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# Probabilistic Construction of Solutions To Some Integral Equations

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## §1. Notations and assumptions

For simplicity, let $D_0 := \mathbb{R}^3 \setminus \{0\}$, and we put $\mathbb{R}_+ := [0,\infty)$. For every $\alpha, \beta \in \mathbb{C}^3$, we use the symbol $\alpha\cdot\beta$ for the inner product, and we define $e_x := x/|x|$ for every $x \in D_0$. In this article we consider the following deterministic nonlinear integral equation:

$$e^{\lambda t|x|^2}u(t,x) = u_0(x) + \frac{\lambda}{2}\int_0^t ds\ e^{\lambda s|x|^2}\int p(s,x,y;u)n(x,y)dy$$
$$+\frac{\lambda}{2}\int_0^t e^{\lambda s|x|^2}f(s,x)ds, \qquad \text{for} \quad \forall(t,x)\in\mathbb{R}_+\times D_0. \tag{1}$$

Here $u \equiv u(t,x)$ is an unknown function : $\mathbb{R}_+ \times D_0 \to \mathbb{C}^3$, $\lambda > 0$, and $u_0 : D_0 \to \mathbb{C}^3$ is the initial data such that $u(t,x)|_{t=0} = u_0(x)$. Moreover, $f(t,x) : \mathbb{R}_+ \times D_0 \to \mathbb{C}^3$ is a given function satisfying $f(t,x)/|x|^2 =: \tilde{f} \in L^1(\mathbb{R}_+)$. The integrand $p$ in (1) is given by

$$p(t,x,y;u) = u(x,y)\cdot e_x\{u(t,x-y) - e_x(u(t,x-y)\cdot e_x)\}. \tag{2}$$

On the other hand, we consider a Markov kernel $K : D_0 \to D_0 \times D_0$. Actually, for every $z \in D_0$, $K_z(dx,dy)$ lies in the space $\mathcal{P}(D_0\times D_0)$ of all probability measures on a product space $D_0 \times D_0$. When the kernel $k$ is given by $k(x,y) = i|x|^{-2}n(x,y)$, then we define $K_z$ as a Markov kernel satisfying that for any positive measurable function $h = h(x,y)$ on $D_0 \times D_0$,

$$\iint h(x,y)K_z(dx,dy) = \int h(x,z-x)k(x,z)dx. \tag{3}$$

Moreover, we assume that for every measurable functions $f, g > 0$ on $\mathbb{R}^+$,

$$\int h(|z|)\nu(dz)\int g(|x|)K_z(dx,dy) = \int g(|z|)\nu(dz)\int h(|y|)K_z(dx,dy) \tag{4}$$

holds, where the measure $\nu$ is given by $\nu(dz) = |z|^{-3}dz$.

## §2. Main result

In this section we shall state our main result, which asserts the existence and uniqueness of solutions to the nonlinear integral equation (1). As a matter of fact, the solution $u(t, x)$ can be expressed as the expectation of a star-product functional, which is nothing but a probabilistic solution constructed by making use of the below-mentioned branching particle systems and branching models. Let

$$M_{\bigstar}^{\langle u_0,f\rangle}(\omega) = \prod \bigstar_{[x_{\tilde{m}}]}\Xi_{m_2.m_3}^{m_1}[u_0, f](\omega), \tag{5}$$

be a probabilistic representation in terms of tree-based star-product functional with weight $(u_0, f)$. For the details of the definition, see the succeeding sections. On the other hand, $M_*^{\langle U,F\rangle}(\omega)$ denotes the corresponding $*$-product functional with weight $(U, F)$. In fact, as to be seen in what follows, in a similar manner as the case of a star-product functional we can construct a $(U, F)$-weighted tree-based $*$-product functional $M_*^{\langle U,F\rangle}(\omega)$, which is indexed by the nodes $(x_m)$ of a binary tree. Here we suppose that $U = U(x)$ (resp. $F = F(t, x)$) is a non-negative measurable function on $D_0$ (resp. $\mathbb{R}_+ \times D_0$) respectively, and also that $F(\cdot, x) \in L^1(\mathbb{R}_+)$ for each $x$. Indeed, in construction of the $*$-product functional, the product in question is taken as ordinary multiplication $*$ instead of the star-product $\bigstar$ in the definition of star-product functional.

**Theorem 1.** *Suppose that* $|u_0(x)| \leqslant U(x)$ *for* $\forall x$ *and* $|\tilde{f}(t, x)| \leqslant F(t, x)$ *for* $\forall t, x$, *and also that for some* $T > 0$ *(* $T >> 1$ *sufficiently large),*

$$E_{T,x}[M_*^{\langle U,F\rangle}] < \infty, \quad \text{a.e.} - x \tag{6}$$

*Then there exists a* $(u_0, f)$*-weighted tree-based star* $\bigstar$*-product functional* $M_{\bigstar}^{\langle u_0,f\rangle}(\omega)$, *indexed by a set of node labels accordingly to the tree structure which a binary critical branching process* $Z^{K_x}(t)$ *determines. Furthermore, the function*

$$u(t, x) = E_{t,x}[M_{\bigstar}^{\langle u_0,f\rangle}] \tag{7}$$

*gives a unique solution to the integral equation* (1). *Here* $E_{t,x}$ *denotes the expectation with respect to a probability measure* $P_{t,x}$ *as the time-reversed law of* $Z^{K_x}(t)$.

## §3. Branching model

In this section we consider a continuous time binary critical branching process $Z^{K_x}(t)$ on $D_0$, whose branching rate is given by a parameter $\lambda|x|^2$, whose branching mechanism is binary with equi-probability, and whose descendant branching particle behavior (or distribution) is determined by the kernel $K_x$. Next, taking

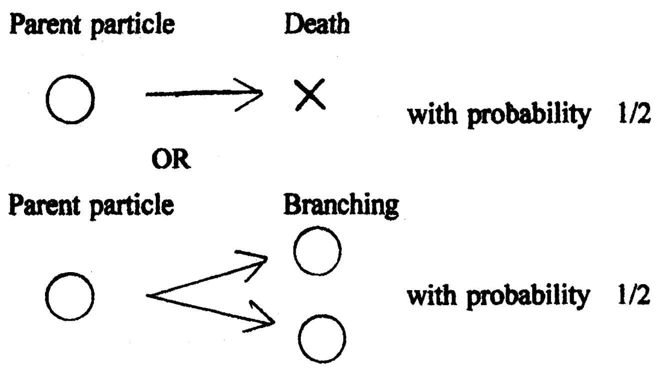


図 1: Binary Branching

notice of the tree structure which the process $Z^{K_x}(t)$ determines, we denote the space of marked trees

$$\omega = (t, (t_m), (x_m), (\eta_m), m \in \mathcal{V}) \tag{8}$$

by $\Omega$. Furthermore, we write the time-reversed law of $Z^{K_x}(t)$ being a probability measure on $\Omega$ as $P_{t,x} \in \mathcal{P}(\Omega)$. Here $t$ denotes the birth time of common ancestor, and the particle $x_m$ dies when $\eta_m = 0$, while it generates two descendants $x_{m1}, x_{m2}$ when $\eta_m = 1$. On the other hand,

$$\mathcal{V} = \bigcup_{\ell \geq 0} \{1, 2\}^{\ell}$$

is a set of all labels, namely, finite sequences of symbols with length $\ell$, which describe the whole tree structure given. For $\omega \in \Omega$ we denote by $\mathcal{N}(\omega)$ the totality of nodes being branching points of tree, and let $N_+(\omega)$ be the set of all nodes $m$ being a member of $\mathcal{V} \setminus \mathcal{N}(\omega)$, whose direct predecessor lies in $\mathcal{N}(\omega)$ and which satisfies the condition $t_m(\omega) > 0$, and let $N_-(\omega)$ be the same set as described above, but satisfying $t_m(\omega) \leqslant 0$. Finally we put

$$N(\omega) = N_+(\omega) \cup N_-(\omega). \tag{9}$$

## §4. Star-product functional and $*$-product functional

In what follows we shall intoduce a tree-based star-product functional in order to construct a probabilistic solution to the class of integral equations (1). First of all, we denote by the symbol $\mathrm{Proj}^z(\cdot)$ a projection of the objective element onto its orthogonal part of the $z$ component in $\mathbb{C}^3$, and we define a $\bigstar$-product of $\beta, \gamma$ for $z \in D_0$ as

$$\beta \bigstar_{[z]} \gamma = -i(\beta \cdot e_z)\mathrm{Proj}^z(\gamma). \tag{10}$$

We shall define $\Theta^m(\omega)$ for each $\omega \in \Omega$ realized as follows. When $m \in N_+(\omega)$, then $\Theta^m(\omega) = \tilde{f}(t_m(\omega), x_m(\omega))$, while $\Theta^m(\omega) = u_0(x_m(\omega))$ if $m \in N_-(\omega)$. Then we define

$$\Xi^{m_1}_{m_2.m_3}(\omega) \equiv \Xi^{m_1}_{m_2,m_3}[u_0, f](\omega) := \Theta^{m_2}(\omega) \bigstar_{[x_{m_1}]} \Theta^{m_3}(\omega), \tag{11}$$

where as for the product order in the star-product $\bigstar$, when we write $m \prec m'$ lexicographically with respect to the natural order $\prec$, the term $\Theta^m$ labelled by $m$ necessarily occupies the left-hand side and the other $\Theta^{m'}$ labelled by $m'$ occupies the right-hand side by all means. And besides, as abuse of notation we write

$$\Xi^{\emptyset}_{m,\emptyset}(\omega) \equiv \Xi^{\emptyset}_{m,\emptyset}[u_0, f](\omega) := \Theta^m(\omega), \tag{12}$$

especially when $m \in \mathcal{V}$ is a label of single terminal point in the restricted tree structure in question.

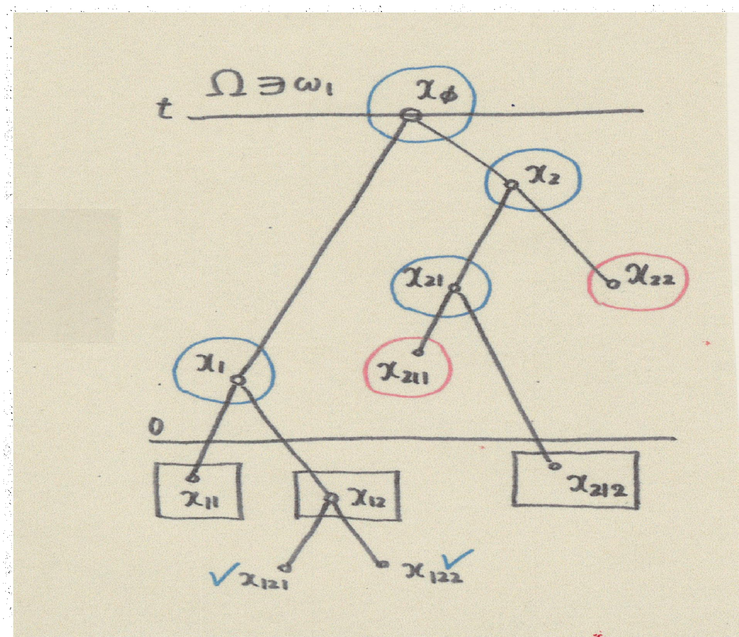


図 2: Example: A realized Tree

Under these circumstances, we consider a random quantity which obtained by executing the star-product $\bigstar$ inductively at each node in $\mathcal{N}(\omega)$, and we call it a tree-based $\bigstar$-product functional, and we express it symbolically as

$$M^{\langle u_0, f\rangle}_{\bigstar}(\omega) = \prod \bigstar_{[x_{\tilde{m}}]} \Xi^{m_1}_{m_2.m_3}[u_0, f](\omega), \tag{13}$$

where $m_1 \in \mathcal{N}(\omega)$ and $m_2, m_3 \in N(\omega)$, and by the symbol $\prod \bigstar$ (as a product relative to the star-product) we mean that the star-products $\bigstar$'s should be succeedingly executed in a lexicographical manner with respect to $x_{\tilde{m}}$ such that $\tilde{m} \in \mathcal{N}(\omega) \cap \{|\tilde{m}| = \ell - 1\}$ when $|m_1| = \ell$.

**Example 1.** Now let us suppose that a tree structure $\omega_1$ ($\in \Omega$) has been realized here (see Figure 2).

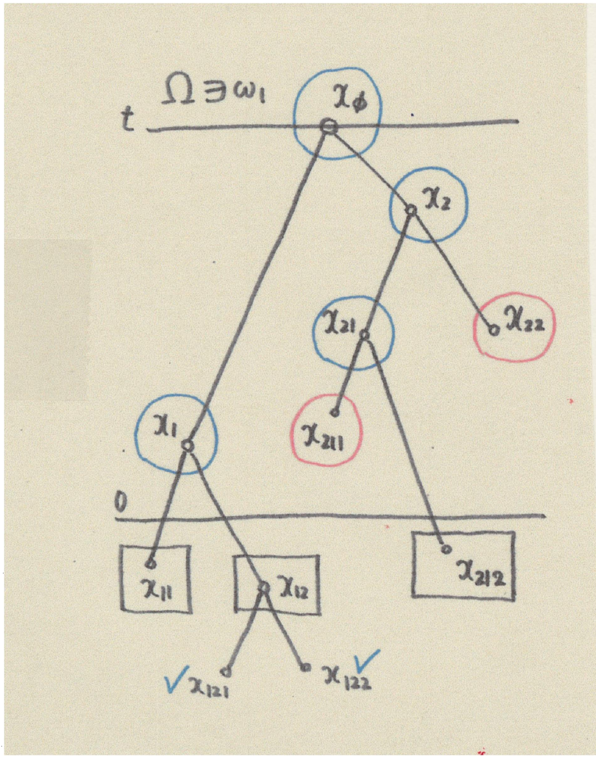


図 3: Classification of Nodes

Next we shall classify those nodes in the realized tree $\omega_1$. As a matter of fact, as to those two particles located at $x_{11}$ and $x_{12}$ with nodes of the level $|m| = \ell = 2$ accompanied by the pivoting node $x_1$, we can construct

$$\Xi^1_{11,12}(\omega_1) = \Theta^{11}(\omega_1) \bigstar_{[x_1]} \Theta^{12}(\omega_1)$$

by a star-product $u_0(x_{11}(\omega_1)) \bigstar_{[X_1]} u_0(x_{12}(\omega_1))$ in accordance with the rule, because both $m_1 = 11$ and $m_2 = 12$ lie in $N_-(\omega)$. As to the node $x_{21}$, how to construct $\Xi(\omega_1)$ is the almost same thing as described above. In fact, it goes similarly because $x_{211}$ lies in $N_+(\omega_1)$ and $x_{212}$ lies in $N_-(\omega_1)$. According to the rule, it follows that

$$\Theta^{211}(\omega_1) = \tilde{f}(t_{211}(\omega_1), x_{211}(\omega_1)) \quad \text{and} \quad \Theta^{212}(\omega_1) = u_0(x_{212}(\omega_1)),$$

hence $\Xi^{21}_{211,212}(\omega_1)$ is given by $\tilde{f}(t_{211}(\omega_1), x_{211}(\omega_1))\bigstar_{[x_{21}]}u_0(x_{212}(\omega_1))$, see Figure 3. Consequently, we obtain finally

$$M_{\bigstar}^{\langle u_0,f\rangle}(\omega_1) = \big(u_0(x_{11})\bigstar_{[x_1]}u_0(x_{12})\big)\bigstar_{[x_\phi]}\\ \Big\{\Big(\tilde{f}(t_{211}, x_{211})\bigstar_{[x_{21}]}u_0(x_{212})\Big)\bigstar_{[x_2]}\tilde{f}(t_{22}, x_{22})\Big\}. \tag{14}$$

□

## §5. Outline of proof: tree-based star-product functional as a solution

In this section we are first going to construct a $(U, F)$-weighted tree-based $*$-product functional $M_*^{\langle U,F\rangle}(\omega)$, which is indexed by the nodes $(x_m)$ of a binary tree. Here recall that $U = U(x)$ (resp. $F = F(t,x)$) is a non-negative measurable function on $D_0$ (resp. $\mathbb{R}_+ \times D_0$) respectively, and also that $F(\cdot, x) \in L^1(\mathbb{R}_+)$ for each $x$. Moreover, in construction of the functional, the product is taken as ordinary multiplication $*$ instead of the star-product $\bigstar$.

In what follows we shall give an outline of the proof of Theorem 1. We need the following technical lemma, which plays an essential role in the proof.

**Lemma 2.** *For $0 \leqslant t \leqslant T$ and $x \in D_0$, the function $V(t,x) = E_{t,x}[M_*^{\langle U,F\rangle}(\omega)]$ satisfies*

$$e^{\lambda t|x|^2}V(t,x) = U(x) + \int_0^t ds\frac{\lambda|x|^2}{2}e^{\lambda s|x|^2}\left\{F(s,x) + \int V(s,y)V(s,z)K_x(dy,dz)\right\}. \tag{15}$$

As a matter of fact, the mapping : $[0,T] \ni t \mapsto e^{\lambda|x|^2 t}V(t,x) \in \bar{\mathbb{R}}_+$ is non-decreasing, so that, it proves to be that

$$E_{t,x}[M_*^{\langle U,F\rangle}(\omega)] < \infty \tag{16}$$

holds for $\forall t \in [0,T]$ and $x \in E_c$, where $E_c$ is a measurable set on which the validity of $E_{t,x}[M_*^{\langle U,F\rangle}] < \infty$ may be kept. Another important aspect for the proof consists in establishment of the following $M_*$-control inequality. That is to say, we have

$$|M_{\bigstar}^{\langle u_0,f\rangle}(\omega)| \leqslant |M_*^{\langle U,F\rangle}(\omega)| \tag{17}$$

because of the validity of a simple inequality

$$|w\bigstar_{[x]}v| \leqslant |w|\cdot|v| \quad \text{for} \quad w,v \in \mathbb{C}^3 \quad \text{and} \quad x \in D_0.$$

On the other hand, it is derived that the space of solutions to (1) is formed by the condition

$$\int_0^T ds \int |u(s,y)|\cdot|u(s,z)|K_x(dy,dz) < \infty \qquad \text{for} \quad x \in E_c.$$

A similar discussion as above leads to

$$u(t,x) = E_{t,x}[M_\bigstar^{\langle u_0,f\rangle}(\omega)] = e^{-\lambda t|x|^2}u_0(x) + \int_0^t ds\ \lambda|x|^2 e^{-\lambda(t-s)|x|^2} \times$$
$$\times \frac{1}{2}\left\{\tilde{f}(s,x) + \iint E_{s,x_1}[M_\bigstar]\bigstar_{[x]}E_{s,x_2}[M_\bigstar]K_x(dx_1,dx_2)\right\}. \qquad (18)$$

Finally we can deduce that $u(t,x) = E_{t,x}[M_\bigstar^{\langle u_0,f\rangle}(\omega)]$ satisfies the integral equation (1), and this $u(t,x)$ is a solution lying in the space $\mathcal{D}$. Actually, $\mathcal{D}$ is a space of all functions $\varphi : \mathbb{R}_+ \times D_0 \to \mathbb{C}^3$, being continuous in $t$ and measurable such that

$$\int_0^\infty ds \int |p(s,x,y;\varphi)|K_x(dy,dz) < \infty, \quad \text{a.e.} - x.$$

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