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Spectral properties of a quantum waveguide with Neumann window

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Abstract

In this document we review some results dealing with the study of the spectral properties of quantum waveguide. We consider a quantum waveguide with Neumann window. We present the effect of such a window on the spectrum of the free Laplacian. Then we study the behavior of the discrete spectrum on the presence of a magnetic filed.
We end by presenting a work in progress with P. Briet in which we consider quantum waveguide with random Neumann windows [9].

1 Introduction

The task of finding eigenenergies E_n and corresponding eigenfunctions $f_n(\mathbf{r})$, $n = 1, 2, ...$ of the Laplacian in the two- (2D) and three-dimensional (3D) domain Ω with mixed Dirichlet

$$f_n(\mathbf{r})|_{\partial\Omega_D} = 0 \tag{1.1}$$

and Neumann

$$\mathbf{n}\nabla f_n(\mathbf{r})|_{\partial\Omega_N} = 0, \tag{1.2}$$

boundary conditions on its confining surface (for 3D) or line (for 2D) $\partial\Omega = \partial\Omega_D \cup \partial\Omega_N$ ($\mathbf{n}$ is a unit normal vector to $\partial\Omega$) is commonly referred to as Zaremba problem [37], it is a known mathematical problem science. Apart from the purely mathematical interest, an analysis of such solutions is of a large practical significance as they describe miscellaneous physical systems.

The study of quantum waves on quantum waveguide has gained much interest and has been intensively studied during the last years for their important physical consequences. The main reason is that they represent an interesting physical effect with important applications in nanophysical devices, but also in flat electromagnetic waveguide. See the monograph [18] and the references therein.

Exner et al. have done seminal works in this field. They obtained results in different contexts, we quote

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[6, 12, 16, 17]. Also in [19, 21, 28] research has been conducted in this area; the first is about the discrete case and the two others for deals with the random quantum waveguide.

It should be noticed that the spectral properties essentially depends on the geometry of the waveguide, in particular, the existence of a bound states induced by curvature [10, 12, 14, 16] or by coupling of straight waveguides through windows [16, 18] were shown. The waveguide with Neumann boundary condition were also investigated in several papers [23, 27]. A possible next generalization are waveguides with combined Dirichlet and Neumann boundary conditions on different parts of the boundary. The presence of different boundary conditions also gives rise to nontrivial properties like the existence of bound states.

2 The model

The system we are going to study is given in Fig 1. We consider a Schrödinger particle whose motion is confined to a pair of parallel plans of width d. For simplicity, we assume that they are placed at $z = 0$ and $z = d$. We shall denote this configuration space by Ω

$$\Omega = \mathbb{R}^2 \times [0, d].$$

Let $\gamma(a)$ be a disc of radius a, without loss of generality we assume that the center of $\gamma(a)$ is the point $(0, 0, 0)$;

$$\gamma(a) = \{(x, y, 0) \in \mathbb{R}^3;\ x^2 + y^2 \leq a^2\}. \tag{2.3}$$

We set $\Gamma = \partial\Omega \diagdown \gamma(a)$. We consider Dirichlet boundary condition on Γ and Neumann boundary condition in $\gamma(a)$.

2.1 The Hamiltonian

Let us define the self-adjoint operator on $L^2(\Omega)$ corresponding to the particle Hamiltonian H. This is will be done by the mean of quadratic forms. Precisely, let q_0 be the quadratic form

$$q_0(f, g) = \int_\Omega \nabla f \cdot \overline{\nabla g} d^3x, \text{ with domain } \mathcal{Q}(q_0) = \{f \in H^1(\Omega);\ f\lceil\Gamma = 0\}, \tag{2.4}$$

where $H^1(\Omega) = \{f \in L^2(\Omega) | \nabla f \in L^2(\Omega)\}$ is the standard Sobolev space and we denote by $f\lceil\Gamma$, the trace of the function f on Γ. It follows that q_0 is a densely defined, symmetric, positive and closed quadratic form. We denote the unique self-adjoint operator associated to q_0 by H and its domain by $D(\Omega)$. It is the hamiltonian describing our system. From [33] (page 276), we infer that the domain $D(\Omega)$ of H is

$$D(\Omega) = \left\{f \in H^1(\Omega);\ -\Delta f \in L^2(\Omega), f\lceil\Gamma = 0, \frac{\partial f}{\partial z}\lceil\gamma(a) = 0\right\}$$

and

$$Hf = -\Delta f, \ \ \forall f \in D(\Omega).$$

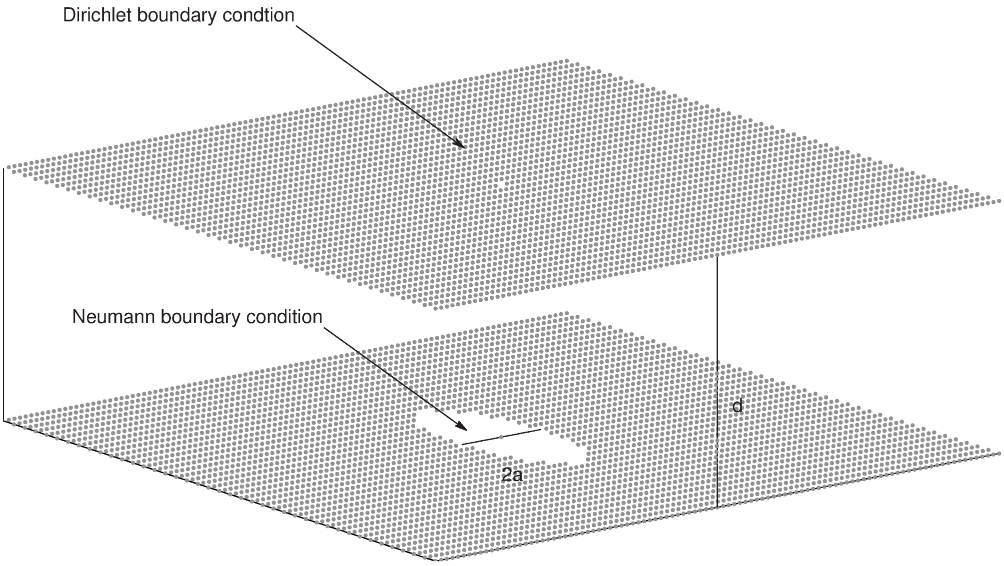


Figure 1: The waveguide with a disc window and two different boundaries conditions

2.2 Some known facts

Let us start this subsection by recalling that in the particular case when $a = 0$, we get H^0, the Dirichlet Laplacian, and $a = +\infty$ we get H^∞, the Dirichlet-Neumann Laplacian. Since

$$H = (-\Delta_{\mathbb{R}^2}) \otimes I \oplus I \otimes (-\Delta_{[0,d]}), \text{on } L^2(\mathbb{R}^2) \otimes L^2([0,d]),$$

(see [33]) we get that the spectrum of H^0 is $[(\frac{\pi}{2d})^2, +\infty[$. Consequently, we have

$$\Big[(\frac{\pi}{d})^2, +\infty\Big[\subset \sigma(H) \subset \Big[(\frac{\pi}{2d})^2, +\infty\Big[.$$

Using the property that the essential spectra is preserved under compact perturbation, we deduce that the essential spectrum of H is

$$\sigma_{ess}(H) = \Big[(\frac{\pi}{d})^2, +\infty\Big].$$

An immediate consequence is the discrete spectrum lies in $\Big[(\frac{\pi}{2d})^2, (\frac{\pi}{d})^2\Big]$.

2.3 Preliminary: Cylindrical coordinates

Let us notice that the system has a cylindrical symmetry, therefore, it is natural to consider the cylindrical coordinates system (r, θ, z). Indeed, we have that

$$L^2(\Omega, dxdydz) = L^2(]0, +\infty[\times[0, 2\pi[\times[0, d], rdrd\theta dz),$$

We note by $\cdot\langle\cdot,\rangle_r$, the scaler product in $L^2(\Omega, dxdydz) = L^2(]0, +\infty[\times[0, 2\pi[\times[0, d], rdrd\theta dz)$ given by

$$\langle f, g\rangle_r = \int_{]0,+\infty[\times[0,2\pi[\times[0,d]} fgrdrd\theta dz.$$

We denote the gradient in cylindrical coordinates by ∇_r. While the Laplacian operator in cylindrical coordinates is given by

$$\Delta_{r,\theta,z} = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{d^2}{dz^2}. \tag{2.5}$$

Therefore, the eigenvalue equation is given by

$$-\Delta_{r,\theta,z} f(r, \theta, z) = Ef(r, \theta, z). \tag{2.6}$$

Since the operator is positive, we set $E = k^2$. The equation (2.6) is solved by separating variables and considering $f(r, \theta, z) = \varphi(r) \cdot \psi(\theta)\chi(z)$. Plugging the last expression in equation (2.6) and first separate χ by putting all the z dependence in one term so that $\frac{\chi''}{\chi}$ can only be constant. The constant is taken as $-s^2$ for convenience. Second, we separate the term $\frac{\psi''}{\psi}$ which has all the θ dependance. Using the fact that the problem has an axial symmetry and the solution has to be 2π periodic and single value in θ, we obtain $\frac{\psi''}{\psi}$ should be a constant $-n^2$ for $n \in \mathbb{Z}$. Finally, we get the following equation for φ

$$\varphi''(r) + \frac{1}{r}\varphi'(r) + [k^2 - s^2 - \frac{n^2}{r^2}]\varphi(r) = 0. \tag{2.7}$$

We notice that the equation (2.7), is the Bessel equation and its solutions could be expressed in terms of Bessel functions. More explicit solutions could be given by considering boundary conditions.

3 Results on discrete spectrum

3.1 One Neumann Window

The first result we give is the following Theorem.

Theorem 3.1 *[29] The operator H has at least one isolated eigenvalue in $\left[(\frac{\pi}{2d})^2, (\frac{\pi}{d})^2\right]$ for any $a > 0$.*

Moreover for a big enough, if $\lambda(a)$ is an eigenvalue of H less then $\frac{\pi^2}{d^2}$, then we have.

$$\lambda(a) = \left(\frac{\pi}{2d}\right)^2 + o\left(\frac{1}{a^2}\right). \tag{3.8}$$

Proof. Let us start by proving the first claim of the Theorem. To do so, we define the quadratic form $\mathcal{Q}_0$,

$$\mathcal{Q}_0(f,g) = \langle \nabla f, \nabla g\rangle_r = \int_{]0,+\infty[\times[0,2\pi[\times[0,d]} (\partial_r f\overline{\partial_r g} + \frac{1}{r^2}\partial_\theta f\overline{\partial_\theta g} + \partial_z f\overline{\partial_z g}) r dr d\theta dz, \tag{3.9}$$

with domain

$$\mathcal{D}_0(\Omega) = \left\{f \in L^2(\Omega, rdrd\theta dz); \nabla_r f \in L^2(\Omega, rdrd\theta dz); f\lceil\Gamma = 0\right\}.$$

Consider the functional q defined by

$$q[\Phi] = \mathcal{Q}_0[\Phi] - (\frac{\pi}{d})^2\|\Phi\|^2_{L^2(\Omega,rdrd\theta dz)}. \tag{3.10}$$

Since the essential spectrum of H starts at $(\frac{\pi}{d})^2$, if we construct a trial function $\Phi \in \mathcal{D}_0(\Omega)$ such that $q[\Phi]$ has a negative value then the task is achieved. Using the quadratic form domain, Φ must be continuous inside Ω but not necessarily smooth. Let χ be the first transverse mode, i.e.

$$\chi(z) = \begin{cases} \sqrt{\frac{2}{d}}\sin(\frac{\pi}{d}z) & \text{if} \quad z \in (0,d) \\ 0 & \text{otherwise.} \end{cases} \tag{3.11}$$

For $\Phi(r,\theta,z) = \varphi(r)\chi(z)$, we compute

$$\begin{aligned} q[\Phi] &= \langle \nabla_r\varphi\chi, \nabla_r\varphi\chi\rangle - (\frac{\pi}{d})^2\|\varphi\chi\|^2_{L^2(\Omega,rdrd\theta dz)}, \\ &= 2\pi\|\varphi'\|^2_{L^2([0,+\infty[,rdr)} \end{aligned}$$

Now let us consider an interval $J = [0,b]$ for a positive $b > a$ and a function $\varphi \in \mathcal{S}([0,+\infty[)$ such that $\varphi(r) = 1$ for $r \in J$. We also define a family $\{\varphi_\tau : \tau > 0\}$ by

$$\varphi_\tau(r) = \begin{cases} \varphi(r) & \text{if} \quad r \in (0,b) \\ \varphi(b+\tau(\ln r - \ln b)) & \text{if} \quad r \geq b. \end{cases} \tag{3.12}$$

Let us write

$$\begin{aligned} \|\varphi'_\tau\|_{L^2([0,+\infty),rdr)} &= \int_{(0,\infty)} |\varphi'_\tau(r)|^2 r dr, \\ &= \int_{(b,+\infty)} \tau^2|\varphi'(b+\tau(\ln r - \ln b))|^2 r dr, \\ &= \tau\int_{(0,+\infty)} |\varphi'(s)|^2 ds = \tau\|\varphi'\|^2_{L^2((0,+\infty))}. \end{aligned} \tag{3.13}$$

Let j be a localization function from $C_0^\infty(0,a)$ and for $\tau, \varepsilon > 0$ we define

$$\Phi_{\tau,\varepsilon}(r,z) = \varphi_\tau(r)[\chi(z) + \varepsilon j(r)^2] = \varphi_\tau(r)\chi(z) + \varphi_\tau\varepsilon j^2(r) = \Phi_{1,\tau,\varepsilon}(r,z) + \Phi_{2,\tau,\varepsilon}(r). \tag{3.14}$$

$$\begin{aligned} q[\Phi] &= q[\Phi_{1,\tau,\varepsilon} + \Phi_{2,\tau,\varepsilon}] \\ &= \mathcal{Q}_0[\Phi_{1,\tau,\varepsilon} + \Phi_{2,\tau,\varepsilon}] - (\frac{\pi}{d})^2\|\Phi_{1,\tau,\varepsilon} + \Phi_{2,\tau,\varepsilon}\|^2_{L^2(\Omega,rdrd\theta dz)}. \\ &= \mathcal{Q}_0[\Phi_{1,\tau,\varepsilon}] - (\frac{\pi}{d})^2\|\Phi_{1,\tau,\varepsilon}\|^2_{L^2(\Omega,rdrd\theta dz)} + \mathcal{Q}_0[\Phi_{2,\tau,\varepsilon}] - (\frac{\pi}{d})^2\|\Phi_{2,\tau,\varepsilon}\|^2_{L^2(\Omega,rdrd\theta dz)} \\ &+ 2\langle \nabla_r\Phi_{1,\tau,\varepsilon}, \nabla_r\Phi_{2,\tau,\varepsilon}\rangle_r - (\frac{\pi}{d})^2\langle \Phi_{1,\tau,\varepsilon}, \Phi_{2,\tau,\varepsilon}\rangle_r. \end{aligned}$$

Using the properties of χ, noting that the supports of φ and j are disjoints and taking into account equation (3.13), we get

$$q[\Phi] = 2\pi\tau\|\varphi'\|_{L^2(0,+\infty)} - 8\pi d\varepsilon\|j^2\|^2_{L^2(0,+\infty)} + 2\varepsilon^2\pi\{2\|jj'\|^2_{(L^2(0,\infty),rdr)} - (\frac{\pi}{d})^2\|j^2\|^2_{(L^2(0,\infty),rdr)}\}. \quad (3.15)$$

Firstly, we notice that only the first term of the last equation depends on τ. Secondly, the linear term in ε is negative and could be chosen sufficiently small so that it dominates over the quadratic one. Fixing this ε and then choosing τ sufficiently small the right hand side of (3.15) is negative. This ends the proof of the first claim.

The proof of the second claim is based on bracketing argument. Let us split $L^2(\Omega, rdrd\theta dz)$ as follows, $L^2(\Omega, rdrd\theta dz) = L^2(\Omega_a^-, rdrd\theta dz) \oplus L^2(\Omega_a^+, rdrd\theta dz)$, with

$$\begin{aligned} \Omega_a^- &= \{(r,\theta,z) \in [0,a] \times [0,2\pi[\times [0,d]\}, \\ \Omega_a^+ &= \Omega \backslash \Omega_a^-. \end{aligned}$$

Therefore

$$H_a^{-,N} \oplus H_a^{+,N} \leq H \leq H_a^{-,D} \oplus H_a^{+,D}. \quad (3.16)$$

Here we index by D and N depending on the boundary conditions considered on the surface $r = a$. The min-max principle leads to

$$\sigma_{ess}(H) = \sigma_{ess}(H_a^{+,N}) = \sigma_{ess}(H_r^{+,D}) = \Big[(\frac{\pi}{d})^2, +\infty\Big[.$$

Hence if $H_r^{-,D}$ exhibits a discrete spectrum below $\frac{\pi^2}{d^2}$, then H do as well. We mention that this is not a necessary condition. If we denote by $\lambda_j(H_a^{-,D}), \lambda_j(H_a^{-,N})$ and $\lambda_j(H)$, the j-th eigenvalue of $H_a^{-,D}$, $H_a^{-,N}$ and H respectively then, again the minimax principle yields the following

$$\lambda_j(H_a^{-,N}) \leq \lambda_j(H) \leq \lambda_j(H_a^{-,D}) \quad (3.17)$$

and for $2 \geq j$

$$\lambda_{j-1}(H_a^{-,D}) \leq \lambda_j(H) \leq \lambda_j(H_a^{-,D}). \quad (3.18)$$

$H_a^{-,D}$ has a sequence of eigenvalues [2, 36], given by

$$\lambda_{k,n,l} = \left(\frac{(2k+1)\pi}{2d}\right)^2 + \left(\frac{x_{n,l}}{a}\right)^2.$$

Where $x_{n,l}$ is the l-th positive zero of Bessel function of order n (see [2, 36]) . The condition

$$\lambda_{k,n,l} < \frac{\pi^2}{d^2}, \quad (3.19)$$

yields that $k = 0$, so we get

$$\lambda_{0,n,l} = \left(\frac{\pi}{2d}\right)^2 + \left(\frac{x_{n,l}}{a}\right)^2.$$

This yields that the condition (3.19) to be fulfilled, will depends on the value of $\left(\frac{x_{n,l}}{a}\right)^2$.

We recall that $x_{n,l}$ are the positive zeros of the Bessel function J_n. So, for any $\lambda(a)$, eigenvalue of H, there exists, $n, l, n', l' \in \mathbb{N}$, such that

$$\frac{\pi^2}{4d^2} + \frac{x_{n,l}^2}{a^2} \leq \lambda(a) \leq \frac{\pi^2}{4d^2} + \frac{x_{n',l'}^2}{a^2}. \tag{3.20}$$

The proof of (3.22) is completed by observing by that $x_{n,l}$ and $x_{n',l'}$ are independent from a. $\boxdot$

3.2 Two Neumann Windows

We consider a Schrödinger particle whose motion is confined to a pair of parallel planes separated by the width d. For simplicity, we assume that they are placed at $z = 0$ and $z = d$. We shall denote this configuration space by Ω

Let $\gamma(a)$ be a disc of radius a with its center at $(0, 0, 0)$ and $\gamma(b)$ be a disc of radius b centered at $(0, 0, d)$; without loss of generality we assume that $0 \leq b \leq a$.

$$\gamma(a) = \{(x, y, 0) \in \mathbb{R}^3;\ x^2 + y^2 \leq a^2\};\ \gamma(b) = \{(x, y, d) \in \mathbb{R}^3;\ x^2 + y^2 \leq b^2\}. \tag{3.21}$$

We set $\Gamma = \partial\Omega \setminus (\gamma(a) \cup \gamma(b))$. We consider Dirichlet boundary condition on Γ and Neumann boundary condition in $\gamma(a)$ and $\gamma(b)$.

Theorem 3.2 *[30] The operator H has at least one isolated eigenvalue in $\left[0, (\frac{\pi}{d})^2\right]$ for any a and b such that $a + b > 0$.*

Moreover for a big enough, if $\lambda(a)$ is an eigenvalue of H less then $\frac{\pi^2}{d^2}$, then we have

$$\lambda(a, b) \in \left(\frac{1}{a^2}, \frac{1}{b^2}\right). \tag{3.22}$$

1. The first claim of Theorem 3.5 is valid for more general shape of bounded surface $\mathcal{S}$, with Neumann boundary condition, not necessarily a disc; (see Figure 2) it suffice that the surface contains a disc of radius $a > 0$.

2. For more general shape $\mathcal{S}$; using discs of radius a and a', such that

$$\gamma(a) \subset \mathcal{S} \subset \gamma(a'); \tag{3.23}$$

In [1] Assel and Ben Salah considered the case of square window.

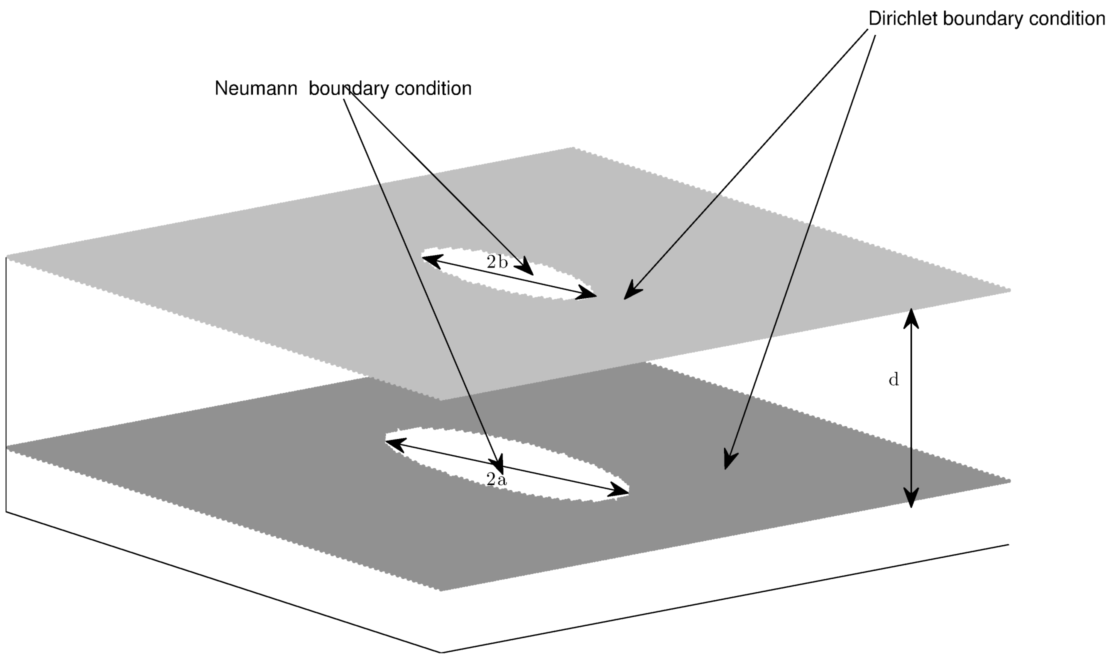


Figure 2: Dirichlet wave guide with two concentric Neumann disc windows on the opposite walls with (in general) different radii a and b.

$$\Omega = \mathbb{R}^2 \times [0, d].$$

When b is big enough, we get the result.

Proposition 3.3 *[30] When the radius a is equal to a critical value a_l at which a new bound state emerges from the continuum, equation (2.6) with $E = \frac{\pi^2}{d^2}$ has a solution $f_l^{(0)}(r, \theta, z)$, unique to a multiplicative constant which at infinity behaves like (valid for both configurations of the boundary conditions)*

$$f_l^{(0)}(r,\theta,z) = \frac{e^{im\theta}}{\sqrt{2\pi}} \left[\frac{\sqrt{2}\sin \pi z}{r^{|m|}} + \beta_l \frac{e^{-\pi\sqrt{3}r}}{\sqrt{r}} \sin 2\pi z + \mathcal{O}\left(\frac{e^{-\pi\sqrt{8}r}}{\sqrt{r}} \right) \right], \quad r \to \infty \tag{3.24}$$

with some constants β_l. Here the two quantum numbers n and m are compacted into the single index l: $l \equiv (n, m)$.

Remark: Compared to the corresponding equation for the quasi-one-dimensional wave guide [5, 6, 17], this asymptotic has a different form what is explained by the additional degree of the in-plane motion.

3.3 Magnetic filed effect

Results on the discrete spectrum of a magnetic Schrödinger operator in waveguide-type domains are scarce. A planar quantum waveguide with constant magnetic field and a potential well is studied in [13], where it was proved that if the potential well is purely attractive, then at least one bound state will appear for any value of the magnetic field. Stability of the bottom of the spectrum of a magnetic Schrödinger operator was also studied in [35]. Magnetic field influence on the Dirichlet-Neumann structures was analyzed in [7, 26], the first dealing with a smooth compactly supported field as well as with the Aharonov-Bohm field in a two dimensional strip and second with perpendicular homogeneous magnetic field in the quasi dimensional.

Despite numerous investigations of quantum waveguides during last few years, many questions remain to be answered, this concerns, in particular, effects of external fields. Most attention has been paid to magnetic fields, either perpendicular to the waveguides plane or threaded through the tube, while the influence of the Aharonov-Bohm magnetic field alone remained mostly untreated.

In their celebrated 1959 paper [4] Aharonov and Bohm pointed out that while the fundamental equations of motion in classical mechanics can always be expressed in terms of field alone, in quantum mechanics the canonical formalism is necessary, and as a result, the potentials cannot be eliminated from the basic equations. They proposed several experiments and showed that an electron can be influenced by the potentials even if no field acts upon it. More precisely, in a field-free multiply-connected region of space, the physical properties of a system depend on the potentials through the gauge-invariant quantity $\oint \mathbf{A}dl$, where $\mathbf{A}$ represents the vector potential. Moreover, the Aharonov-Bohm effect only exists in **the multiply-connected region of space**. The Aharonov-Bohm experiment allows in principle to measure the decomposition into homotopy classes of the quantum mechanical propagator.

A possible next generalization are waveguides with combined Dirichlet and Neumann boundary conditions on different parts of the boundary with an Aharonov-Bohm magnetic field with the flux $2\pi\alpha$. The presence of different boundary conditions and Aharonov-Bohm magnetic field also gives rise to nontrivial properties like the existence of bound states. This question is the main object of the paper. The rest of the paper is organized as follows, in Section 2, we define the model and recall some known results. In section 3, we present the main result of this note followed by a discussion. Section 4 is devoted for numerical computations.

3.3.1 The model

et H_{AB} be the Aharonov-Bohm Schrödinger operator in $\mathrm{L}^2(\Omega)$, defined initially on the domain $\mathrm{C}_0^\infty(\Omega)$, and given by the expression

$$H_{AB} = (i\nabla + \mathbf{A})^2, \tag{3.25}$$

where $\mathbf{A}$ is a magnetic vector potential for the Aharonov-Bohm magnetic field $\mathbf{B}$, and given by

$$\mathbf{A}(x,y,z) = (A_1, A_2, A_3) = \alpha\Big(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}, 0\Big), \quad \alpha \in \mathbb{R}\setminus\mathbb{Z}. \tag{3.26}$$

The magnetic field $\mathbf{B} : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$\mathbf{B}(x,y,z) = curl\mathbf{A} = 0 \tag{3.27}$$

outside the z-axis and

$$\int_{\varrho} \mathbf{A} = 2\pi\alpha, \tag{3.28}$$

where ϱ is a properly oriented closed curve which encloses the z-axis. It can be shown that H_{AB} has a four-parameter family of self-adjoint extensions which is constructed by means of von Neumann 's extension theory [8]. Here we are only interested in the Friedrichs extension of H_{AB} on $\mathrm{L}^2(\Omega)$ which can be constructed by means of quadratic forms. We get that the domain $D(\Omega)$ of H is

$$D(\Omega) \;=\; \{u \in \mathrm{H}^1(\Omega);\quad (i\nabla + \mathbf{A})^2 u \in \mathrm{L}^2(\Omega), u\lceil\Gamma = 0, \nu.(i\nabla + \mathbf{A})u\lceil\gamma(a) = 0\},$$

where ν the normal vector and

$$Hu = (i\nabla + \mathbf{A})^2 u, \quad \forall u \in D(\Omega). \tag{3.29}$$

Let's start by recalling that in the particular case when $a = 0$, we get H^0, the magnetic Dirichlet Laplacian, and when $a = +\infty$ we get H^∞, the magnetic Dirichlet-Neumann Laplacian.

Proposition 3.4 *The spectrum of* H^0 *is* $[(\frac{\pi}{d})^2, +\infty[$, *and the spectrum of* H^∞ *coincides with* $[(\frac{\pi}{2d})^2, +\infty[$.

Proof. We have

$$H \;=\; (i\nabla + \widetilde{\mathbf{A}})^2 \otimes I \oplus I \otimes (-\Delta_{[0,d]}), \quad \text{on} \quad \mathrm{L}^2(\mathbb{R}^2\setminus\{0\}) \otimes \mathrm{L}^2([0,d]).$$

where $\widetilde{\mathbf{A}} := \alpha\left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}\right)$. Consider the quadratic form

$$\begin{aligned}\widetilde{q}[u] &= \int_{\mathbb{R}^2} \mid (i\nabla + \widetilde{\mathbf{A}})u \mid^2 dxdy \\ &= \int_{\mathbb{R}^2} \left|\left(i\partial_x + \alpha\frac{y}{x^2+y^2}\right)u\right|^2 dxdy + \int_{\mathbb{R}^2} \left|\left(i\partial_y - \alpha\frac{x}{x^2+y^2}\right)u\right|^2 dxdy.\end{aligned} \tag{3.30}$$

By using polar coordinates we get

$$r = \sqrt{x^2+y^2}; \qquad \frac{x}{r} = \cos\theta, \qquad \frac{y}{r} = \sin\theta,$$

and

$$\frac{\partial\theta}{\partial x} = \frac{-y}{r^2}, \quad \frac{\partial\theta}{\partial y} = \frac{x}{r^2}, \quad \partial_x = \cos\theta\frac{\partial}{\partial r} - \frac{y}{r^2}\frac{\partial}{\partial\theta}, \quad \partial_y = \sin\theta\frac{\partial}{\partial r} + \frac{x}{r^2}\frac{\partial}{\partial\theta}.$$

Hence (3.30) becomes

$$\widetilde{q}[u] \quad = \quad \int \left(\mid \partial_r u \mid^2 + \frac{1}{r^2} \mid (i\partial_\theta u - \alpha u) \mid^2 \right) r dr d\theta. \tag{3.31}$$

Expanding u into Fourier series with respect to θ

$$u(r,\theta) = \sum_{k=-\infty}^{\infty} u_k(r) \frac{e^{ik\theta}}{\sqrt{2\pi}}.$$

we get

$$\int_{\mathbb{R}^2} \mid (i\nabla + \widetilde{\mathbf{A}})u \mid^2 dxdy \geq \min_k \mid k+\alpha \mid^2 \int \frac{1}{x^2+y^2} \mid u(x,y) \mid^2 dxdy. \tag{3.32}$$

Here the form in the right hand side is considered on the function class $\mathrm{H}^1(\mathbb{R}^2)$, obtained by the completion of the class $\mathcal{C}_0^\infty(\mathbb{R}^2\backslash\{0\})$. Inequality (3.32) is the Hardy inequality in two dimensions with Aharonov-Bohm vector potential [3]. This yields that $\sigma\left((i\nabla + \widetilde{\mathbf{A}})^2\right) \subset [0,+\infty[$.

Since $\sigma(-\Delta) = \sigma_{ess}(-\Delta) = [0,+\infty[$, then there exists a Weyl sequences $\{h_n\}_{n=1}^\infty$ for the operator $-\Delta$ in $\mathrm{L}^2(\mathbb{R}^2)$ at $\lambda \geq 0$. Construct the functions

$$\varphi_n(x,y) = \begin{cases} h_n & \text{if } x > n \text{ and } y > n, \\ 0 & \text{if not.} \end{cases}$$

Let us compute

$$\begin{aligned} \parallel \left((i\nabla + \widetilde{\mathbf{A}})^2 - \lambda\right) \varphi_n \parallel \quad &\leq \quad \parallel (\Delta - \lambda)\varphi_n \parallel + \parallel \widetilde{\mathbf{A}}^2 \varphi_n \parallel + \parallel \widetilde{\mathbf{A}} \nabla \varphi_n \parallel \\ &\leq \quad \parallel (\Delta - \lambda)\varphi_n \parallel + \frac{c}{n}. \end{aligned}$$

Where c is positive.

Therefore, the functions $\psi_n = \dfrac{\varphi_n}{\parallel \varphi_n \parallel}$ is Weyl sequence for $(i\nabla + \widetilde{\mathbf{A}})^2$ at $\lambda \geq 0$, thus $[0,+\infty[\subset \sigma_{ess}\left((i\nabla + \widetilde{\mathbf{A}})^2\right) \subset \sigma\left((i\nabla + \widetilde{\mathbf{A}})^2\right)$.

Then we get that the spectrum of $(i\nabla + \widetilde{\mathbf{A}})^2$ is $[0,+\infty[$, we know that the spectrum of $-\Delta^0_{[0,d]}$ and $-\Delta^\infty_{[0,d]}$ is $\{(\frac{j\pi}{d})^2, \ j \in \mathbb{N}^\star\}$ and $\{(\frac{(2j+1)\pi}{2d})^2, \ j \in \mathbb{N}\}$ respectively. Therefore we have the spectrum of H^0 is $[(\frac{\pi}{d})^2, +\infty[$. And the spectrum of H^∞ coincides with $[(\frac{\pi}{2d})^2, +\infty[$. ■

Consequently, we have

$$\left[(\frac{\pi}{d})^2, +\infty\right[\subset \sigma(H) \subset \left[(\frac{\pi}{2d})^2, +\infty\right[.$$

Using the property that the essential spectra is preserved under compact perturbation, we deduce that the essential spectrum of H is

$$\sigma_{ess}(H) = \left[(\frac{\pi}{d})^2, +\infty\right[.$$

Theorem 3.5 *[31] Let H be the operator defined on (3.29) and $\alpha \in \mathbb{R} \setminus \mathbb{Z}$. There exist $a_0 > 0$ such that for any $0 < \frac{a}{d} < a_0$, we have*

$$\sigma_d(H) = \emptyset.$$

There exist $a_1 > 0$, such that $\frac{a}{d} > a_1$, we have

$$\sigma_d(H) \neq \emptyset.$$

The presence of magnetic field in three dimensional straight strip of width d with the Neumann boundary condition on a disc window of radius $0 < \frac{a}{d} < a_0$ and Dirichlet boundary conditions on the remained part of the boundary, destroys the creation of discrete eigenvalues below the essential spectrum. If $\frac{a}{d} > a_1$, the effect of the magnetic field is reduced. This result is still true for more general Neumann window containing some disc. To get the optimal result of a_0 and a_1, we need explicit calculation.

Proof. The proof follow the same steps as in the previous two subsections. The main difference is By introducing the magnetic filed we get a new Bessel equation we obtain $\frac{1}{P}(i\frac{\partial}{\partial\theta} + \alpha)^2 P$ should be a constant $-(m-\alpha)^2 = -\nu^2$ for $m \in \mathbb{Z}$.

Finally, we get the new equation for R

$$R''(r) + \frac{1}{r}R'(r) + [\lambda - k_z^2 - \frac{\nu^2}{r^2}]R(r) = 0. \tag{3.33}$$

We notice that the equation (3.33), is a Bessel equation which by the introduction of the term α is different from equation (2.7). The solutions of (3.33) could be expressed in terms of Bessel functions. More explicit solutions could be given by considering boundary conditions.

The solution of the equation (3.33) is given by $R(r) = cJ_\nu(\beta r)$, where $c \in \mathbb{R}^\star$, $\beta^2 = \lambda - k_z^2$ and J_ν is the Bessel function of first kind of order ν.

We assume that

$$\begin{aligned} R'(a) = 0 \quad &\Leftrightarrow \quad J_\nu(\beta a) = 0 \\ &\Leftrightarrow \quad a\beta = x'_{\nu,n}. \end{aligned} \tag{3.34}$$

Where $x'_{\nu,n}$ is the $n-$th positive zero of the Bessel function J'_ν.

Using the same notations as the last section,

$$H_a^{-,N} \oplus H_a^{+,N} \leq H \leq H_a^{-,D} \oplus H_a^{+,D}. \tag{3.35}$$

By equation (3.34), $H_a^{-,N}$ has a sequence of eigenvalues given by

$$\begin{aligned} \lambda_{j,\nu,n} &= \frac{x'^2_{\nu,n}}{a^2} + k_z^2 \\ &= \frac{x'^2_{\nu,n}}{a^2} + \left(\frac{(2j+1)\pi}{2d}\right)^2. \end{aligned}$$

As we are interested for discrete eigenvalues which belongs to $[\left(\frac{\pi}{2d}\right)^2, \left(\frac{\pi}{d}\right)^2)$ only $\lambda_{0,\nu,n}$ intervenes.

If

$$\left(\frac{\pi}{d}\right)^2 \leq \lambda_{0,\nu,n}, \tag{3.36}$$

then H does not have a discrete spectrum. We recall that $\nu^2 = (m-\alpha)^2$ and it is related to magnetic flux, also recall that $x'_{\nu,n}$ are the positive zeros of the Bessel function J'_n and $\forall \nu > 0$, $\forall n \in \mathbb{N}^\star$; $0 < x'_{\nu,n} < x'_{\nu,n+1}$ (see [2]). So, for any eigenvalue of $H_a^{-,N}$,

$$\frac{x'^2_{\nu,1}}{a^2} + \left(\frac{\pi}{2d}\right)^2 < \frac{x'^2_{\nu,n}}{a^2} + \left(\frac{\pi}{2d}\right)^2 = \lambda_{0,\nu,n}.$$

An immediate consequence of the last inequality is that to satisfy (3.36) it is sufficient to have

$$3\left(\frac{\pi}{2d}\right)^2 < \frac{x'^2_{\nu,1}}{a^2},$$

therefore

$$\frac{\sqrt{3}\pi}{2d} < \frac{x'_{\nu,1}}{a},$$

then

$$\frac{a}{d} < \frac{2x'_{\nu,1}}{\sqrt{3}\pi}.$$

We have(see [2, 36])

$$\nu + \alpha_n \nu^{1/3} < x'_{\nu,n},$$

where $\alpha_n = 2^{-1/3}\beta_n$ and β_n is the $n-$th positive root of the equation

$$J_{\frac{2}{3}}\left(\frac{2}{3}x^{3/2}\right) - J_{\frac{-2}{3}}\left(\frac{2}{3}x^{3/2}\right) = 0.$$

For $n = 1$, we have $\alpha_n \nu^{1/3} \approx 0.6538$ (see [2]), then

$$c_0 := 0.6538 + \alpha < 0.6538 + \nu < x'_{\nu,1}. \tag{3.37}$$

Then we get that for d and a positives such that $\frac{a}{d} < a_0 := \frac{2c_0}{\sqrt{3}\pi}$,

$$\sigma_d(H) = \emptyset.$$

This ends the proof of the first result of the theorem 3.5.

By the min-max principle and (3.35), we know that if $H_a^{-,D}$ exhibits a discrete spectrum below $(\frac{\pi}{d})^2$, then H do as well.

$H_a^{-,D}$ has a sequence of eigenvalues [29, 30, 36], given by

$$\lambda_{j,\nu,n} = \left(\frac{x_{\nu,n}}{a}\right)^2 + \left(\frac{(2j+1)\pi}{2d}\right)^2.$$

Where $x_{\nu,n}$ is the $n-$th positive zero of Bessel function of order ν (see [2]). As we are interested for discrete eigenvalues which belongs to $[(\frac{\pi}{2d})^2, (\frac{\pi}{d})^2)$ only for $\lambda_{0,\nu,n}$.

If the following condition

$$\lambda_{0,\nu,n} < \left(\frac{\pi}{d}\right)^2 \tag{3.38}$$

is satisfied, then H have a discrete spectrum.

We recall that $0 < x_{\nu,n} < x_{\nu,n+1}$ for any $\nu > 0$ and any $n \in \mathbb{N}^\star$ (see [2]). So, for any eigenvalue of $H_a^{-,D}$,

$$\frac{x_{\nu,1}^2}{a^2} + \left(\frac{\pi}{2d}\right)^2 < \frac{x_{\nu,n}^2}{a^2} + \left(\frac{\pi}{2d}\right)^2 = \lambda_{0,\nu,n}.$$

An immediate consequence of the last inequality is that to satisfy (3.38) it is sufficient to set then

$$\frac{2x_{\nu,1}}{\sqrt{3}\pi} < \frac{a}{d}.$$

We have

$$\sqrt{\left(n - \frac{1}{4}\right)^2 \pi^2 + \nu^2} < x_{\nu,n},$$

For $n = 1$, we have

$$c_1 := \sqrt{\left(\frac{3\pi}{4}\right)^2 + \alpha^2} < \sqrt{\left(\frac{3\pi}{4}\right)^2 + \nu^2} < x_{\nu,1}. \tag{3.39}$$

Then we get that for d and a positives such that $\frac{a}{d} > a_1 := \frac{2c_1}{\sqrt{3}\pi}$,

$$\sigma_d(H) \neq \emptyset.$$

■

4 Random boundaries conditions

Results on random waveguides are rare and there are still serious open questions on this context. We cote the the following available references [21, 24, 28] for the continuous case, and [19, 20] for discrete model.

In [9] we consider a two dimensional quantum waveguide with mixed and random boundaries conditions. Precisely we are interested on the behavior of the integrated density of states and prove that it decreases exponentially fast at the bottom of the spectrum. Below we recall the definition of such quantities.

4.1 Results and discussion

4.1.1 The model

Let D_0 be the strip $\mathbb{R} \times (0, d)$. Let $(\omega_\gamma)_{\gamma \in \mathbb{Z}}$ be a family of independent and identically distributed random variables taking values in $[0, 1]$. We denote by $(\mathbb{P}, \mathcal{F}, \Omega)$ the corresponding probability space and assume that

(A.1) There exits $0 < c < 1$; such that

$$\mathbb{P}\{\omega_0 = 1\} = c. \tag{4.40}$$

Let $\mathcal{H}_\omega$ be the following quadratic form defined as follow:

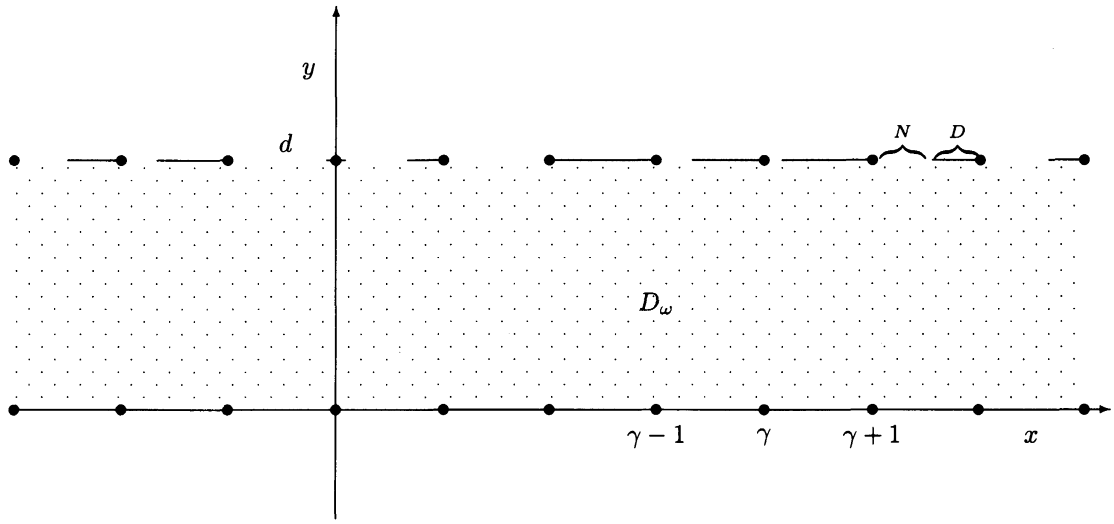
For $u \in \mathcal{D}(\mathcal{H}_\omega) = \{u \in H^1(D_0); u(x,d) = 0;$ for $\gamma \in \mathbb{Z};$ and $x \in [\gamma + \omega_\gamma, \gamma + 1[$ and $u(x,0) = 0, \forall x \in \mathbb{R}\})$

$$\mathcal{H}_\omega[u,u] = \int_{D_\omega} \nabla u(x)\overline{\nabla u(x)}dx. \tag{4.41}$$

So on the boundary $y = d$ we have mixed boundaries conditions; precisely for $\gamma \in \mathbb{Z}, y = d$ and $x \in [\gamma, \gamma + \omega_\gamma[$ we consider Neumann boundary condition and for $x \in [\gamma + \omega_\gamma, \gamma + 1[$ we consider Dirichlet condition, with the convention that when $\omega_\gamma = 1$, $[\gamma + 1, \gamma + 1[= \varnothing$ and get Neumann boundary condition on $[\gamma, \gamma + 1[$.

On the boundary $y = 0$ we consider only Dirichlet boundaries conditions.

For a fixed realization, the following picture will help in visualizing the domain



Picture 1.

Notice that here we have a family of quadratic forms acting on different domains. There is a family of random maps (φ_ω) that transform these different domains D_ω to the non-random domain, D_0 by dilatation (a change of variables). This transforms the randomness from the domain say to the measure which we denote by μ_ω. Thus a random medium will be modeled by an ergodic random self-adjoint operator. Indeed the family of maps yield an equivalent quadratic form with domain $H_0^1(D_0)$

$$\mathcal{H}\omega[u,u] = \int_{D_0} \nabla u(x)\overline{\nabla u(x)}d\mu_\omega.$$

$\mathcal{H}_\omega$ is a symmetrical, closed and positive quadratic form. Let H_ω be the self-adjoint operator associated to $\mathcal{H}_\omega$ [33]. Consequently if we consider τ_γ the shift function i.e $(\tau_\gamma u)(x,y) = u(x - \gamma, y)$. This ensures that H_ω is a measurable family of self-adjoint operators and ergodic [22, 32]. Indeed, $(\tau_\gamma)_{\gamma \in \mathbb{Z}}$ is a group of unitary operators on $L^2(D_0)$ and for $\gamma \in \mathbb{Z}$ we have

$$\tau_\gamma H_\omega \tau_{-\gamma} = H_{\tau_\gamma \omega}.$$

According to [22, 32] we know that there exists $\Sigma, \Sigma_{pp}, \Sigma_{ac}$ and Σ_{sc} closed and non-random sets of $\mathbb{R}$ such that Σ is the spectrum of H_ω with probability one and such that if σ_{pp} (respectively σ_{ac} and σ_{sc}) design the pure point spectrum (respectively the absolutely continuous and singular continuous spectrum) of H_ω, then $\Sigma_{pp} = \sigma_{pp}, \Sigma_{ac} = \sigma_{ac}$ and $\Sigma_{sc} = \sigma_{sc}$ with probability

The following Lemma gives the precise location of the spectrum.

Lemma 4.1 *The spectrum* Σ, *of* H_ω *is* $[\frac{\pi^2}{4d^2}, +\infty[$ *with probability one.*
We set $E_0 = \frac{\pi^2}{4d^2}$.

Proof: Let us denote by $H^{DN}_{\mathbb{R}\times[0,d]}$, the Laplace operator $-\Delta$ defined on $L^2(\mathbb{R} \times [0,d])$ with Dirichlet boundary conditions on $y=0$ and Neumann boundaries conditions on $y=d$. We denote this domain by D^{DN}. We set $\Lambda_k = [-\frac{k}{2}, \frac{k}{2}] \times [0,d]$. First let us notice that for any $\omega \in \Omega$, we have

$$H_\omega \geq H^{DN}_{\mathbb{R}\times[0,d]}. \tag{4.42}$$

This gives that

$$\Sigma \subset \sigma(H^{DN}_{\mathbb{R}\times[0,d]}\cdot) = [\frac{\pi^2}{4d^2}, +\infty[.$$

So one needs to show the opposite inclusion, i.e

$$[\frac{\pi^2}{4d^2}, +\infty[\subset \Sigma \text{ for } \mathbb{P} - \text{almost every } \omega \in \Omega. \tag{4.43}$$

For this, let $\widetilde{\Omega}$, be the following events

$$\widetilde{\Omega} = \Big\{\omega \in \Omega: \quad \text{for any } \mathrm{k} \in \mathbb{N}, \text{there exists } \Lambda_{\mathrm{k}}^{(\omega)} \subset \mathbb{R} \times [0, \mathrm{d}], \text{suchthat } \mathrm{D}^{\omega}_{\Lambda_{\mathrm{k}}^{(\omega)}} = \mathrm{D}^{\mathrm{DN}}_{\Lambda_{\mathrm{k}}^{(\omega)}}\Big\}. \tag{4.44}$$

Here $A_{\Lambda_k^{(\omega)}}$ is the set of points which are both in A and Λ_k^ω. In (4.44) we asked that all sites inside $\Lambda_k^{(\omega)}$ to be equal to one. Let $E \in [\frac{\pi^2}{4d^2}, +\infty[= \sigma(H^{DN}_{\mathbb{R}\times[0,d]})$ be arbitrarily fixed. Using Weyl criterion, we know that there exists a Weyl sequence $(\varphi_{E,n})_{n\in\mathbb{N}} \subset L^2(\mathbb{R} \times [0,d])$, for $-\Delta$. Thus $\|\varphi_{E,n}\| = 1$, for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \|(\Delta + E \cdot \mathbb{I})\varphi_{E,n}\| = 0. \tag{4.45}$$

Notice that for any $i \in \mathbb{Z}$, $(T_i\varphi_{E,n})_{n\in\mathbb{N}}$ is also a Weyl sequence. Without loss of generality, we assume that the sequence $(\varphi_{E,n})_{n\in\mathbb{N}}$ is compactly supported. So for any $\omega \in \widetilde{\Omega}$, there exists a Weyl sequences $(\varphi^\omega_{E,n})_{n\in\mathbb{N}}$ for $H^{DN}_{\mathbb{R}\times[0,d]}$ on $\mathbb{R} \times [0,d]$ with the property that all the supports are contained inside the cubes of (4.44). So for every $\omega \in \widetilde{\Omega}$ and any $n \in \mathbb{N}$ there exists an integer k_n^ω and a cube $\Lambda^{(\omega)}_{k_n^\omega}$ and $\varphi^\omega_{E,n}$ as in (4.44) such that $supp(\varphi^\omega_{E,n})$ is contained in $\Lambda^{(\omega)}_{k_n^\omega}$. That is,

$$\min\{| x - y |: x \in supp\varphi^\omega_{E,n};\ y \in (\mathbb{R} \times [0,d]) \backslash \Lambda^{(\omega)}_{k_n^\omega}\} > 0.$$

So, for any $n \in \mathbb{N}$ and $\omega \in \widetilde{\Omega}$, we get

$$\|(H_\omega - E\mathbb{I})\varphi^\omega_{E,n})\| = \|(\Delta + E \cdot \mathbb{I})\varphi^\omega_{E,n}\|. \tag{4.46}$$

Hence, $(\varphi^\omega_{E,n})_{n\in\mathbb{N}}$ is also a Weyl sequence for H_ω. So we get (4.43) for any $\omega \in \widetilde{\Omega}$. Now it suffices to check that $\mathbb{P}(\widetilde{\Omega}) = 1$. For this let λ be an integer bigger than 2. $(\Lambda_{k,\lambda})_{\lambda\in\mathbb{N}} \subset \mathbb{R} \times [0, d]$ be a sequence of disjoint cubes in $\mathbb{R} \times [0, d]$ i.e $\Lambda_{k,\lambda_1} \cap \Lambda_{k,\lambda_2} = \emptyset$ whenever $\lambda_1 \neq \lambda_2$. We set $\Omega_{k,\lambda} = \{\omega \in \Omega : D^\omega_{\Lambda_{k,\lambda}} = D^{ND}_{\Lambda_{k,\lambda}}\}$. So $(\Omega_{k,\lambda})_{\lambda\in\mathbb{N}}$, is a sequence of two by two statistically independent sets, with non-zero probability and independent of $\lambda \in \mathbb{N}$. So, using the Borel-Cantelli lemma, we get that $\mathbb{P}(\Omega_k) = 1$ for any $k \in \mathbb{N} + 2$, where

$$\Omega_k = \limsup_{\lambda\to\infty} \Omega_{k,\lambda}.$$

The proof of Lemma 4.1 is ended by noting that

$$\cap_{k\in\mathbb{N}+2}\Omega_k \subset \widetilde{\Omega}.$$

□

Theorem 4.2 *Let H_ω be the operator defined in section 4.1.1 . Assume that (A.1) is satisfied. There exists $\varepsilon_0 > 0$ such that:*

1. $\Sigma \cap [\frac{\pi^2}{4d^2}, \frac{\pi^2}{4d^2} + \varepsilon_0) = \Sigma_{pp} \cap [\frac{\pi^2}{4d^2}, \frac{\pi^2}{4d^2} + \varepsilon_0)$.
2. *an eigenfunction corresponding to an eigenvalue in* $[\frac{\pi^2}{4d^2}, \frac{\pi^2}{4d^2} + \varepsilon_0)$ *decays exponentially.*

The result of Theorem , is based on multiscale analysis [15, 34]. The proof can be related to the behavior of the so called the integrated density of states [21, 28, 24].

We recall that the integrated density of states is defined as follows: we note by H_{ω,Λ_L} the restriction of H_ω to $\Lambda_L = D_\omega \cap [-\frac{L}{2}, \frac{L}{2}] \times [0, d]$ with self-adjoint boundary conditions. As H_ω is elliptic, the resolvent of H_{ω,Λ_L} is compact and, consequently, the spectrum of H_{ω,Λ_L} is discrete and is made of isolated eigenvalues of finite multiplicity [?]. We define

$$N_{\Lambda_L}(E) = \frac{1}{|\Lambda_L|} \cdot \#\{\text{eigenvalues of } \mathrm{H}_{\omega,\Lambda} \leq \mathrm{E}\}. \tag{4.47}$$

Here $\mathrm{vol}(\Lambda_L)$ is the volume of Λ_L in the Lebesgue sense and $\#E$ is the cardinal of E.

It is shown that the limit of $N_{\Lambda_L}(E)$ when Λ_L tends to $\mathbb{R}^2$ exists almost surely. It is called the **integrated density of states** of H_ω . See [22, 32]. The behavior of N is considered [9].

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