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SOME RESULTS ON SET-VALUED STOCHASTIC INTEGRALS WITH RESPECT TO POISSON JUMP IN AN M-TYPE 2 BANACH SPACE

JINPING ZHANG*, ITARU MITOMA, AND YOSHIAKI OKAZAKI

1. Introduction

Probability theory is an important tool of modeling randomness in a practical problem. But besides randomness, in the real world, there exists other kind of uncertainties such as impreciseness or vagueness. Set-valued functions are employed to model the impreciseness in applied field such as in Economics, control theory (see for example [1]). Integrals of set-valued functions have been received much attention with widespread applications, see for example [2, 7, 9, 10] etc. Recently, stochastic integrals for set-valued stochastic processes with respect to the Brownian motion and martingales have been received much attention, e.g. see [12, 13, 18, 23, 32, 37]. Correspondingly, the set-valued stochastic differential equations are studied, e.g. see [23, 25, 33, 34, 35, 36]. Michta (2011) [22] extended the integrator to a larger class: semimartingales. But the integrably boundedness of the corresponding set-valued stochastic integrals are not obtained since the semimartingales may not be of finite variation. In such cases, the set-valued stochastic integrals may not be well defined as Ogura pointed out [25].

The Poisson stochastic processes are special. They play important roles both in the random mathematics (c.f. [11, 8, 17]) and in applied fields, for example, in the financial mathematics [17]. If the characteristic measure ν of a stationary Poisson process $\mathbf{p}$ is finite, then both of the Poisson random measure $N(dsdz)$ (where $z \in Z$, the state space of $\mathbf{p}$) and the compensated Poisson random measure $\tilde{N}(dsdz)$ are of finite variation a.s. We will give some results (without giving proof since the page limitation) on the set-valued stochastic integrals with respect to the Poisson random measure $N(dsdz)$, $\tilde{N}(dsdz)$. For the detail proof, the reader can refer to [31, 38]. For example, the stochastic integrals for set-valued $\mathscr{S}$-predictable (see Definition 3.2) processes with respect to $N(dsdz)$ and $\tilde{N}(dsdz)$ are L^2-integrably bounded. For Brownian or Martingale integrator with continuous part, the integrable boundedness are not obtained until now. Furthermore, if the σ-algebra $\mathcal{F}$ is separable, then the integral $\{I_t(F)\}$ of convex set-valued stochastic process will not become a set-valued martingale, which is very different from single valued case. We would like to pointed out that there is a gap in the proof of Theorem 3.7 in [31] about the set-valued martingale property of set-valued stochastic integral with respect to the compensated Poisson measure, which is corrected and proven in [38].

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This paper is organized as follows: In Section 2 we give the notations and the preliminaries in the set-valued theory. Section 3 is on the definitions and results of stochastic integrals for set-valued $\mathscr{S}$-predictable processes with respect to $N(dsdz)$ and $\tilde{N}(dsdz)$.

2. Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, $\{\mathcal{F}_t\}_{t\geq 0}$ a filtration satisfying the usual conditions, that is: $\mathcal{F}_0$ includes all P-null sets in $\mathcal{F}$, the filtration is non-decreasing and right continuous. Let $\mathcal{B}(E)$ be the Borel field of a topological space E, $(X, \|\cdot\|)$ a separable Banach space equipped with the norm $\|\cdot\|$ and $\mathbf{K}(\mathbf{X})$ (resp. $\mathbf{K_b}(\mathbf{X})$, $\mathbf{K_c}(\mathbf{X})$) the family of all nonempty closed (resp. closed bounded, closed convex) subsets of X. Let $1 \leq p < +\infty$ and $L^p(\Omega, \mathcal{F}, P; X)$ (denoted briefly by $L^p(\Omega; X)$) be the Banach space of equivalence classes of X-valued $\mathcal{F}$-measurable functions $f : \Omega \to X$ such that the norm

$$\|f\|_p = \left\{ \int_\Omega \|f(\omega)\|^p dP \right\}^{1/p}$$

is finite. An X-valued function f is called *L^p-integrable* if $f \in L^p(\Omega; X)$.

A set-valued function $F : \Omega \to \mathbf{K}(X)$ is said to be *measurable* if for any open set $O \subset X$, the inverse $F^{-1}(O) := \{\omega \in \Omega : F(\omega) \cap O \neq \emptyset\}$ belongs to $\mathcal{F}$. Such a function F is called a *set-valued random variable.* Let $\mathcal{M}(\Omega, \mathcal{F}, P; \mathbf{K}(X))$ be the family of all set-valued random variables, which is briefly denoted by $\mathcal{M}(\Omega; \mathbf{K}(X))$.

For any open subset $O \subset X$, set

$$Z_O := \{E \in \mathbf{K}(X) : E \cap O \neq \emptyset\},$$

$$\mathcal{C} := \{Z_O : O \subset X,\ O \textit{ is open}\},$$

and let $\sigma(\mathcal{C})$ be the σ-algebra generated by $\mathcal{C}$.

A set-valued function $F : \Omega \to \mathbf{K}(\mathbf{X})$ is measurable if and only if F is $\mathcal{F}/\sigma(\mathcal{C})$-measurable.

For $A, B \in 2^X$ (the power set of X), $H(A, B) \geq 0$ is defined by

$$H(A, B) := \max\{\sup_{x\in A} \inf_{y\in B} ||x - y||, \sup_{y\in B} \inf_{x\in A} ||x - y||\}.$$

$H(A, B)$ for $A, B \in \mathbf{K}_b(X)$ is called the *Hausdorff metric.* It is well-known that $\mathbf{K_b}(\mathbf{X})$ equipped with the H-metric denoted by $((\mathbf{K}_b(X), H))$ is a complete metric space.

The following results are also well-known. (see e.g. [9], [19], [24]).

Proposition 2.1. (i) *For $A, B, C, D \in \mathbf{K}(\mathbf{X})$, we have*

$$H(A + B, C + D) \leq H(A, C) + H(B, D),$$

$$H(A \oplus B, C \oplus D) = H(A + B, C + D),$$

where $A \oplus B := cl\{a + b; a \in A,\ b \in B\}$.
(ii) *For $A, B \in \mathbf{K}(\mathbf{X})$, $\mu \in \mathbb{R}$, we have*

$$H(\mu A, \mu B) = |\mu| H(A, B).$$

For $F \in \mathcal{M}(\Omega, \mathbf{K}(X))$, the family of all L^p-integrable selections is defined by

$$S_F^p(\mathcal{F}) := \{f \in L^p(\Omega, \mathcal{F}, P; X) : f(\omega) \in F(\omega) \ a.s.\}.$$

In the following, $S_F^p(\mathcal{F})$ is denoted briefly by S_F^p. If S_F^p is nonempty, F is said to be L^p-*integrable.* F is called L^p*-integrably bounded* if there exits a function $h \in L^p(\Omega, \mathcal{F}, P; \mathbb{R})$ such that $\|x\| \leq h(\omega)$ for any x and ω with $x \in F(\omega)$. It is equivalent to that $\|F\|_{\mathbf{K}} \in L^p(\Omega; \mathbb{R})$, where $\|F(\omega)\|_{\mathbf{K}} := \sup_{a \in F(\omega)} \|a\|$. The family of all measurable $\mathbf{K(X)}$-valued L^p-integrably bounded functions is denoted by $L^p(\Omega, \mathcal{F}, P; \mathbf{K}(X))$. Write it for brevity as $L^p(\Omega; \mathbf{K}(X))$.

The *integral (or expectation)* of a set-valued random variable F was defined by Aumann in 1965 ([2]):

$$E[F] := \{E[f] : f \in S_F^1\}.$$

PROPOSITION 2.2. *([35]) Let $F \in \mathcal{M}(\Omega; X)$, $1 \leq p < +\infty$. Then F is L^p-integrably bounded if and only if S_F^p is nonempty and bounded in $L^p(\Omega; X)$.*

Let $\mathbb{R}_+$ be the set of all nonnegative real numbers and $\mathcal{B}_+ := \mathcal{B}(\mathbb{R}_+)$. $\mathbb{N}$ denotes the set of natural numbers. An X-valued stochastic process $f = \{f_t : t \geq 0\}$ (or denoted by $f = \{f(t) : t \geq 0\}$) is defined as a function $f : \mathbb{R}_+ \times \Omega \longrightarrow X$ with the $\mathcal{F}$-measurable section f_t, for $t \geq 0$. We say f is *measurable* if f is $\mathcal{B}_+ \otimes \mathcal{F}$-measurable. The process $f = \{f_t : t \geq 0\}$ is called $\mathcal{F}_t$*-adapted* if f_t is $\mathcal{F}_t$-measurable for every $t \geq 0$. $f = \{f_t : t \geq 0\}$ is called *predictable* if it is $\mathcal{P}$-measurable, where $\mathcal{P}$ is the σ-algebra generated by all left continuous and $\mathcal{F}_t$-adapted stochastic processes.

In a fashion similar to the X-valued stochastic process, a *set-valued stochastic process* $F = \{F_t : t \geq 0\}$ is defined as a set-valued function $F : \mathbb{R}_+ \times \Omega \longrightarrow \mathbf{K(X)}$ with $\mathcal{F}$-measurable section F_t for $t \geq 0$. It is called *measurable* if it is $\mathcal{B}_+ \otimes \mathcal{F}$-measurable, and $\mathcal{F}_t$*-adapted* if for any fixed t, $F_t(\cdot)$ is $\mathcal{F}_t$-measurable. $F = \{F_t : t \geq 0\}$ is called *predictable* if it is $\mathcal{P}$-measurable.

DEFINITION 2.3. (see [9]) An integrable bounded convex set-valued $\mathcal{F}_t$-adapted stochastic process $\{F_t, \mathcal{F}_t : t \geq 0\}$ is called a *set-valued* $\mathcal{F}_t$*-martingale* if for any $0 \leq s \leq t$ it holds that $E[F_t|\mathcal{F}_s] = F_s$ in the sense of $S^1_{E[F_t|\mathcal{F}_s]}(\mathcal{F}_s) = S^1_{F_s}(\mathcal{F}_s)$.

It is called a *set-valued submartingale (supermartingale)* if for any $0 \leq s \leq t$, $E[F_t|\mathcal{F}_s] \supset F_s$ (resp. $E[F_t|\mathcal{F}_s] \subset F_s$) in the sense of $S^1_{E[F_t|\mathcal{F}_s]}(\mathcal{F}_s) \supset S^1_{F_s}(\mathcal{F}_s)$ (resp. $S^1_{E[F_t|\mathcal{F}_s]}(\mathcal{F}_s) \subset S^1_{F_s}(\mathcal{F}_s)$).

3. STOCHASTIC INTEGRALS WITH RESPECT TO POISSON POINT PROCESSES

3.1. Single Valued Stochastic Integrals w.r.t. Poisson Point Processes. Let X be a separable Banach space and Z be another separable Banach space with σ-algebra $\mathcal{B}(Z)$. A *point function* $\mathbf{p}$ on Z means a mapping $\mathbf{p} : \mathbf{D_p} \to Z$, where the domain $\mathbf{D_p}$ is a countable subset of $[0, T]$. $\mathbf{p}$ defines a counting measure $N_{\mathbf{p}}(dtdz)$ on $[0, T] \times Z$ (with the product σ-algebra $\mathcal{B}([0, T]) \otimes \mathcal{B}(Z)$) by

$$N_{\mathbf{p}}((0, t], U) : = \#\{\tau \in \mathbf{D_p} : \tau \leq t, \mathbf{p}(\tau) \in U\}, \quad t \in (0, T], \ U \in \mathcal{B}(Z). \tag{3.1}$$

For $0 \leq s < t \leq T$,

$$N_{\mathbf{p}}((s, t], U) := N_{\mathbf{p}}((0, t], U) - N_{\mathbf{p}}((0, s], U). \tag{3.2}$$

In the following, we also write $N_{\mathbf{p}}((0,t],U)$ as $N_{\mathbf{p}}(t,U)$.

A *point process* is obtained by randomizing the notion of point functions. If there is a continuous $\mathcal{F}_t$-adapted increasing process $\hat{N}_{\mathbf{p}}$ such that for $U \in \mathcal{B}(Z)$ and $t \in [0,T]$, $\tilde{N}_{\mathbf{p}}(t,U) := N_{\mathbf{p}}(t,U) - \hat{N}_{\mathbf{p}}(t,U)$ is an $\mathcal{F}_t$-martingale, then the random measure $\{\hat{N}_{\mathbf{p}}(t,U)\}$ is called the *compensator* of the point process $\mathbf{p}$ (or $\{N_{\mathbf{p}}(t,U)\}$) and the process $\{\tilde{N}_{\mathbf{p}}(t,U)\}$ is called the *compensated* point process.

A point process $\mathbf{p}$ is called the *Poisson Point Process* if $N_{\mathbf{p}}(dtdz)$ is a Poisson random measure on $[0,T]\times Z$. A Poisson point process is stationary if and only if its intensity measure $\nu_{\mathbf{p}}(dtdz) = E[N_{\mathbf{p}}(dtdz)]$ is of the form

$$\nu_{\mathbf{p}}(dtdz) = dt\nu(dz) \tag{3.3}$$

for some measure $\nu(dz)$ on $(Z,\mathcal{B}(Z))$. $\nu(dz)$ is called the *characteristic measure of* $\mathbf{p}$.

Let ν be a σ- finite measure on $(Z,\mathcal{B}(Z))$, (i.e. there exists $U_i \in \mathcal{B}(Z), i \in \mathbb{N}$, pairwise disjoint such that $\nu(U_i) < \infty$ for all $i \in \mathbb{N}$ and $Z = \cup_{i=1}^{\infty} U_i$), $\mathbf{p} = (\mathbf{p}_t)$ be the $\mathcal{F}_t$-adapted stationary Poisson point process on Z with the characteristic measure ν such that the compensator $\hat{N}_{\mathbf{p}}(t,U) = E[N_{\mathbf{p}}(t,U)] = t\nu(U)$ (non-random).

The above definitions and notations of Poisson point processes come from [11] and [30].

For convenience, we will omit the subscript $\mathbf{p}$ in the above notations.

PROPOSITION 3.1. ([31]) *Assume $\nu(Z)$ is finite. Then for any $U \in \mathcal{B}(Z)$, both $\{N(t,U), t \in [0,T]\}$ and $\{\tilde{N}(t,U), t \in [0,T]\}$ are stochastic processes with finite variation a.s.*

For convenience, from now on, we suppose ν is a finite measure in the measurable space $(Z,\mathcal{B}(Z))$.

DEFINITION 3.2. An X-valued mapping $f(t,z,\omega)$ defined on $[0,T] \times Z \times \Omega$ is called $\mathscr{S}$-predictable if the mapping $(t,z,\omega) \to f(t,z,\omega)$ is $\mathscr{S}/\mathcal{B}(X)$-measurable, where $\mathscr{S}$ is the smallest σ-algebra on $[0,T]\times Z\times\Omega$ with respect to which all mappings $g : [0,T]\times Z\times\Omega \to X$ satisfying (i) and (ii) below are measurable:

(i) for each $t \in [0,T]$, the mapping $(z,\omega) \to g(t,z,\omega)$ is $\mathcal{B}(Z)\otimes\mathcal{F}_t$-measurable;

(ii) for each $(z,\omega) \in Z\times\Omega$, the mapping $t \to g(t,z,\omega)$ is left continuous.

REMARK 3.3. (see e.g. [30]) $\mathscr{S} = \mathcal{P}\otimes\mathcal{B}(Z)$, where $\mathcal{P}$ denotes the σ-field on $[0,t]\times\Omega$ generated by all left continuous and $\mathcal{F}_t$-adapted processes.

Set

$$\mathscr{L} = \Big\{ f(t,z,\omega) : f \textit{ is } \mathscr{S}\textit{-predictable and } E\Big[\int_0^T \int_Z \|f(t,z,\omega)\|^2 \nu(dz)dt\Big] < \infty \Big\}$$

equipped with the norm

$$\|f\|_{\mathscr{L}} := \Big(E\Big[\int_0^T \int_Z \|f(t,z,\omega)\|^2 \nu(dz)dt\Big]\Big)^{1/2}.$$

Let $\mathbb{S}$ be the subspace of those $f \in \mathscr{L}$ for which there exists a partition $0 = t_0 < t_1 < \cdots < t_n = T$ of of $[0,T]$ such that

$$f(t,z,\omega) = f(0,z,\omega)\chi_{\{0\}}(t) + \sum_{i=1}^{n} \chi_{(t_{i-1},t_i]}(t) f(t_{i-1},z,\omega).$$

Let f be in $\mathbb{S}$ and

$$f(t,z,\omega) = f(0,z,\omega)\chi_{\{0\}}(t) + \sum_{i=1}^{n} \chi_{(t_{i-1},t_i]}(t) f(t_{i-1},z,\omega), \tag{3.4}$$

where $0 = t_0 < t_1 < \cdots < t_n = T$ is a partition of $[0,T]$. Define

$$\begin{aligned} J_T(f) &= \int_0^{T+} \int_Z f(s-,z,\omega) N(dtdz) \\ &:= \sum_{i=1}^{n} \int_Z f(t_{i-1},z,\omega) N((t_{i-1},t_i],dz), \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} I_T(f) &= \int_0^{T+} \int_Z f(s-,z,\omega) \tilde{N}(dtdz) \\ &:= \sum_{i=1}^{n} \int_Z f(t_{i-1},z,\omega) \tilde{N}((t_{i-1},t_i],dz), \end{aligned} \tag{3.6}$$

where $\int_Z f(t_{i-1},z,\omega)N((t_{i-1},t_i],dz)$ and $\int_Z f(t_{i-1},z,\omega)\tilde{N}((t_{i-1},t_i]dz)$ are the Bochner integrals. The notation '$\int_0^{T+}$ ' means ' $\int_{(0,T]}$' .

For any integer $0 \le k \le n$, let

$$M_k = \sum_{i=1}^{k} \int_Z f(t_{i-1},z,\omega) \tilde{N}((t_{i-1},t_i],dz)$$

then M_k is $\mathcal{F}_{t_k}$-measurable, $E[M_k] = 0$, $E[I_T(f)] = E[M_n] = 0$ and

$$\begin{aligned} E[M_k|\mathcal{F}_{t_{k-1}}] &= E[(M_{k-1} + \int_Z f(t_{k-1},z\omega)\tilde{N}((t_{k-1},t_k],dz)|\mathcal{F}_{t_{k-1}}] \\ &= M_{k-1} + E[\int_Z f(t_{k-1},z,\omega)\tilde{N}((t_{k-1},t_k],dz)|\mathcal{F}_{t_{k-1}}] \\ &= M_{k-1} + \int_Z f(t_{k-1},z,\omega) E[\tilde{N}((t_{k-1},t_k],dz)] = M_{k-1}. \end{aligned} \tag{3.7}$$

For any $t \in (0,T]$, define

$$\begin{aligned} J_t(f) &= \int_0^{t+} \int_Z f(s-,z,\omega) N(dzds) \\ &:= \sum_{i=1}^{n} \int_Z f(t_{i-1},z,\omega) N((t_{i-1} \wedge t, t_i \wedge t],dz), \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} I_t(f) &= \int_0^{t+} \int_Z f(s-,z,\omega) \tilde{N}(dzds) \\ &:= \sum_{i=1}^{n} \int_Z f(t_{i-1},z,\omega) \tilde{N}((t_{i-1} \wedge t, t_i \wedge t],dz). \end{aligned} \tag{3.9}$$

LEMMA 3.4. *([31]) For any $f \in \mathbb{S}$, both $\{I_t(f)\}$ and $\{J_t(f)\}$ are $\mathcal{F}_t$-adapted integrable processes. Moreover, $\{I_t(f)\}$ is an X-valued right continuous martingale. And for any $t \in (0,T]$,*

$$E[\int_0^{t+}\int_Z f(s-,z,\omega)\tilde{N}(dsdz)] = 0, \tag{3.10}$$

$$E[\int_0^{t+}\int_Z f(s-,z,\omega)N(dsdz)] = \int_0^{t+}\int_Z E[f(s-,z,\omega)]ds\nu(dz), \tag{3.11}$$

In order to extend the integrand from the step function which belongs to $\mathbb{S}$ to a more general case (belongs to $\mathcal{L}$), it is necessary to add some assumption in the Banach space X. Now we assume X is of M-type 2 below.

DEFINITION 3.5. ([5]) A Banach space $(X, \|\cdot\|)$ is called M-type 2 if and only if there exists a constant $C_X > 0$ such that for any X-valued martingale $\{\mathbf{M_k}\}$, it holds that

$$\sup_k E[\|\mathbf{M}_k\|^2] \leq C_X \sum_k E[\|\mathbf{M}_k - \mathbf{M}_{k-1}\|^2]. \tag{3.12}$$

THEOREM 3.6. *([31]) Let X be of M-type 2 and $(Z, \mathcal{B}(Z))$ a separable Banach space with finite measure ν. Let $\mathbf{p}$ be a stationary Poisson process with the characteristic measure ν and let f be in $\mathbb{S}$. Then there exists a constant C such that*

$$\begin{aligned} &E\Big[\sup_{0<s\leq t}\Big\|\int_0^{s+}\int_Z f(\tau-,z,\omega)\tilde{N}(d\tau dz)\Big\|^2\Big] \\ &\leq C\int_0^t\int_Z E[\|f(s,z,\omega)\|^2]ds\nu(dz), \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} &E\Big[\sup_{0<s\leq t}\Big\|\int_0^{s+}\int_Z f(\tau-,z,\omega)N(d\tau dz)\Big\|^2\Big] \\ &\leq C\int_0^t\int_Z E[\|f(s,z,\omega)\|^2]ds\nu(dz), \end{aligned} \tag{3.14}$$

where C depends on the constant C_X in Definition 3.5.

LEMMA 3.7. *([31]) $\mathbb{S}$ is dense in $\mathcal{L}$ with respect to the norm $\|\cdot\|_{\mathcal{L}}$.*

By Lemma 3.7, for any $f \in \mathcal{L}$, there exist a sequence $\{f^n : n \in \mathbb{N}\}$ in $\mathbb{S}$ such that $\{f^n\}$ converges to f with respect to $\|\cdot\|_{\mathcal{L}}$ and the sequence

$$\Big\{\int_0^{t+}\int_Z f^n(s-,z,\omega)\tilde{N}(dsdz), n \in \mathbb{N}\Big\}$$

converges to a limit in L^2-sense. We denote the limit by

$$I_t(f) = \int_0^{t+}\int_Z f(s-,z,\omega)\tilde{N}(dsdz),$$

which is called the *stochastic integral of f* with respect to the compensated Poisson random measure $\tilde{N}(dsdz)$. Similarly, we can define the *stochastic integral of f* with respect to the Poisson random measure $N(dsdz)$, denoted by

$$J_t(f) = \int_0^{t+}\int_Z f(s-,z,\omega)N(dsdz).$$

Similarly, for any $0 < s < t < T$,

$$\int_s^t \int_Z f(\tau-, z, \omega)\tilde{N}_{\mathbf{p}}(d\tau dz)$$

and

$$\int_s^t \int_Z f(\tau-, z, \omega)N(d\tau dz)$$

can be well defined.

REMARK 3.8. When the measure ν is finite, for any $U \in \mathcal{B}(Z)$, the processes $\{N(t, U)\}$ and $\{\tilde{N}(t, U\}$ are both of finite variation a.s. Then the stochastic integrals coincide with the Lebesgue-Stieltjes integrals.

COROLLARY 3.9. *([31]) Let X be of M-type 2 and $(Z, \mathcal{B}(Z))$ a separable Banach space with finite measure ν. Let $\mathbf{p}$ be a stationary Poisson process with the characteristic measure ν and let f be in $\mathscr{L}$. Then there exists a constant C such that*

$$\begin{aligned} &E\Big[\sup_{0<s\leq t}\Big\|\int_0^{s+}\int_Z f(\tau-, z, \omega)\tilde{N}(d\tau dz)\Big\|^2\Big] \\ &\leq C\int_0^t\int_Z E[\|f(s, z, \omega)\|^2]ds\nu(dz), \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} &E\Big[\sup_{0<s\leq t}\Big\|\int_0^{s+}\int_Z f(\tau-, z, \omega)N(d\tau dz)\Big\|^2\Big] \\ &\leq C\int_0^t\int_Z E[\|f(s, z, \omega)\|^2]ds\nu(dz), \end{aligned} \tag{3.16}$$

where C depends on the constant C_X in Definition 3.5.

COROLLARY 3.10. *([31]) For any $f \in \mathscr{L}$, both $\{I_t(f)\}$ and $\{J_t(f)\}$ are $\mathcal{F}_t$-adapted square-integrable processes. Moreover, $\{I_t(f)\}$ is an X-valued right continuous martingale. And for any $t \in (0, T]$,*

$$E\Big[\int_0^{t+}\int_Z f(s-, z, \omega)\tilde{N}(dsdz)\Big] = 0, \tag{3.17}$$

$$E\Big[\int_0^{t+}\int_Z f(s-, z, \omega)N(dsdz)\Big] = \int_0^t\int_Z E[f(s, z, \omega)]ds\nu(dz), \tag{3.18}$$

3.2. **Set-Valued Stochastic Integrals w.r.t. Poisson Point Processes.** A set-valued stochastic process $F = \{F_t\} : [0, T] \times Z \times \Omega \to \mathbf{K}(X)$ is called *$\mathscr{S}$-predictable* if $F(z, t, \omega)$ is $\mathscr{S}/\sigma(\mathcal{C})$-measurable.

Set

$$\begin{aligned} \mathscr{M} = \Big\{&F(t, z, \omega) : F \text{ is } \mathscr{S}\text{−predictable and} \\ &E\Big[\int_0^T\int_Z \|F(t, z, \omega)\|^2_{\mathbf{K}}dt\nu(dz)\Big] < \infty\Big\} \end{aligned}$$

Given a set-valued stochastic process $\{F(t,z,\omega)\}$, the X-valued stochastic process $\{f(t,z,\omega)\}$ is called an *$\mathscr{S}$-selection* if $f(t,z,\omega) \in F(t,z,\omega)$ for all (t,z,ω) and $f \in \mathscr{S}$. By Proposition ??, for $F \in \mathscr{M}$, the $\mathscr{S}$-selection exists and satisfies $E\Big[\int_0^T \int_Z \|f(t,z,\omega)\|^2 dt\nu(dz)\Big] < \infty$ since

$$E\Big[\int_0^T \int_Z \|f(t,z,\omega)\|^2 dt\nu(dz)\Big] \le E\Big[\int_0^T \int_Z \|F(t,z,\omega)\|^2_{\mathbf{K}} dt\nu(dz)\Big] < \infty,$$

which means $f \in \mathscr{L}$. The family of all f which belongs to $\mathscr{L}$ and satisfies $f(t,z,\omega) \in F(t,z,\omega)$ *for a.e.* (t,z,ω) is denoted by $S(F)$, that is

$$S(F) = \{f \in \mathscr{L} : f(t,z,\omega) \in F(t,z,\omega) \ for \ a.e. \ (t,z,\omega)\}.$$

Set

$$\tilde{\Gamma}_t := \{\int_0^{t+} \int_Z f(s-,z,\omega)\tilde{N}(dsdz) : (f(t))_{t\in[0,T]} \in S(F)\},$$

$$\Gamma_t := \{\int_0^{t} \int_Z f(s-,z,\omega)N(dsdz) : (f(t))_{t\in[0,T]} \in S(F)\}.$$

REMARK 3.11. It is easy to see for any $t \in [0,T]$, $\tilde{\Gamma}_t$ and Γ_t are the subsets of $L^2[\Omega,\mathcal{F}_t,P;X]$. Furthermore, if $\{F_t,\mathcal{F}_t : t \in [0,T]\}$ is convex, then so are $\tilde{\Gamma}_t$ and Γ_t.

Let $de\tilde{\Gamma}_t$ (resp. $de\Gamma_t$) denote the decomposable set of $\tilde{\Gamma}_t$ (resp. Γ_t) with respect to $\mathcal{F}_t$, $\overline{de}\tilde{\Gamma}_t$ (resp. $\overline{de}\Gamma_t$)the decomposable closed hull of $\tilde{\Gamma}_t$ (resp. Γ_t)with respect to $\mathcal{F}_t$, where the closure is taken in $L^1(\Omega,X)$. That is to say, for any $g \in \overline{de}\tilde{\Gamma}_t$ (resp. $\overline{de}\Gamma_t$)and any given $\epsilon > 0$, there exists a finite $\mathcal{F}_t$-measurable partition $\{A_1,...,A_m\}$ of Ω and $(f^1(t))_{t\in[0,T]},...,(f^m(t))_{t\in[0,T]} \in S(F)$ such that

$$\|g - \sum_{k=1}^m \chi_{A_k} \int_0^{t+} \int_Z f^k(s-,z,\omega)\tilde{N}(dsdz)\|_{L^1} < \epsilon.$$

$$(resp. \ \|g - \sum_{k=1}^m \chi_{A_k} \int_0^{t+} \int_Z f^k(s-,z,\omega)N(dsdz)\|_{L^1} < \epsilon)$$

Similar to Theorem 4.1 in [32], we have

THEOREM 3.12. *Let $\{F_t,\mathcal{F}_t : t \in [0,T]\} \in \mathscr{M}$, then for any $t \in [0,T]$, $\overline{de}\Gamma_t \subset L^1(\Omega,\mathcal{F}_t,P;X)$. Moreover, there exists a set-valued random variable $J_t(F) \in \mathcal{M}(\Omega,\mathcal{F}_t,P;\mathbf{K}(X))$ such that $S^1_{J_t(F)}(\mathcal{F}_t) = \overline{de}\Gamma_t$. Similarly, there exists a set-valued random variable $I_t(F) \in \mathcal{M}(\Omega,\mathcal{F}_t,P;\mathbf{K}(X))$ such that $S^1_{I_t(F)}(\mathcal{F}_t) = \overline{de}\tilde{\Gamma}_t$. If F is convex, then so are $S^1_{I_t(F)}(\mathcal{F}_t)$ and $S^1_{J_t(F)}(\mathcal{F}_t)$.*

DEFINITION 3.13. The set-valued stochastic processes $(J_t(F))_{t\in[0,T]}$ and $(I_t(F))_{t\in[0,T]}$ defined as above are called the stochastic integrals of $\{F_t,\mathcal{F}_t : t \in [0,T]\} \in \mathscr{M}$ with respect to the Poisson random measure $N(ds,dz)$ and the compensated random measure $\tilde{N}(dsdz)$ respectively. For each t, we denote $I_t(F) = \int_0^{t+} \int_Z F(s-,z,\omega)\tilde{N}(dsdz)$, $J_t(F) = \int_0^{t+} \int_Z F(s-,z,\omega)N(dsdz)$. Similarly, for $0 < s < t$,we also can define the set-valued random variable $I_{s,t}(F) = \int_s^t \int_Z F(\tau-,z,\omega)\tilde{N}(d\tau dz)$, $J_{s,t}(F) = \int_s^t \int_Z F(\tau-,z,\omega)N(d\tau dz)$.

For brevity, the integral $\int_0^{t+} \int_Z h(s-,z,\omega)\tilde{N}(dsdz)$ ($\int_0^{t+} \int_Z h(s-,z,\omega)\tilde{N}(dsdz)$) also will be denoted by $\int_0^{t+} \int_Z h_{s-}\tilde{N}(dsdz)$ ($\int_0^{t+} \int_Z h_{s-}\tilde{N}(dsdz)$ resp.), where h is an X-valued or $\mathbf{K}(\mathbf{X})$-valued integrand.

PROPOSITION 3.14. *([38]) Assume set-valued stochastic processes $\{F_t, \mathcal{F}_t : t \in [0,T]\}$ and $\{G_t, \mathcal{F}_t : t \in [0,T]\} \in \mathscr{M}$. Then*

$$J_t(F+G) = cl\{J_t(F) + J_t(G)\} \ a.s \text{ and } I_t(F+G) = cl\{I_t(F) + I_t(G)\} \ a.s.,$$

where the cl *stands for the closure in X.*

THEOREM 3.15. *([31]) Assume a set-valued stochastic process $\{F_t, \mathcal{F}_t : t \in [0,T]\} \in \mathscr{M}$. Then $\{J_t(F)\}$ and $\{I_t(F)\}$ are integrably bounded.*

THEOREM 3.16. *([31, 38]) Let a convex set-valued stochastic process $\{F_t, \mathcal{F}_t : t \in [0,T]\} \in \mathscr{M}$, then the stochastic integral $\{I_t(F), \mathcal{F}_t : t \in [0,T]\}$ is a set-valued submartingale but not a set-valued martingale.*

REMARK 3.17. With the assumption of $\mathcal{F}$ being separable with respect to the probability measure P, Theorem 3.7 in [31] pointed out that the integral $\{I_t(F)\}$ is a set-valued martingale. But unfortunately, now we found there is a gap in the proof. In fact, $\{I_t(F)\}$ is not a set-valued martingale except for special case (the singletons). The counterexample and rigorous proof are given in [38].

THEOREM 3.18. *([38]) Assume a set-valued stochastic process $\{F_t, \mathcal{F}_t : t \in [0,T]\} \in \mathscr{M}$. Then both $\{J_t(F)\}$ and $\{I_t(F)\}$ are L^2-integrably bounded.*

THEOREM 3.19. *([31])(Castaing representation of set-valued stochastic integral)*
Assume $\mathcal{F}$ is separable with respect to the probability measure P. Then for a set-valued stochastic process $\{F_t, \mathcal{F}_t : t \in [0,T]\} \in \mathscr{M}$, there exists a sequence $\{(f_t^i)_{t\in[0,T]} : i = 1, 2, ...\} \subset S(F)$ such that for each $t \in [0,T], z \in Z$, $F(t,z,\omega) = cl\{(f_t^i(z,\omega)) : i = 1, 2, ...\}$ a.s., and

$$I_t(F)(\omega) = cl\{\int_0^{t+}\int_Z f_{s-}^i(z,\omega)\tilde{N}(dsdz)(\omega) : i = 1, 2, ...\} \ a.s.$$

and

$$J_t(F)(\omega) = cl\{\int_0^{t+}\int_Z f_{s-}^i(z,\omega)N(dsdz)(\omega) : i = 1, 2, ...\} \ a.s.$$

THEOREM 3.20. *([38]) Assume $\mathcal{F}$ is separable with respect to P. Let $\{F_t\}_{t\in[0,T]}$ and $\{G_t\}_{t\in[0,T]}$ be set-valued stochastic processes in $\mathscr{M}$. Then for all t, it follows that*

$$\begin{aligned} & E\big[H\Big(\int_0^{t+}\int_Z F(s-,z,\omega)N(dsdz), \int_0^{t+}\int_Z G(s-,z,\omega)N(dsdz)\Big)\big] \\ & \le E\big[\int_0^{t+}\int_Z H(F(s-,z,\omega), G(s-,z,\omega))N(dsdz)\big] \\ & = E\big[\int_0^{t}\int_Z H(F(s,z,\omega), G(s,z,\omega))ds\nu dz\big] \end{aligned} \tag{3.19}$$

and

$$\begin{aligned} & E\big[H^2\Big(\int_0^{t+}\int_Z F(s-,z,\omega)N(dsdz), \int_0^{t+}\int_Z G(s-,z,\omega)N(dsdz)\Big)\big] \\ & \le CE\big[\int_0^{t+}\int_Z H^2(F(s-,z,\omega), G(s-,z,\omega))N(dsdz)\big] \\ & = CE\big[\int_0^{t}\int_Z H^2(F(s,z,\omega), G(s,z,\omega))ds\nu(dz)\big] \end{aligned} \tag{3.20}$$

where C is the constant appearing in Corollary 3.9.

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Mathematical Department
North China Electric Power University
Beijing 102206
China
E-mail address: `zhangjinping@ncepu.edu.cn`

Mathematical Department
Saga University
Saga 840-8502
Japan
E-mail address: `mitoma@ms.saga-u.ac.jp`

Department of Systems Design and Informatics
Kyushu Institute of Technology
Iizuka 820-8502
Japan
E-mail address: `okazaki@ces.kyutech.ac.jp`