数理解析研究所講究録
第 1906 巻 2014 年 95-102

A Proof of Anscombe and Aumann's Theorem

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March 31, 2014

1 Introduction

The cerebrated theorem by Savage (1954) states that if the decision-maker's behavior complies with some set of reasonable axioms, then her behavior can be described as if she tries to maximize the expected utility with respect to some subjective probability. While the proof of Savage's theorem is quite lengthy (for example, see Fishburn (1970)), Anscombe and Aumann (1963) largely simplified the story by introducing a randomizing devise which generates an objective probability. However, their proof assumes that the state space is finite. (See also Kreps (1988).) This note extends their result to a general state space which is not necessarily finite by the use of Liesz representation theorem. The proof is simple and quite easy to follow.

2 Preliminaries

2.1 Probability Charge

We call a family of subsets Σ of a set S an *algebra* if it satisfies the three conditions: (1) $\phi \in \Sigma$, (2)[1] $A \in \Sigma \Rightarrow A^c \in \Sigma$ and (3) $A, B \in \Sigma \Rightarrow A \cup B \in \Sigma$, and call a pair of a set and an algebra defined on that set, (S, Σ), a *measurable space*. Given a measurable space (S, Σ), a set function $p : \Sigma \to [0, +\infty]$ which satisfies the following two conditions is called a *finitely-additive measure* or a *charge*:

$$p(\phi) = 0 \tag{1}$$

$$(\forall A, B \in \Sigma) \quad A \cap B = \phi \;\Rightarrow\; p(A \cup B) = p(A) + p(B)\,. \tag{2}$$

Condition (2) is called a *finite additivity*. A charge p which also satisfies $p(S) = 1$ is called a *probability charge*.

2.2 Preference Order

Let X be a set of alternatives. We call any subset $\succ$ of $X \times X$ a *binary relation* on X and write as $p \succ q$ when $(p, q) \in \succ$. A binary relation $\succ$ is said to be *asymmetric* if

$$(\forall p, q \in X) \quad p \succ q \;\Rightarrow\; q \not\succ p\,,$$

[1]Here, A^c denotes the complement of A in S.

where $q \not\succ p$ means that $(q,p) \notin \succ$, and it is said to be *negatively transitive* if

$$(\forall p,q,r \in X) \quad p \not\succ q \text{ and } q \not\succ r \;\Rightarrow\; p \not\succ r\,.$$

A binary relation $\succ$ is called a *preference order* or a *preference relation* when it is asymmetric and negatively transitive. Given a preference order $\succ$, the binary relation $\succeq$ is defined by

$$(\forall p,q \in X) \quad p \succeq q \;\Leftrightarrow\; q \not\succ p$$

and the binary relation $\sim$ is defined by

$$(\forall p,q \in X) \quad p \sim q \;\Leftrightarrow\; p \not\succ q \text{ and } q \not\succ p\,.$$

A function $u : X \to \mathbb{R}$ is said to *represent* a preference order $\succ$ if it holds that

$$(\forall p,q \in X) \quad p \succ q \;\Leftrightarrow\; u(p) > u(q)\,.$$

3 Herstein and Milnor's Mixture Space Theorem

A set Φ is called a *mixture space* if there exists a function $h : [0,1] \times \Phi \times \Phi \to \Phi$ which satisfies the following three conditions:

M1. $h_1(\phi,\rho) = \phi$

M2. $h_a(\phi,\rho) = h_{1-a}(\rho,\phi)$

M3. $h_a(h_b(\phi,\rho),\rho) = h_{ab}(\phi,\rho)$

Here, the first argument is denoted by a subscript. For instance, the set of probability charges on a measurable space (S,Σ) considered in the previous section is a mixture space by defining $h_a(p,q) := ap + (1-a)q$. Consider the following three axioms with respect to a binary relation $\succ$ defined on a mixture space:

A1	**(Ordering)**	$\succ$ is a preference order
A2	**(Independence)**	$\phi \succ \rho \;\Rightarrow\; (\forall a \in (0,1])(\forall \mu)\;\; h_a(\phi,\mu) \succ h_a(\rho,\mu)$
A3	**(Continuity)**	$\phi \succ \rho$ and $\rho \succ \mu$ $\Rightarrow\; (\exists a,b \in (0,1))\;\; h_a(\phi,\mu) \succ \rho$ and $\rho \succ h_b(\phi,\mu)$

Then, the following theorem holds. For its proof, see, for instance, Kreps (1988).

Theorem 1 (Herstein and Milnor, 1953). *A binary relation $\succ$ defined on a mixture space Φ satisfies Axioms A1, A2 and A3 if and only if there exists a function $F : \Phi \to \mathbb{R}$ which satisfies*

Representation $\quad \phi \succ \rho \Rightarrow F(\phi) > F(\rho)$ *and*

Affinity $\quad (\forall a, \phi, \rho) \quad F(h_a(\phi, \rho)) = aF(\phi) + (1-a)F(\rho)\,.$

Furthermore, F is unique up to a positive affine transformation.

4 Riesz Representation Theorem

In what follows, we fix a measurable space (S, Σ) consisting of a set S and an algebra Σ defined on it. We denote by $B(S, \Sigma)$ or more simply by B the set of all Σ-measurable and bounded real-valued functions defined on the measurable space (S, Σ). Also, we denote the subset of $B(S, \Sigma)$ consisting of all the simple functions by $B_0(S, \Sigma)$ or B_0, where a function is *simple* if its range is a finite set.

For a functional $I : B \to \mathbb{R}$ on a measurable space $B(S, \Sigma)$, it is said to be *norm-continuous* if it holds that, for any sequence $\langle a_n \rangle_{n=1}^{\infty} \subseteq B$ and for any element $a \in B$,

$$\|a - a_n\| \to 0 \quad \Rightarrow \quad |I(a) - I(a_n)| \to 0\,,$$

where $\|\cdot\|$ is the sup norm.

Theorem 2 (Riesz Representation Theorem). *For a linear functional $I : B \to \mathbb{R}$ which is norm-continuous and satisfies that $I(\chi_S) = 1$, it holds that*

$$(\forall a \in B) \quad I(a) = \int_S a(s)\, dp(s)\,. \tag{3}$$

Here, p is a probability charge on (S, Σ) defined by $(\forall A \in \Sigma)\ p(A) = I(\chi_A)$.[2]

The integral with respect to a probability charge was developed by Dunford and Schwartz (1988), and then, Rao and Rao (1983). For a proof of Riesz Representation Theorem, see Rao and Rao (1983, p.135, Theorem 4.7.4).

For a functional $I : B \to \mathbb{R}$ on a measurable space $B(S, \Sigma)$, it is said to be *additive* if it holds that

$$(\forall a, b \in B) \quad I(a + b) = I(a) + I(b)$$

and it is said to be *monotonic* if it holds that

$$(\forall a, b \in B) \quad a \geq b \Rightarrow I(a) \geq I(b)\,.$$

Note that the additivity implies that $I(0) = I(\chi_\phi) = 0$. (Let $a = b = \chi_\phi$ in the definition of the additivity.)

[2] χ denotes the indicator function.

Lemma 1. *If a functional $I : B \to \mathbb{R}$ is additive, I satisfies the homogeneity for rational numbers:*

$$(\forall a \in B)(\forall r \in \mathbb{Q}) \quad I(ra) = rI(a).$$

Proof. Let $a \in B$. First, let $r \in \mathbb{Q}_+$ be a positive rational number such that $r = m/n$ $(m, n \in \mathbb{N})$. The additivity of I implies that $nI((m/n)a) = I(n(m/n)a) = I(ma) = mI(a)$, and hence, $I(ra) = rI(a)$. Second, let r be a negative rational number. Note that $I(a) = -I(-a)$ holds because $0 = I(a-a) = I(a)+I(-a)$ by the additivity. Therefore, $I(ra) = I(-|r|a) = |r|I(-a) = -|r|I(a) = rI(a)$, which completes the proof. □

Lemma 2. *If a functional $I : B \to \mathbb{R}$ is additive and monotonic and satisfies that $I(\chi_S) = 1$, then I is norm-continuous.*

Proof. Let $\varepsilon > 0$ and let $\langle a_n \rangle_{n=1}^{\infty}$ be convergent to a in the norm topology. Also, let δ be a positive rational number such that $0 < \delta < \varepsilon$ and let $N \in \mathbb{N}$ be such that $(\forall n \geq N)$ $\|a - a_n\| < \delta$. Then, for any $n \geq N$, it holds that

$$\begin{aligned} I(a) - I(a_n) &= I(a) + I(-a_n) \\ &= I(a - a_n) \\ &\leq I(\|a - a_n\|\chi_S) \\ &\leq I(\delta\chi_S) \\ &\leq \delta I(\chi_S) = \delta < \varepsilon, \end{aligned}$$

where the first equality holds since $0 = I(a_n - a_n) = I(a_n) + I(-a_n)$ by the additivity; the second equality holds by the additivity; the first inequality holds by the definition of the norm and the monotonicity; the second inequality holds by the assumption and the monotonicity; and the third inequality holds by Lemma 1. Similarly, we can show that $I(a_n) - I(a) < \varepsilon$. The lemma then follows. □

Lemma 3. *If a functional $I : B \to \mathbb{R}$ is additive and monotonic and satisfies that $I(\chi_S) = 1$, then I is homogeneous.*

Proof. Let $\lambda \in \mathbb{R}$ and let $\langle r_n \rangle_n$ be a sequence of rational numbers which converges to λ. Then, for any $a \in B$, $r_n a$ converges to λa in the norm topology. Therefore, it holds that $I(\lambda a) = \lim_{n\to\infty} I(r_n a) = \lim_{n\to\infty} r_n I(a) = \lambda I(a)$, where the first equality holds by the norm-continuity (Lemma 2), the second equality holds by Lemma 1 and the last equation holds by the assumption.

□

Let K be a convex set which satisfies that $[-1,1] \subseteq K \subseteq \mathbb{R}$ and denote the subset of B (or B_0) consisting of K-valued functions by $B(K)$ (or $B_0(K)$). Then, the next proposition holds.

Proposition 1. *For a functional $I : B(K) \to \mathbb{R}$ which is additive and monotonic and satisfies that $I(\chi_S) = 1$, (3) holds.*

Proof. By the additivity, the monotonicity and the assumption that $I(\chi_S) = 1$, I is norm-continuous (Lemma 2) and homogeneous (Lemma 3). By the homogeneity, I can be extended to B. Since I thus extended is a norm-continuous linear functional on B, the result follows from Theorem 2. □

5 Anscombe and Aumann's Theorem

5.1 Lottery Act

In this section, we assume that the state space is given by a measurable space (S, Σ) and that an outcome space is given by a mixture space Y. For example, let X be a space of prizes and let Y be the space of all simple probability charges on X. Here, a probability charge p is said to be *simple* if its support is a finite set, *i.e.*: if there exists a subset $\{x_1, x_2, \ldots, x_n\} \subseteq X$ such that $\sum_{x \in \{x_1,\ldots,x_n\}} p(\{x\}) = 1$ holds. Then, Y is a mixture space.

A function from S into Y is called *Anscombe-Aumann (A-A) act* or *lottery act*. A lottery act which is Σ-measurable and whose range is a finite set is called *simple* lottery act. The set of all simple lottery acts is denoted by L_0. Also, the set of simple lottery acts whose range is a singleton is denoted by L_c. Suppose that a binary relation $\succ$ is defined on the set L_0. We induce the binary relation on Y from a binary relation $\succ$ on L_0 as follows and denote it by the same symbol $\succ$, *i.e.*: $(\forall y, y' \in Y)\ y \succ y' \ \Leftrightarrow\ f \succ g$ where $(\forall s)\ f(s) = y$ and $g(s) = y'$.

We can now construct a mixture of two simple lottery acts. Let $f, g \in L_0$ and let h be a function which makes Y a mixture space. Then, use h to define the mixture of f and g by $s \mapsto h_a(f(s), g(s))$. Then, it can be easily seen that the set L_0 becomes a mixture space by the operation thus defined. We write the mixture of f and g by $af + (1-a)g$.

5.2 Axioms and Representation Theorem

We consider some axioms on a binary relation $\succ$ defined on L_0. Here, f, g, h are any element of L_0 and λ is any real number such that $\lambda \in (0, 1]$.

B1 (Ordering)	$\succ$ is a preference order on L_0
B2 (Independence)	$f \succ g \Rightarrow \lambda f + (1-\lambda)h \succ \lambda g + (1-\lambda)h$
B3 (Continuity)	$f \succ g$ and $g \succ h$ $\Rightarrow (\exists \alpha, \beta \in (0,1))\ \alpha f + (1-\alpha)h \succ g$ and $g \succ \beta f + (1-\beta)h$
B4 (Monotonicity)	$(\forall f, g \in L_0)\ [(\forall s \in S)\ f(s) \succeq g(s)] \Rightarrow f \succeq g$
B5 (Nondegeneracy)	$(\exists f, g \in L_0)\ f \succ g$

Note the similarity of Axioms B1-B3 to Axioms A1-A3. The next theorem is an extension of Anscombe and Aumman (1963) to a (not necessarily finite) general state space.

Theorem 3. *A binary relation $\succ$ defined on the set L_0 satisfies B1, B2, B3, B4 and B5 if and only if there exist a unique probability charge p on (S, Σ) and an affine function u on Y which is unique up to a positive affine transformation such that*

$$f \succ g \Leftrightarrow \int_S u(f(s))\, dp(s) > \int_S u(g(s))\, dp(s)\,.$$

5.3 Proof of Theorem 3

We show that Axioms B1-B5 implies the representation. Other claims are easy to verify and their proofs are omitted. Since the set L_0 is a mixture space and a binary relation $\succ$ satisfies A1-A3 by B1-B3, Theorem 1 implies that there exists an affine function on L_0 which represents $\succ$. We denote this function by J. We further define the affine function u on Y by $(\forall y \in Y)\ u(y) = J(y)$, where y in the right-hand side is understood to be a constant lottery act which always takes on y. B4 and B5 imply the existence of $y^*, y_* \in Y$ such that $y^* \succ y_*$. To see this, suppose that $(\forall y, y' \in Y)\ \ y \sim y'$. Then, by B4, for any pair of simple lottery acts f, g, it holds that $f \sim g$, which contradicts B5. Hence, by applying an appropriate affine transformation to J, we can normalize u so that $u(y^*) = 1$ and $u(y_*) = -1$.

Let $K := u(Y)$. Then K is convex by the affinity of u and satisfies $[-1, 1] \subseteq K$ by the previous paragraph. Define a function $U : L_0 \to B_0(K)$ by

$$(\forall f)(\forall s) \quad U(f)(s) = u(f(s))\,.$$

Then, U is surjective since $K = u(Y)$ and U satisfies that $U(f) = U(g) \Rightarrow f \sim g$ by B4 and the fact that u is a representation on Y. Therefore, U is bijective. Furthermore, the affinity of u implies $(\forall \alpha \in [0,1])\ U(\alpha f + (1-\alpha)g) = \alpha U(f) + (1-\alpha)U(g)$. Now, define a functional I on $B_0(K)$ by

$$(\forall a \in B_0(K)) \quad I(a) = J(U^{-1}(a))\,.$$

Clearly, it holds that $(\forall f \in L_0)\ \ I(U(f)) = J(f)$. Also, note that $I(0)\ (= I(\chi_\phi)) = 0$. To see this, let $f := h_{1/2}(y^*, y_*)$ be a constant lottery act. Then, $U(f) = u(h_{1/2}(y^*, y_*)) = (1/2)u(y^*) + (1/2)u(y_*) = 0$ by the affinity of u. Hence, $I(0) = J(f) = u(f) = 0$.

This paragraph proves that it holds that

$$(\forall a, b \in B_0(K))(\forall \alpha \in [0,1]) \quad I(\alpha a + (1-\alpha)b) = \alpha I(a) + (1-\alpha)I(b)\,. \tag{4}$$

Let $a, b \in B_0(K)$ and let $\alpha \in [0,1]$. Also, let $f, g \in L_0$ be such that $U(f) = a$ and $U(g) = b$. Since U is surjective, such f and g exist. Then, by the previous paragraph,

$$\begin{aligned} I(\alpha a + (1-\alpha)b) &= J(U^{-1}(\alpha a + (1-\alpha)b)) \\ &= J(U^{-1}(\alpha U(f) + (1-\alpha)U(g))) \\ &= J(U^{-1}(U(\alpha f + (1-\alpha)g))) \\ &= J(\alpha f + (1-\alpha)g) \\ &= \alpha J(f) + (1-\alpha)J(g) \\ &= \alpha I(a) + (1-\alpha)I(b)\,, \end{aligned}$$

where the fifth equality holds because J is an affine function.

This paragraph proves that the functional I satisfies all the assumptions of Proposition 1. (i) $I(\chi_S) = 1$. Let $f \in L_0$ be a constant lottery act such that $(\forall s)\ f(s) = y^*$. Then $U(f) = u(y^*) = \chi_S$ by the definition of U, and hence, $I(\chi_S) = J(U^{-1}(\chi_S)) = J(f) = J(y^*) = u(y^*) = 1$. (ii) Additivity. In Equation (4), letting $b = 0$ shows

$$(\forall a \in B_0(K))(\forall \alpha \in [0,1]) \quad I(\alpha a) = \alpha I(a)$$

since $I(0) = 0$. (See the end of the second paragraph.) The additivity then follows from this and by setting $\alpha = 1/2$ in Equation (4). (iii) Monotonicity. Let $a, b \in B_0(K)$ and let $f, g \in L_0$ be such that $U(f) = a$ and $U(g) = b$. Then, it follows that

$$\begin{aligned} & a \geq b \;\Rightarrow\; U(f) \geq U(g) \;\Rightarrow\; (\forall s)\ \ u(f(s)) \geq u(g(s)) \\ \Rightarrow\; & (\forall s)\ \ f(s) \succeq g(s) \;\Rightarrow\; f \succeq g \;\Rightarrow\; J(f) \geq J(g) \;\Rightarrow\; I(a) \geq I(b)\,, \end{aligned}$$

where the third implication holds by the fact that u is a representation on Y; the fourth holds by B4; and the fifth holds by the fact that J is a representation on L_0. The monotonicity was thus proved.

This paragraph applies Proposition 1 to I and then completes the proof. As shown in the previous paragraph, I satisfies all the assumptions of Proposition 1, and hence, if we define a probability charge p on (S, Σ) by $(\forall A \in \Sigma)\ \ p(A) := I(\chi_A)$, it holds that

$$(\forall a \in B_0) \quad I(a) = \int_S a(s)\, dp(s)\,.$$

Then, for any $f, g \in L_0$, it follows that

$$f \succ g \;\Leftrightarrow\; J(f) > J(g) \;\Leftrightarrow\; I(U(f)) > I(U(g))$$

$$\Leftrightarrow\; \int_S U(f)(s)\, dp(s) > \int_S U(g)(s)\, dp(s) \;\Leftrightarrow\; \int_S u(f(s))\, dp(s) > \int_S u(g(s))\, dp(s)\,,$$

which completes the proof of Theorem 3.

References

Anscombe, F. J. and R. J. Aumann (1963): "A Definition of Subjective Probability," *Annals of Mathematical Statistics* 34, 199-205.

Dunford, N. and J. T. Schwartz (1988): *Linear Operator Part I: General Theory*, Wiley Classics Library.

Fishburn, P. C. (1970): *Utility Thoery for Decision Making*. Wiley, New York.

Herstein, I. N. and J. Milnor (1953): " An Axiomatic Approach to Measurable Utility," *Econometrica* 21, 291-297.

Kreps, D. (1988): *Notes on the Theory of Choice*. Boulder, Colorado: Westview Press.

Rao, K. P. S. B. and M. B. Rao (1983): *Theory of Charges*, Academic Press.

Savage, L. (1954): *The Foundations of Statistics*. John Wiley, New York.