Q)$, with $\eta_0 = const\mathbf{1}_Q$.

Therefore, conditional on $\widetilde{V}_Q$, the sample mean ξ_Q admits (Gaussian) bounded probability density. Given two $3NL$-distant cubes, such a decomposition, with a proper choice of the subset Q depending upon $\mathbf{x}, \mathbf{y}$ and L, gives rise to a simple representation of eigenvalues of the operators $\mathbf{H}_{\mathbf{B}_L(\mathbf{x})}(\omega)$, $\mathbf{H}_{\mathbf{B}_L(\mathbf{y})}(\omega)$, and ultimately to the upper bound (3.3).

3.2 The scaling scheme

The exponential decay of the EFs with eigenvalues in the localization energy band $(-\infty, E^*]$, with sufficiently large negative E^*, can be proved with the help of the MPMSA procedure simpler that the one used in [6,10], with scale lengths defined by the recursive equation $L_{k+1} = [L_k^\alpha]$, $1 < \alpha < 2$.

More precisely, instead of the variable-energy scaling analysis with $L_{k+1} = [L_k^\alpha]$, used in [6, 10] and essentially going back to its single-particle variant developed by von Dreifus and Klein [11], we carry our first the fixed-energy MSA induction with a multiplicative growth of the scale lengths: $L_{k+1} = YL_k$, $\mathbb{N} \ni Y \ge 2$. The advantages of the multiplicative length scale sequence have been demonstrated by Germinet and Klein [16]; in particular, it leads easily to a sub-exponential decay of the EF correlators.

On the other hand, a second MSA induction is still required, with exponential growth of the scales, $L_{k+1} = [L_k^\alpha]$, to prove exponential decay of the localized

eigenfunctions, for the multiplicative scheme proves only the sub-exponential decay of EFs.

Note that Klein and Nguyen [18, 19], adapting to the multi-particle systems the bootstrap MSA developed by Germinet and Klein [16] for the single-particle models, proved both exponential decay of eigenfunctions and sub-exponential decay of EF correlators in the localization energy zone, in lattice and continuous systems. As usual in the bootstrap MSA, one starts with fairly weak assumptions on the localization properties in a finite cube of size L_0, and then infers from them much stronger properties at arbitrary large scales L_k. This requires the total of four interconnected multi-scale analyses. However, the EVC bounds, and as a result, the decay bounds are obtained in [18, 19] with respect to the Hausdorff and not the norm distance.

It is worth noticing that Klein and Nguyen addressed in [19] a hard analytical problem related to the lack of the so-called complete covering condition for the alloy-type external random potential (considered in their work); such a condition was assumed in [6], and this made substantially simpler the crucial EVC bound. Klein and Nguyen adapted to the multi-particle setting the quantitative unique continuation principle, thus extending the MPMSA technique to a much larger class of alloy potentials than in [6]. In the present work, we simply do not encounter such a functional-analytical problem. Pictorially, the Gaussian r.f. with continuous argument is "omnipresent" in the configuration space, while an alloy-type potential may affect only a subset thereof. The representation (3.4) provides a very clear formalization of this informal argument.

In our work, we do not aim at the sharpest probabilistic decay estimates but focus on more efficient decay bounds in terms of the (symmetrized) norm-distance. This allows us to make use of only two multi-scale analyses – one with the multiplicative scales $L_{k+1} = YL_k$ and another one with the faster growing scales $L_{k+1} = [L_k^\alpha]$.

The main EVC bound provided by Theorem 5 and valid for all $3NL$-distant cubes (in the symmetrized norm-distance) has another advantage: it simplifies the multi-particle scale induction, which becomes much closer to its single-particle counterpart than in [6, 10, 21] and in [18, 19].

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