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Non-ruin probability of an insurer under the Lundberg model

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Abstract. In the risk theory for an insurer, the Lundberg model is the most basic one, and non-ruin probability $\phi(u)$ is obtained by solving an integral-differential equation. But the equation was solved only when probability distributions $F_X(x)$ of claim amounts are Gamma distributions of particular parameters. In this report, we announce that $\phi(u)$ is obtained as an explicit function which is an exact solution if $F_X(x)$ is an archive data of an insurer and a good approximation even if $F_X(x)$ is any other probability distribution. We shall state more precious arguments elsewhere.

1 Introduction

"Non-ruin problem" is one of the main themes in the risk theory for a non-life insurer. Following to Lundberg [3, 1], we define the *surplus* U_t at time t of an insurer by

$$U_t = u + \kappa t - S_t, \tag{1.1}$$

where

$$\begin{cases} u \text{ is a non-negative constant denoting the initial surplus,} \\ \kappa \text{ is a positive constant denoting premium per a time,} \\ \{S_t\} \text{ is a compound Poisson process corresponding to insurances up to } t. \end{cases} \tag{1.2}$$

When the surplus U_t arrive at the domain $\{x : x < 0\}$ in some time t, the insurer ruins. "Non-ruin problem' is to calculate such probability $\phi(u)$ that the insurer does not ruin forever (see (2.2) for the precise definition).

In the Lundberg model (1.1), the claim amount $\{S_t\}$ is written as

$$S_t = \sum_{k=0}^{N_t} X_k \quad \text{with } X_0 \equiv 0 \tag{1.3}$$

where $\{N_t\}$ is a Poisson process with parameter $\gamma > 0$, and $X_1, X_2, \cdots$ are non-negative valued $i.i.d$ random variables with a distribution function F_X. Then the non-ruin probability $\phi(u)$ is obtained by a solution of the integral-differential equation (2.4). However, the equation was not solved without assuming that $F_X(x)$ is exponential probability distribution, Gamma ones of particular parameters, or their linear combinations.

To solve the integral-differential equation (2.4) for general $F_X(x)$, we use the Laplace transform as mentioned in §3. Using it, we prove the main result (Theorem 4.2) which is summarized as the non-ruin probability $\phi(u)$ is given by the explicit function (4.3) which is the exact solution of the integral-differential equation (2.4) if $F_X(x)$ is an archived data of an insurer and "a good approximation" for any other $F_X(x)$.

In this note, we only outline the proof of Theorem 4.2 and the precise proof will be stated elsewhere. We apply the theorem to some $F_X(dx)$'s in §5 and illustrate graphs of $\phi(u)$ in order to show relation between $\phi(u)$ and u .

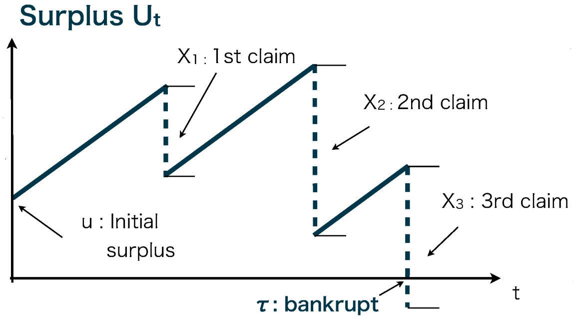
2 Non-ruin probability $\phi(u)$

At first, we define the *non-ruin probability* $\phi(u)$. Let U_t is the surplus process given by (1.1) and τ be *the first hitting time* of U_t to the domain $\{x : x < 0\}$, that is

(2.1) $$\tau(\omega) \equiv \begin{cases} \inf\{t > 0 : U_t(\omega) < 0\} \\ \infty \qquad \text{if the above } \{\cdots\} \text{ is empty} \end{cases}$$

If the initial surplus is u, then the non-ruin probability $\phi(u)$ is defined by

(2.2) $$\phi(u) = \mathbf{P}_u[\tau = \infty] \equiv \mathbf{P}[\tau = \infty | U_0 = u], \quad \text{(from now on, we use this notation).}$$



In (1.3), we assume that $\mathbf{E}[X_1] \equiv \mu$ exists through this report. Since $\mathbf{E}[N_t] = \gamma\, t$, the expected value of claims in a unit time is $\gamma\,\mu$, what derives that "the fair premium" of a unit time should be $\gamma\,\mu$.

But an insurer adds θ times of the "the fair premium" as "*a safety margin*" that is

(2.3) $$(1+\theta)\,\gamma\,\mu \equiv \kappa,$$

then κ comes to be an actual premium of a unit time and we arrive at the Lundberg model (1.1). For insurers, it is important to know non-ruin probability $\phi(u)$, especially to obtain the functional relation between initial surplus u, safety margin θ in (2.3), and $\phi(u)$.

By using Markov property of $\{U_t\}$, we may derive that the non-ruin probability $\phi(u)$ is a continuous function of $u \geq 0$ and that it is a solution of the following *integral-differential equation**1.

Proposition 2.1 (Integral-differential equation of the Lundberg model). (i) If $\phi(u) \in C^1$ at a neighborhood of u, then it fulfills

(2.4) $$\phi'(u) = \rho\,\phi(u) - \rho \int_0^u \phi(u-x)\,dF_X(x), \quad \rho \equiv \frac{\gamma}{\kappa} = \frac{1}{(1+\theta)\,\mu}.$$

*1 The proof of (2.5) may be found in Iwasawa's book [2].

(ii) $\phi(u)$ satisfies the following boundary conditions.

$$\phi(\infty)=1, \quad \phi(0)=\frac{\theta}{1+\theta}, \quad \phi'(0)=\frac{\theta}{(1+\theta)^2\,\mu}. \quad \diamond \tag{2.5}$$

3 The integral-differential equation and Laplace transform

We use Laplace transform in order to solve (2.4). Since $\phi(u)$ is bounded and continuous with respect to u, there exists its Laplace transform

$$\widehat{\phi}(\lambda) \equiv \int_0^\infty du\; e^{-\lambda u}\,\phi(u) \quad \text{if } \Re(\lambda)>0, \tag{3.1}$$

and from (2.4) we have

$$\widehat{\phi}(\lambda)=\frac{\phi(0)}{\lambda-\rho+\rho\,\widehat{F_X}(\lambda)} \quad \text{with } \widehat{F_X}(\lambda)\text{, the Laplace transform of } dF_X(x). \tag{3.2}$$

Since the abscissa of convergence in (3.1) is 0,

$$\phi(u)=\lim_{R\to\infty}\frac{1}{2\pi i}\int_{c-iR}^{c+iR} d\lambda\;\frac{\phi(0)\;e^{\lambda u}}{\lambda-\rho+\rho\,\widehat{F_X}(\lambda)} \tag{3.3}$$

for arbitrary*[2] $c>0$. If we can achieve the above calculation, $\phi(u)$ will be obtained. However it is not easy without assuming that $d\,F_X(\lambda)$ is a Gamma distribution of a particular parameter (including an exponential distribution).

Here we present a useful Laplace transform, which will play an essential role in this report.

Lemma 3.1. $I_A(\,\cdot\,)$ denotes the indicator function of a set A. Let n be non-negative integer, $b\geq 0$, and c be a real constant. If $\lambda>\max\{c,0\}$, then

$$\frac{e^{-\lambda b}}{(\lambda-c)^{n+1}}=\int_0^\infty du\; e^{-\lambda u}\,\Big\{e^{cu}\,(u-b)^n\,I_{\{u>b\}}(u)\;\frac{e^{-bc}}{n!}\Big\}. \quad \diamond \tag{3.4}$$

From now on, we denote by $\delta_a(dx)$ the delta distribution with a point mass at the point a.

Example 3.2 (A case of constant claim amount). Suppose that all claim amount X_k's equal to a constant $a>0$, that is $dF_X(x)=\delta_a(dx)$, and $\mathbf{E}[X_1]=\mu=a$. Then

$$\phi(u)=\phi(0)\;e^{\rho u}\sum_{k=0}^{\infty}(-1)^k\,\frac{\rho^k}{k!}\;e^{-k\,a\,\rho}\,(u-k\,a)^k\;I_{\{u:\,u>k\,a\}}(u), \tag{3.5}$$

where note that the above summation is finite.

Proof of Example 3.2. In this case $\widehat{F_X}(\lambda)=e^{-\lambda a}$, and (3.2) implies that

$$\widehat{\phi}(\lambda)=\frac{\phi(0)}{\lambda-\rho+\rho\,\exp\{-\lambda a\}}. \tag{3.6}$$

*2 Note that c may be take as large as need.

In the Laplace inversion formula (3.3), we may take c as large as we need, we can assume that

$$|\frac{\rho\,\exp\{-\lambda\,a\}}{(\lambda-\rho)}| < 1 \quad \text{in (3.3)}. \tag{3.7}$$

Generally if $|x| < 1$, then we have

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-1)^k x^k + \cdots . \tag{3.8}$$

Owing to (3.7) and (3.8), we may expand (3.6) into

$$\widehat{\phi}(\lambda) = \phi(0) \sum_{k=0}^{\infty} (-1)^k \frac{\rho^k}{(\lambda-\rho)^{k+1}}\, e^{-k\lambda\,a}.$$

The above series converges uniformly with respect to λ on the integration path of (3.3). So we may calculate (3.3) term-wise, and Lemma 3.1 implies that

$$\begin{aligned} \frac{\rho^k}{(\lambda-\rho)^{k+1}}\, e^{-k\,\lambda\,a} &= \int_0^\infty du\; e^{-\lambda u} \times \\ &\times \Big\{ e^{\rho u}\, \frac{\rho^k}{k\,!}\, e^{-k\,a\,\rho}\, (u-k\,a)^k\, I_{\{u>k\,a\}}(u) \Big\}, \quad k = 0, 1, \cdots . \end{aligned} \tag{3.9}$$

Since each $\{\quad\}$ of (3.9) is continuous and (3.5) converges uniformly with respect to u in a compact set, we obtain (3.5) by the uniqueness of Laplace transform. □

Now we can present much simpler proof of the following well-known result*3 in the risk theory by using the analogous argument as Example 3.2.

Example 3.3 (Mikosch [4]). Let $\widetilde{X}_k, k = 1, 2, \cdots$, be *i.i.d* random variables following to *the integrated tail distribution* $dF_{\widetilde{X}}(x)$ that is

$$dF_{\widetilde{X}}(x) = \frac{\mathbf{P}\{X_k > x\}}{\mathbf{E}[X_k]} dx, \quad k = 1, 2, \ldots, \tag{3.10}$$

where X_k's are claim amounts in (1.3). Then it holds that

$$\phi(u) = \phi(0) \left[1 + \sum_{n=1}^{\infty} (1+\theta)^{-n} \mathbf{P}\{\widetilde{X}_1 + \cdots + \widetilde{X}_n \le u\} \right] \quad \text{for} \quad u > 0. \quad \diamond \tag{3.11}$$

Remark 3.4. Compared to our main theorem 4.2, summation in (3.11) is essentially infinite, and it is not easy to calculate integrated tail distribution (3.10) in general, cf. Remark 4.3.

Proof of Example 3.3. By (3.7) and (3.8), we expand (3.2) into

$$\begin{aligned} \widehat{\phi}(\lambda) &= \frac{\phi(0)}{\lambda} \frac{1}{1 - \rho\Big(1 - \mathbf{E}[\exp\{-\lambda X\}]\Big)/\lambda} \\ &= \frac{\phi(0)}{\lambda} \Big\{ 1 + \sum_{n=1}^{\infty} \rho^n \Big(\mathbf{E}\,\big[\frac{1-\exp\{-\lambda X\}}{\lambda}\big]\Big)^n \Big\}. \end{aligned} \tag{3.12}$$

*3 Example 3.3 and its proof are owing to Takaoka [5].

Here note that
$$\frac{1-\exp\{\lambda X\}}{\lambda}=\int_0^X dx\ \exp\{-\lambda x\}$$
and that X_k's are mutually independent. From (3.12) we see that

$$(3.13)\qquad \widehat{\phi}(u)=\phi(0)\left(\frac{1}{\lambda}+\sum_{n=1}^{\infty}\rho^n\ \mathbf{E}\Big[\int_0^{X_1}dx_1\cdots\int_0^{X_n}dx_n\ \frac{\exp\{-\lambda(x_1+\cdots+x_n)\}}{\lambda}\Big]\right).$$

By Lemma 3.1, we know that
$$\frac{\exp\{-\lambda(x_1+\cdots+x_n)\}}{\lambda}=\int_0^\infty du\ e^{-\lambda u}\ I_{\{u:u>x_1+\cdots+x_n\}}(u).$$
So calculating the Laplace inversion in (3.13), we have
$$\begin{aligned}\phi(u)&=\phi(0)\ \Big(1+\sum_{n=1}^{\infty}\rho^n\ \mathbf{E}\Big[\int_0^{X_1}dx_1\cdots\int_0^{X_n}dx_n\ I_{\{u:u>x_1+\cdots+x_n\}}(u)\Big]\Big)\\ &=\phi(0)\ \Big(1+\sum_{n=1}^{\infty}\rho^n\int_0^\infty dx_1\cdots\int_0^\infty dx_n\Big\{\prod_{j=1}^{n}\mathbf{P}[x_j<X_j]\Big\}\ I_{\{u:u>x_1+\cdots+x_n\}}(u)\Big)\\ &=\phi(0)\ \Big(1+\sum_{n=1}^{\infty}\{\rho\ \mu^n\ \mathbf{P}\left[\widetilde{X}_1+\cdots+\widetilde{X}_n\le u\right]\Big).\quad\square\end{aligned}$$

4 Non-ruin probability under general setting

In this section, we will obtain the approximation of non-ruin probability $\phi(u)$ for arbitrary claim amount distribution $F_X(x)$.

Fix an arbitrary non-negative integer M, and consider sequences of positive numbers such that

$$(4.1)\qquad \begin{aligned}&0<a_1<a_2<\cdots<a_M,\\ &0\le p_j<1,\ j=1,\cdots,M,\quad\text{with}\quad p_1+\cdots+p_j+\cdots+p_M=1,\end{aligned}$$

Then we define a mixed delta distribution

$$(4.2)\qquad dF_X(x)\equiv\sum_{j=1}^{M}p_j\ \delta_{a_j}(dx).$$

Firstly we introduce necessary notations to state the main result.

Definition 4.1. (i) When $\{p_1,\cdots,p_M\}$ and $\{a_1,\cdots,a_m\}$ be a M non-negative numbers as in (4.1), we regard them as M-dimensional vectors, i.e.
$$\mathbf{p}\equiv(p_1,\cdots,p_M),\quad \mathbf{a}\equiv(a_1,\cdots,a_M).$$
(ii) Let $\mathbf{k}\equiv(k_1,k_2,\cdots,k_M)$ be a M-dimensional vector whose components are *non-negative integers*. Then we denote the whole of them by $\mathbb{Z}_+^M$ and use the following notations:
$$|\mathbf{k}|\equiv\sum_{j=1}^{M}k_j,\qquad \mathbf{k}!\equiv\prod_{j=1}^{M}k_j!.\ \text{ where we set } 0!\equiv 1,$$
and an inner product $\mathbf{k}\cdot\mathbf{a}=\sum_{j=1}^{M}k_j\,a_j$ for a M-dimensional vector $\mathbf{a}$.

(iii) Finally if $\mathbf{p} = (p_1, \cdots, p_M)$ is a M-dimensional vector as in (4.1), then we define

$$\mathbf{p}^{\mathbf{k}} \equiv \prod_{j=1}^{M} {p_j}^{k_j}, \quad \text{where} \quad 0^{\ell} \equiv \begin{cases} 1 & \text{if} \quad \ell = 0 \\ 0 & \text{if} \quad \ell \geq 1. \end{cases} \quad \diamond$$

The main result

Theorem 4.2. Let F_X be the mixture delta distribution (4.2). Then the following (4.3) is the solution of the integral-differential equation (2.4):

$$(4.3) \qquad \phi(u) = \phi(0)\, e^{\rho\, u} \sum_{\mathbf{k} \in \mathbb{Z}_+^M} (-1)^{|\mathbf{k}|} \frac{\mathbf{p}^{\mathbf{k}}\, \rho^{|\mathbf{k}|}}{\mathbf{k}!}\, e^{-\rho\, \mathbf{a} \cdot \mathbf{k}} \left(u - \mathbf{a} \cdot \mathbf{k}\right)^{|\mathbf{k}|} I_{\{u:\, u > \mathbf{a} \cdot \mathbf{k}\}}(u),$$

where $\mu = \mathbf{E}[X_1]$ and $\rho = \dfrac{1}{(1+\theta)\,\mu}$.

Remark 4.3. (i) Note that the summation in (4.3) is finite for each u.

(ii) Usually an insurer collect their archived data as the form of (4.2). So we assert that (4.3) is the non-ruin probability based on real data.

(iii) As far as we know, non-ruin probability $\phi(u)$ does not obtained in explicit form unless $F_X(x)$ is gamma distributions of particular parameters (including exponential dones), or their mixed ones. However we can approximate any distribution $G_X(x)$ of claim amount X, by mixed delta distribution $F_X(x)$ of (4.2), and the latter gives an explicit form of $\phi(u)$.

For instance, choosing suitable non-negative integer M and positive constants $a_1, \cdots, a_M$, we set: Define $a_0 \equiv 0$, and

$$p_j \equiv \int_{a_{j-1}}^{a_j} dG_X(x) \quad \text{if } j = 1, \cdots, M-1, \quad p_M \equiv \int_{a_{M-1}}^{\infty} dG_X(x),$$

$$d\,F_X(x) \equiv \sum_{j=1}^{M} p_j\, \delta_{a_j}(dx) \quad \text{with} \quad p_1 + p_2 + \cdots + p_M = 1,$$

which is a good approximation for any G_X.

Sketch of the proof of Theorem 4.2. The Laplace transform of (4.2) is $\widehat{F_X}(\lambda) = \sum_{j=1}^{M} p_j \exp\{-a_j\,\lambda\}$. Then we have

$$\widehat{\phi}(\lambda) = \frac{\phi(0)}{\lambda - \rho + \rho\,\widehat{F_X}(\lambda)} = \frac{\phi(0)}{\lambda - \rho + \rho \sum_{j=1}^{M} p_j \exp\{-a_j\,\lambda\}}$$

$$= \frac{\phi(0)}{\lambda - \rho} \left\{ 1 + \frac{p_1\, e^{-a_1 \lambda} + p_2\, e^{-a_2 \lambda} + \cdots + p_M\, e^{-a_M \lambda}}{\lambda - \rho} \right\}^{-1}.$$

In Laplace inversion formula (3.3), we may take $c = \Re\lambda > 0$ as large as we need. If c is large enough, then

$$\left| \frac{p_1\, e^{-a\,\lambda} + p_2\, e^{-2\,a\,\lambda} + \cdots + p_M\, e^{-M\,a\,\lambda}}{\lambda - \rho} \right| < 1 \quad \text{for } \lambda = c + iy,$$

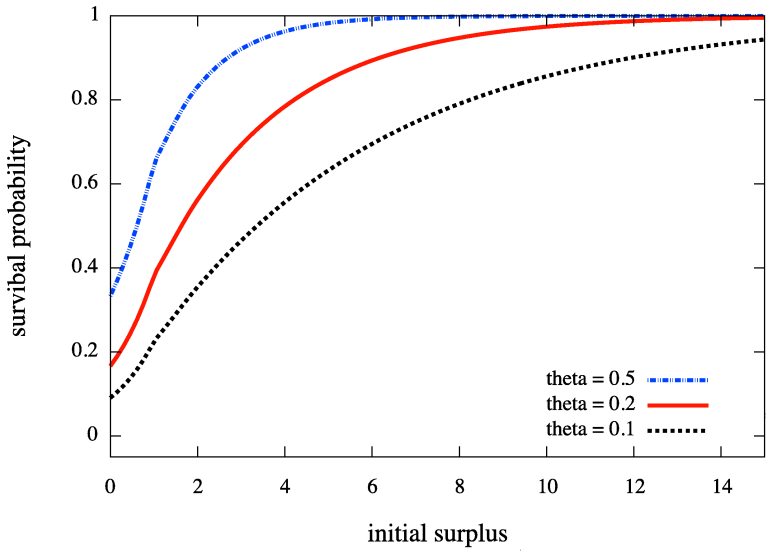
where y is real. Now by the same arguments as in Example 3.2, we obtain the theorem. $\square$.

5 Examples

In this section, we apply Theorem 4.2 in order to calculate non-ruin probabilities for some claim amount distributions.

Example 5.1 (Case of constant claim). Let $dF_X(x) = \delta_1(dx)$, and we calculated $\phi(u)$ given by (4.3) for three different values of safety margin θ.

From the figure, we see that it should need as much initial surplus about 8 times and 15 times of the unit claim amount for $\theta = 0.2$ and $\theta = 0.1$ respectively.



Example 5.2 (Gamma distribution case). When F_X is gamma distribution $\Gamma(2, 2/\mu)$, that is

$$dF_X(x) = (\frac{2}{\mu})^2 \, x \, e^{-2x/\mu} \, dx,$$

non-ruin probability $\phi(u)$ also has simple form[*4]:

$$\phi(u) = Ae^{-Cu} + Be^{-Du} + 1. \tag{5.1}$$

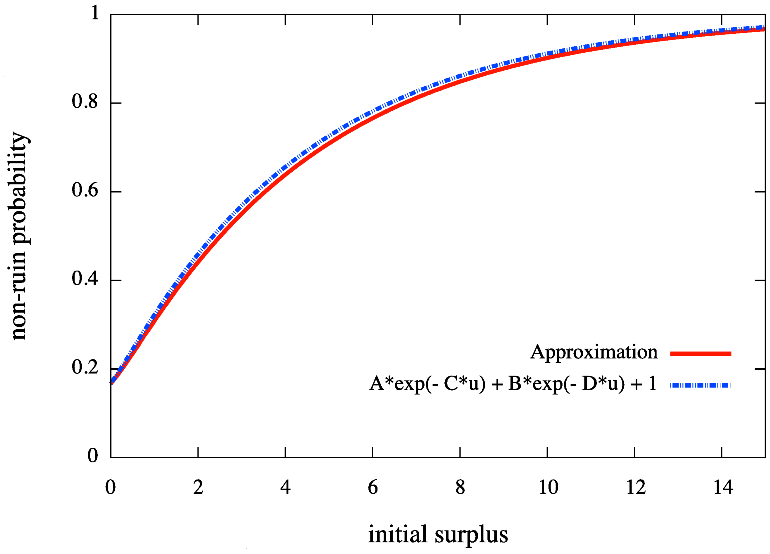
where

$$C \equiv \frac{-3\mu - 4\theta\mu - \sqrt{9+8\theta}\mu}{2\mu^2\,(1+\theta)}, \quad D \equiv \frac{-3\mu - 4\theta\mu + \sqrt{9+8\theta}\mu}{2\mu^2\,(1+\theta)},$$

$$A \equiv \frac{D\,\mu^2 - \theta}{(C-D)(1+\theta)\,\mu^2}, \quad B \equiv -\frac{C\,\mu^2 - \theta}{(C-D)(1+\theta)\,\mu^2}.$$

Similar to the previous example, we put $\mu = 1$, $\theta = 0.2$, $M = 9$ and $a =$(the 99$^{\text{th}}$ percentile point of $\Gamma(2, 2/\mu)/9$. We compare our approximation (4.3) and the exact solution (5.1) in the following figure.

[*4] See Iwasawa' book [2].



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