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Stochastic differential equations with set-valued solutions

by

Michał Kisielewicz

Faculty of Mathematics Computer Science and Econometrics, University of Zielona Góra, Poland

1. Introduction

The first papers dealing with differential equations with compact convex set-valued solutions due to Francesco De Blasi and others (see [1], [2]). Latter on, such equations have been also investigated by the author of this lecture (see [3]). The present lecture is devoted to set-valued stochastic differential equations of the form

$$(1) \qquad x_t = x_0 + \int_0^t F(\tau, x_\tau)d\tau + \int_0^t \mathcal{G}_G(x)dB_\tau,$$

where $F : [0,T] \times X \to \mathrm{Cl}(\mathbb{R}^d)$ and $G : [0,T] \times X \to \mathrm{Cl}(\mathbb{R}^{d\times m})$ are given convex valued Caratheodory multifunctions, and integrals are defined as some set-valued random variables with values in the space $\mathrm{Cl}(\mathbb{R}^d)$. They are considered on a complete filtered probability space $\mathcal{P}_{\mathbb{F}} = (\Omega, \Sigma_{\mathbb{F}}, \mathbb{F}, P)$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Let us recall that for given $\mathbb{F}$-nonanticipative set-valued process $\Phi = (\Phi_t)_{t\geq 0}$ defined on $\mathcal{P}_{\mathbb{F}}$ with values in the space $\mathrm{Cl}(\mathbb{R}^d)$ of all nonempty closed subsets of $\mathbb{R}^d$, a set-valued stochastic integral $\int_0^t \Phi_\tau d\tau$ is defined to be a set-valued random variable such that $S_{\mathcal{F}_t}(\int_0^t \Phi_\tau d\tau) = \overline{\mathrm{dec}} J_t(S_{\mathbb{F}}(\Phi))$, where $S_{\mathbb{F}}(\Phi))$ denotes the set of all square integrable $\mathbb{F}$-nonanticipative selectors of Φ, $J_t(f)(\omega) = \int_0^t f_\tau(\omega)d\tau$ for every $\omega \in \Omega$ and $f \in S_{\mathbb{F}}(\Phi))$, and $S_{\mathcal{F}_t}(\int_0^t \Phi_\tau d\tau)$ contains all $\mathcal{F}_t$-measurable selectors of $\int_0^t \Phi_\tau d\tau$. In a similar way (see [4]) for an $\mathbb{F}$-nonanticipative set-valued process Ψ with values in $\mathrm{Cl}(\mathbb{R}^{d\times m})$

Address correspondence to Michał Kisielewicz, University of Zielona Góra, Podgórna 50, 65–246 Zielona Góra, Poland; E-mail: M.Kisielewicz@wmie.uz.zgora.pl

and an m-dimensional $\mathbb{F}$-Brownian motion $B = (B_t)_{t\geq 0}$, a set-valued integral $\int_0^t \Psi_\tau dB_\tau$ is defined as a set-valued random variable such that $S_{\mathcal{F}_t}(\int_0^t \Psi_\tau dB_\tau) = \overline{\text{dec}}\mathcal{J}_t(S_{\mathbb{F}}(\Psi))$, where $\mathcal{J}_t(g)(\omega) = (\int_0^t g_\tau dB_\tau)(\omega)$ for every $\omega \in \Omega$ and $g \in S_{\mathbb{F}}(\Psi))$. Unfortunalely, a set-valued integral $\int_0^t \Psi_\tau dB_\tau$ is not in the general case integrably bounded (see [12]). Therefore, we shall apply in (1) a generalized set-valued stochastic integral $\int_0^t \mathcal{G}_G(x) dB_\tau$ defined (see [9]) for a nonempty set $\mathcal{G}_G(x) = \text{co}\{g \circ x : g \in \mathcal{G}\} \subset \mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$, where $\mathcal{G}$ is a nonempty set of Carathèodory selectors of G, and for every $g \in \mathcal{G}$ and an $\mathbb{F}$-nonanticipative process $x = (x_t)_{t\geq 0}$ with values in X, a process $g \circ x$ is defined by $(g \circ x)_t(\omega) = g(t, x_t(\omega))$ for $(t,\omega) \in \mathbb{R}^+ \times \Omega$. A set-valued stochastic integral $\int_0^t \mathcal{G}_G(x) dB_\tau$ is defined as a set-valued random variable such that $S_{\mathcal{F}_t}(\int_0^t \mathcal{G}_G(x) dB_\tau) = \overline{\text{dec}}\mathcal{J}_t(\mathcal{G}_G(x))$. Such set-valued stochastic integrals are in some cases integrably bounded. In particular, it is the case if $\mathcal{G}_G(x)$ is defined by a finite set $\mathcal{G} = \{g^1, ..., g^p\}$ of Carathèodory selectors of an square integrably bounded Carathèodory multifunction G. It can be verified (see [9]) that for every sequence $(g^n)_{n=1}^\infty$ of Carathèodory selectors of an square integrably bounded Carathèodory multifunction G such that $\sum_{n=1}^\infty \|g^n\|^2 < \infty$, a generalized set-valued stochastic integral $\int_0^t \text{co}\{g^n : n \geq 1\} dB_\tau$ is also square integrably bounded. A generalized set-valued stochastic integral $\int_0^t \mathcal{G}_G(x) dB_\tau$ covers with a set valued stochastic integral $\int_0^t \Psi_\tau dB_\tau$, if $\mathcal{G}_G(x)$ is such that $\mathcal{G}_G(x) = S_{\mathbb{F}}(\Psi)$.

2. Properties of set-valued stochastic integrals

Let (X, ρ) be a metric space, and $F : [0,T] \times X \to \text{Cl}(\mathbb{R}^d)$ and $G : [0,T] \times X \to \text{Cl}(\mathbb{R}^{d\times m})$ convex valued Carathèodory multifunctions. For an $\mathbb{F}$-nonanticipative stochastic process $x = (x_t)_{t\geq 0}$ with values in a metric space (X, ρ), we shall consider a set-valued stochastic process $F \circ x$ and a set $\mathcal{G}_G(x)$ defined by $(F \circ x)_t(\omega) = F(t, x_t(\omega))$ and $\mathcal{G}_G(x) = \text{co}\{g \circ x : g \in \mathcal{G}\}$, where $\mathcal{G}$ is a set of Carathèodory selectors of G and $(g \circ x)_t(\omega) = g(t, x_t(\omega))$ for every $g \in \mathcal{G}$ and $(t,\omega) \in \mathbb{R}^+ \times \Omega$. A set-valued integral $\int_0^t (F \circ x)_\tau d\tau$ is defined such as above for $\Phi = F \circ x$. It is denoted by $\int_0^t F(\tau, x_\tau) d\tau$. If F is integrably bounded then (see [10]) the set-valued integral $\int_0^t F(\tau, x_\tau) d\tau$ is integrably bounded and a set-valued process $(\int_0^t F(\tau, x_\tau) d\tau)_{t\geq 0}$ is uniformly integrably bounded and continuous. If G is uniformly square integrably bounded then (see [9]) for every finite set $\mathcal{G} = \{g^1, ..., g^p\}$ of Carathèodory selectors of G the generalized set-valued stochastic integral $\int_0^t \mathcal{G}_G(x) dB_\tau$ is square integrably bounded and a set-valued process $(\int_0^t \mathcal{G}_G(x)dB_\tau)_{t\geq 0}$ is uniformly integrably bounded and continuous set-valued submartingale. More precisely, we have

$$E\bigg\| \int_0^t \mathcal{G}_G(x)dB_\tau \bigg\|^2 \leq p\cdot E\int_0^t \max_{1\leq k\leq p} |g^k(\tau,x_\tau)|^2 d\tau.$$

In particular case, if $\|G(t,z)\| \leq K$ then $E\|\int_0^t \mathcal{G}_G(x)dB_\tau\|^2) \leq p\cdot K$.

Having given two uniformly square integrably bounded Caratheodory multifunctions $G : [0,T]\times X \to \mathrm{Cl}(\mathbb{R}^{d\times m})$ and $\widetilde{G} : [0,T]\times X \to \mathrm{Cl}(\mathbb{R}^{d\times m})$ and families $\mathcal{G} = \{g^1, ..., g^p\}$ and $\widetilde{\mathcal{G}} = \{\widetilde{g}^1, ..., \widetilde{g}^p\}$, of Caratheodory selectors of G and $\widetilde{G}$, respectinely we obtain

$$Eh^2\bigg(\int_0^t \mathcal{G}_G(x)dB_\tau, \int_0^t \mathcal{G}_{\widetilde{G}}(x)dB_\tau\bigg) \leq p\cdot E\int_0^t \max_{1\leq k\leq p} |g^k(\tau,x_\tau) - \widetilde{g}^k(\tau,x_\tau)|^2 d\tau.$$

Similar results can be obtained (see [9]) for every infinite families $\mathcal{G} = \{g^n : n\geq 1\}$ and $\widetilde{\mathcal{G}} = \{\widetilde{g}^n : n\geq 1\}$ of Caratheodory selectors of G and $\widetilde{G}$, respectively such that $\sum_{n=1}^\infty |g^n(t,z)|^2 < \infty$ and $\sum_{n=1}^\infty |\widetilde{g}^n(t,z)|^2 < \infty$ uniformly with respect $(t,z)\in[0,T]\times X$. In particular, in such a case we get

$$Eh^2\bigg(\int_0^t \mathcal{G}_G(x)dB_\tau, \int_0^t \mathcal{G}_{\widetilde{G}}(x)dB_\tau\bigg) \leq E\int_0^t \sup_{k\geq 1} |g^k(\tau,x_\tau) - \widetilde{g}^k(\tau,x_\tau)|^2 d\tau.$$

If the above multifunction $G : [0,T]\times X \to \mathrm{Cl}(\mathbb{R}^{d\times m})$ possesses a finite family $\mathcal{G} = \{g^1, ..., g^p\}$ of Lipschitz continuous with respect to $z\in X$ selectors, then there is a number $D > 0$ such that

$$Eh^2\bigg(\int_0^t \mathcal{G}_G(x)dB_\tau, \int_0^t \mathcal{G}_G(\widetilde{x})dB_\tau\bigg) \leq p\cdot DE\int_0^t \rho^2(x_\tau,\widetilde{x}_\tau)d\tau$$

for every $\mathbb{F}$-nonanticipative processes $x = (x_t)_{t\geq 0}$ and $\widetilde{x} = (\widetilde{x}_t)_{t\geq 0}$ with values in a metric space X. Similar result can be obtained, by some additional assumptions, if G possesses an infinite family of Lipschitz continuous selectors.

3. Generalized stochastic differential equations

Let (X,h) be a complete metric space of all nonempty compact convex subsets of $\mathbb{R}^d$ with the Hausdorff metric h, $F : \mathbb{R}^+ \times X \to \mathrm{Cl}(\mathbb{R}^d)$ and

$G : \mathbb{R}^+ \times X \to \mathrm{Cl}(\mathbb{R}^{d\times m})$ Carathéodory set-valued mappings, and $\mathcal{G}$ a nonempty family of Carathéodory selectors of G. By a stochastic diffrential equation $SDE(F, \mathcal{G}_G)$ with set-valued solutions we mean a relation

$$x_t = x_0 + \int_0^t F(\tau, x_\tau)d\tau + \int_0^t \mathcal{G}_G(x)dB_\tau, \tag{2}$$

which has to be satisfied a.s. for every $t \geq 0$ by a system $(\mathcal{P}_{\mathbb{F}}, x, B)$, called a weak solution of $SDE(F, \mathcal{G}_G)$, consisting of a complete filtered probability space $\mathcal{P}_{\mathbb{F}}$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, an $\mathbb{F}$-adapted continuous set-valued process $x = (x_t)_{t\geq 0}$ with values in the space X and an m-dimensional $\mathbb{F}$-Brownian motion $B = (B_t)_{t\geq 0}$ defined on $\mathcal{P}_{\mathbb{F}}$ such that $S_{\mathbb{F}}(F \circ x) \neq \emptyset$ and $\mathcal{G}_G(x)$ is a nonempty subset of $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$, where $(F \circ x)_t(\omega) = F(t, x_t(\omega))$ and $\mathcal{G}_G(x) = \mathrm{co}\{g \circ x : g \in \mathcal{G}\}$, for every $(t, \omega) \in \mathbb{R}^+ \times \Omega$. A weak solution $(\mathcal{P}_{\mathbb{F}}, x, B)$ of $SDE(F, \mathcal{G}_G)$ is said to be unique in law if for every weak solution $(\widetilde{\mathcal{P}}_{\widetilde{\mathbb{F}}}, \widetilde{x}, \widetilde{B})$ of $SDE(F, \mathcal{G}_G)$ we have $Px^{-1} = P\widetilde{x}^{-1}$, where Px^{-1} and $P\widetilde{x}^{-1}$ denote distributions of set-valued random variables $x : \Omega \to C(\mathbb{R}^+, X)$ and $\widetilde{x} : \Omega \to C(\mathbb{R}^+, X)$. In particular, if apart from the above multifunctions F, G and a family $\mathcal{G}$ of Caratheòdory selectors of G, we have also given a filtered probability space $\mathcal{P}_{\mathbb{F}}$ and an m-dimensional $\mathbb{F}$-Brownian motion $B = (B_t)_{t\geq 0}$ defined on $\mathcal{P}_{\mathbb{F}}$, then an $\mathbb{F}$-non-anticipative continuous set-valued process $x = (x_t)_{t\geq 0}$ with values in the space X such that $(\mathcal{P}_{\mathbb{F}}, x, B)$ is a weak solution of $SDE(F, \mathcal{G}_G)$, is said to be a strong solution of $SDE(F, \mathcal{G}_G)$.

Similarly as in the classical theory of stochastic differential equations we can define initial value problems for $SDE(F, \mathcal{G}_G)$. In particular, for given a filtered probability space $\mathcal{P}_{\mathbb{F}}$, an m-dimensional $\mathbb{F}$-Brownian motion $B = (B_t)_{t\geq 0}$ and an $\mathcal{F}_0$-measurable set-valued random variable $\xi : \Omega \to X$ we can look for a strong solution x for $SDE(F, \mathcal{G}_G)$ such that $x_0 = \xi$ a.s. Such defined problem is written in the differential form

$$\begin{cases} dx_t = F(t, x_t)dt + \mathcal{G}_G(x)dB_t \\ x_0 = \xi. \end{cases} \tag{2}$$

Apart from the existence problems for stochastic differential equations with set-valued solutions we can look for their relations with stochastic differential inclusions $SDI(\Phi, \Psi)$ written as relations of the form

$$z_t - z_s \in \int_s^t \Phi(\tau, z_\tau)d\tau + \int_s^t \Psi(\tau, z_\tau)dB_\tau$$

that has to be satisfied, for given set-valued measurable mappings $\Phi : \mathbb{R}^+ \times \mathbb{R}^d \to \mathrm{Cl}(\mathbb{R}^d)$ and $\Psi : \mathbb{R}^+ \times \mathbb{R}^d \to \mathrm{Cl}(\mathbb{R}^{d\times m})$, by a system $(\mathcal{P}_{\mathbb{F}}, z, B)$, called a weak solution of $SDI(\Phi, \Psi)$, consisting of a complete filtered probability space $\mathcal{P}_{\mathbb{F}}$ with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, an d-dimensional $\mathbb{F}$-adapted continuous process $z = (z_t)_{t\geq 0}$ and an m-dimensional $\mathbb{F}$-Brownian motion $B = (B_t)_{t\geq 0}$ defined on $\mathcal{P}_{\mathbb{F}}$ such that $S_{\mathbb{F}}(\Phi \circ x) \neq \emptyset$ and $S_{\mathbb{F}}(\Psi \circ x) \neq \emptyset$. Solutions of stochastic differential equations with set-valued solutions can be applied in the thory of fuzzy differential equations.

We shall present now the skech of the proof of the existence and uniqueness theorem for an initial value problem (2) with $\mathcal{G}_G(x)$ defined by a finite family $\mathcal{G}$ of Lipschitz continuous selectors of G.

Theorem 1. *Let $T > 0$, and $F : [0,T] \times X \to \mathrm{Cl}(\mathbb{R}^d)$ and $G : [0,T] \times X \to \mathrm{Cl}(\mathbb{R}^{d\times m})$ be Carathéodory set-valued mappings with convex-valued and assume there are numbers $C > 0$ and $D > 0$ such that*

(i) $\|F(t,x)\| + \|G(t,x)\| \leq C(1 + \|x\|)$ *for* $x \in X$ *and* $t \in [0,T]$,

(ii) $h(F(t,x), F(t,y)) \leq Dh(x,y)$ *for* $x, y \in X$ *and* $t \in [0,T]$,

(iii) G possesses a finite Lipschitz continuos with respect to $z \in X$ family $\mathcal{G} = \{g^1, ..., g^p\}$ of selectors with Lipschitz constants $D_1, ..., D_p$ bounded above by D.

If $\mathcal{P}_{\mathbb{F}}$ is a filtered complete separable probability space with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions and $B = (B_t)_{t\geq 0}$ is an m-dimensional $\mathbb{F}$-Brownian motion defined on $\mathcal{P}_{\mathbb{F}}$, then for every $\mathcal{F}_0$-measurable set-valued random variable $\xi : \Omega \to X$ such that $E\|\xi\|^2 < \infty$ there exists exactly one strong solution of the initial value problem (2).

Proof (Skech of the proof). Similarly as in the classical theory of stochastic diffrential equations, in the first step of the proof, we define a sequence $(x^n)_{n=1}^{\infty}$ of successive approximations of the form: $x_t^0 = \xi$ a.s. for every $t \in [0,T]$ and

$$x_t^{n+1} = \xi + \int_0^t F(\tau, x_\tau^n)d\tau + \int_0^t \mathcal{G}_G(x^n)dB_\tau \quad \text{a.s.} \tag{3}$$

for every $t \in [0,T]$ and $n = 1,2,\dots$. It is clear that a sequence $(x^n)_{n=1}^{\infty}$ is well defined, because for $n = 0$ set-valued processes $(F(t,\xi))_{0\le t\le T}$ and $(G(t,\xi))_{0\le t\le T}$ are $\mathbb{F}$-non-anticipative and square integrably bounded by a random variable $k = C(1 + \|\xi\|)$. Therefore, set-valued process $(\int_0^t F(\tau,\xi)d\tau)_{0\le t\le T}$ is continuous uniformly square integrably bounded. By finitness of the family $\mathcal{G}$, the set-valued stochastic process $(\int_0^t \mathcal{G}_G(\xi)dB_\tau)_{0\le t\le T}$ is continuous uniformly square integrably bounded. This, together with convexity of the set-valued stochastic integrals $\int_0^t F(\tau,\xi)d\tau$ and $\int_0^t \mathcal{G}_G(\xi)dB_\tau$ implies that $x_t^1 \in X$ a.s. for every $t \in [0,T]$. Hence also follows that a set-valued process $(x_t^1)_{0\le t\le T}$ is square integrably bounded. By continuity of set-valued processes $(\int_0^t F(\tau,\xi)d\tau)_{0\le t\le T}$ and $(\int_0^t \mathcal{G}_G(\xi)dB_\tau)_{0\le t\le T}$ it follows that a set-valued process $(x_t^1)_{0\le t\ leT}$ is continuous. Immediately from the definition of x^1 it follows that $(x_t^1)_{0\le t\le T}$ is $\mathbb{F}$-adapted, and hence $\mathbb{F}$-non-anticipative. Thus set-valued processes $(F(t,x_t^1))_{0\le t\le T}$ and $(G(t,x_t^1))_{0\le t\le T}$ are $\mathbb{F}$-non-anticipative and square integrably bounded. Then $\mathcal{G}_G(x^1)$ is a nonempty subset of $\mathbb{L}^2(\mathbb{R}^+ \times \Omega, \Sigma_{\mathbb{F}}, \mathbb{R}^{d\times m})$. By the inductive procedure it can be easily verified that all set-valued processes $(x_t^n)_{0\le t\le T}$ are well defined with values in X, and are continuous and uniformly square integrably bounded.

The second steps of the proof deals with the estimations of the $Eh^2(x_t^{n+1}, x_t^n)$ for every $n \ge 1$ and $0 \le t \le T$. By the properties of set-valued stochastic integrals presented above, and the definition of x_t^0 and x_t^1 for $t \in [0,T]$ we get

$$[Eh^2(x_t^1,x_t^0)]^{1/2} \le \left[TE\int_0^t \|F(\tau,\xi)\|^2 d\tau\right]^{1/2} + \left[pE\int_0^t \max_{1\le k\le p} |g^k(\tau,\xi)|^2 d\tau\right]^{1/2}$$

$$\le \left[TC^2E(1+\|\xi\|)^2 t\right]^{1/2} + \left[C^2pE(1+\|\xi\|)^2 t\right]^{1/2} =$$

$$\left(\sqrt{T}+\sqrt{p}\right) C[E(1+\|\xi\|)^2]^{1/2}\sqrt{t},$$

which can be written in the form $Eh^2(x_t^1,x_t^0) \le KL\cdot t$ where $K = (\sqrt{T} + \sqrt{p})^2$ and $L = C^2E(1+\|\xi\|)^2$. By the definition of a sequence $(x^n)_{n=1}^{\infty}$ and properties of set-valued stochastic integrals presented above, for every $t \in [0,T]$ we obtain

$$[Eh^2(x_t^{n+1},x_t^n)]^{1/2} \le \left[Eh^2\left(\int_0^t F(\tau,x_\tau^n)d\tau, \int_0^t F(\tau,x_\tau^{n-1})d\tau\right)\right]^{1/2} +$$

$$\left[Eh^2\left(\int_0^t \mathcal{G}_G(x^n)dB_\tau, \int_0^t \mathcal{G}_G(x^{n-1})dB_\tau\right)\right]^{1/2} \le$$

$$\left[TDE\int_0^t h^2(x_\tau^n, x_\tau^{n-1})d\tau\right]^{1/2} + \left[pDE\int_0^t h^2(x_\tau^n, x_\tau^{n-1})d\tau\right]^{1/2} =$$

$$(\sqrt{TD}+\sqrt{pD})\left[E\int_0^t h^2(x_\tau^n, x_\tau^{n-1})d\tau\right]^{1/2},$$

which can be wrirtten in the form $Eh^2(x_t^{n+1}, x_t^n) \le KDE\int_0^t h^2((x_\tau^n, x_\tau^{n-1})d\tau$.

The third step of the proof is connected with convergence of the sequence $(x^n)_{n=1}^\infty$ with respect to the metric topology of the metric space $(\mathcal{C}, \mathrm{d})$ defined by $\mathcal{C} =: C([0,T], \mathcal{L}^2)$ with $\mathrm{d}(u,v) = \sup_{0\le t\le T}\sqrt{Eh^2(u_t, v_t)}$ for continuous set-valued processes $u = (u_t)_{0\le t\le T}$ and $v = (v_t)_{0\le t\le T}$ with values in the Polish space $\mathcal{L}^2 =: \mathbb{L}^2(\Omega, \mathcal{F}, P, X)$ consisting of all set-valued random variables (equivalence casses of) $z : \Omega \to X$ such that $E\|z\|^2 < \infty$.

Immediately from the results of the above two steps, we obtain

$$\sup_{0\le t\le T} Eh^2(x_t^{n+1}, x_t^n) \le L\frac{K^{n+1}D^{n+1}T^{n+1}}{(n+1)!}$$

for every $n = 0, 1, 2, \ldots$. Hence it follows that $(x^n)_{n=1}^\infty$ is a Cauchy sequence of the complet metric space $(\mathcal{C}, \mathrm{d})$. Then there is $x \in \mathcal{C}$ such that $\mathrm{d}(x^n, x) \to 0$ as $n \to \infty$. Let us observe that x is $\mathbb{F}$-non-anticipative, i.e., that it is $\mathbb{F}$-adaptive and $\beta([0,T])\otimes\mathcal{F}$-measurable. Indeed, $\mathbb{F}$-adaptness follows immediately from $\mathbb{F}$-adaptness of x^n for every $n \ge 1$ and the result $\mathrm{d}(x^n, x) \to 0$ as $n \to \infty$. Let $f : \Omega \times ([0,T] \times \mathcal{C}) \to X$ be defined by $f(\omega, (t,z)) = z_t(\omega)$ for $\omega \in \Omega$ and $(t,z) \in [0,T] \times \mathcal{C}$. It is clear that $f(\cdot, (t,z)) = z_t(\omega)$ is $\mathcal{F}$-measurable for fixed $(t,z) \in [0,T] \times \mathcal{C}$. Furthermore, $f(\cdot, (t,z)) \in \mathcal{L}^2$ and the set-valued mapping $[0,T] \times \mathcal{C} \ni (t,z) \to f(\cdot, (t,z)) \in \mathcal{L}^2$ is continuous, because for every sequence $\{(t_n, z^n)\}_{n=1}^\infty$, such that $t_n \to t_0$ and $\sup_{0\le t\le T} Eh^2(z_t^n, z_t^0) \to 0$ we have

$$Eh^2[f(\cdot, (t_n, z^n)), f(\cdot, (t_0, z^0))] = Eh^2(z_{t_n}^n, z_{t_0}^0) \le Eh^2(z_{t_n}^n, z_{t_n}^0) +$$

$$Eh^2(z_{t_n}^0, z_{t_0}^0) \le \sup_{0\le t\le T} Eh^2(z_t^n, z_t^0) + Eh^2(z_{t_n}^0, z_{t_0}^0).$$

Then $Eh^2[f(\cdot, (t_n, z^n)), f(\cdot, (t_0, z^0))] \to 0$ as $n \to 0$. Therefore, a set-valued mapping $g : [0,T]\times\Omega\times\mathcal{C} \to X$ defined by $g(t, \omega, z) = f(\omega, (t,z))$ for $(t, \omega) \in$

$[0,T]\times\Omega$ and $z\in\mathcal{C}$ is $\mathcal{F}\otimes\beta([0,T]\times\mathcal{C})$ - measurable. But $\mathcal{F}\otimes\beta([0,T]\times\mathcal{C})\subset\mathcal{F}\otimes\beta_T\otimes\beta(\mathcal{C})$, where β_T and $\beta(\mathcal{C})$ denote the Borel σ - algebras on $[0,T]$ and $\mathcal{C}$, respectively. Therefore, $g(\cdot,\cdot,z)$ is $\mathcal{F}\otimes\beta_T$ - measurable, which implies that for every $z\in\mathcal{C}$ a set-valued process $(z_t)_{0\le t\le T}$ with values in X such that $E\|z_t\|^2<\infty$ is $\mathcal{F}\otimes\beta_T$ - measurable, because $z_t(\omega)=g(t,\omega,z)$.

In the fourth step we verify that $Eh^2(x_t,\xi+\int_0^t F(\tau,x_\tau)d\tau+\int_0^t\mathcal{G}_G(x)dB_\tau)=0$ for every $0\le t\le T$. It follows from inequalities

$$Eh^2\left(x_t,\xi+\int_0^t F(\tau,x_\tau)d\tau+\int_0^t\mathcal{G}_G(x)dB_\tau\right)\le 2Eh^2(x_t,x_t^{n+1})+$$

$$2Eh^2\left(\int_0^t F(\tau,x_\tau^n)d\tau+\int_0^t\mathcal{G}_G(x^n)dB_\tau,\int_0^t F(\tau,x_\tau)d\tau+\int_0^t\mathcal{G}_G(x)dB_\tau\right)$$

$$\le 2Eh^2(x_t,x_t^{n+1})+4T^2D^2Eh^2(x_t,x_t^n)+4pD^2Eh^2(x_t,x_t^n)$$

for every $n\ge 1$ and $0\le t\le T$. Immediately from the equality $x_t=\xi+\int_0^t F(\tau,x_\tau)d\tau+\int_0^t\mathcal{G}_G(x)dB_\tau$ a.s. for every $0\le t\le T$ and continuity of the set-valued processes $(\int_0^t F(\tau,x_\tau)d\tau)_{0\le t\le T}$ and $(\int_0^t\mathcal{G}_G(x)dB_\tau)_{0\le t\le T}$ it follows that $(x_t)_{0\le t\le T}$ is continuous.

Similarly as above we can verify that for two continuous set-valued processes $(x_t)_{0\le t\le T}$ and $(y_t)_{0\le t\le T}$ satisfying conditions (2) we obtain $Eh^2(x_t,y_t)=0$. □

Remark 1. *In a similar way we can consider the case with a set $\mathcal{G}$ containing an infinity many Lipschitz continuous selectors of G satisfying some additional conditions implying integrable boundedness of a set-valued integral $\int_0^t\mathcal{G}_G(x)dB_\tau$ and boundedness from above of Lipschitz constants of the above selectors.* □

References

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