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"Dynamical Relativity" vs. Indeterminacy in Dynamics*

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1 Dynamical Relativity in Family of Dynamics

In this report, we discuss the roles and meaning of "dynamical relativity" in the descriptions of physical processes, which is a new concept proposed recently by the author [1]. While the standard sorts of relativity like Einstein's resolves successfully the ***kinematical*** ambiguities due to the *non-uniqueness of reference frames* unavoidable in theoretical descriptions of physical processes, any systematic approaches to the problem of ***indeterminacy in dynamics*** caused by the presence of constraints or a *family of dynamics* do not seem to have so far been attempted in contrast. In our discussion here, the duality relation between the kinematical and dynamical relativities plays essential roles, whose essence in an abstract categorical context can naturally be understood by the following duality [2] between **inductive** $\varinjlim$ & **projective** $\varprojlim$ limits:

$$[\text{Kinematics of } \varinjlim \overset{\swarrow}{\underset{\nwarrow}{\longleftarrow}}\,] \overset{\text{duality}}{\leftrightarrows} [\,\overset{\nwarrow}{\underset{\swarrow}{\longleftarrow}} \varprojlim : \text{Dynamics in projective limit}]$$

due to the adjunction involving the diagonal functor Δ [s.t. $\Delta(c)(j) \equiv c$ for $c \in \mathcal{C}\ \forall j \in J$]:

$$\text{left adjoint} \quad \varinjlim \uparrow \ \overset{\mathcal{C}}{\underset{\mathcal{C}^J}{\Delta\downarrow}} \ \uparrow \varprojlim \quad \text{right adjoint}$$

The essence of the above contents can be explained on the basis of the following points:

1. Usual relativity principle as **kinematical** unification of **many reference frames** on sector-classifying space: 1) Galileian relativity in non-relativistic physics; 2) special relativity arising from electromagnetism

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due to Poincaré and Einstein; 3) Einstein's general relativity controlling gravity.

2. **Dynamical relativity** unifies dynamically a family of dynamics.

3. **Duality** between kinematical & dynamical relativities:
While "coordinate-free" nature of modern geometry is subsumed in Einstein's kinematical relativity, the plurality of ***indeterminate dynamics*** as the essence of dynamical relativity is dual to it, without being absorbed in the former one.

1.1 Sector-Classifying Space in Micro-Macro Duality

To explain a *sector-classifying space* in the above, we consider its roles in terms of the following basic concepts:
1) ***sectors*** as Micro-Macro boundary, which constitutes
2) ***Micro-Macro duality***, whose Macro side is formed through
3) ***emergence processes*** via "forcing" [Macro $\Longleftarrow$ Micro].

1.2 Sectors and Micro-Macro Duality

1) ***Sectors= pure phases*** parametrized by ***order parameters.***

Here order parameters are the spectral values of central observables belonging to the centre $\mathfrak{Z}_\pi(\mathcal{X}) = \pi(\mathcal{X})'' \cap \pi(\mathcal{X})'$ of represented algebra $\pi(\mathcal{X})''$ of physical variables commuting with all other physical variables in a generic representation π of $\mathcal{X}$. Mathematically, a ***sector*** is defined by a ***quasi-equivalence*** class of ***factor*** state (& representation π_γ) of the algebra $\mathcal{X}$ of physical variables, characterized by ***trivial centre*** $\pi_\gamma(\mathcal{X})'' \cap \pi_\gamma(\mathcal{X})' =: \mathfrak{Z}_{\pi_\gamma}(\mathcal{X}) = \mathbb{C}1$ as a *minimal unit* of representations classified by ***quasi-equivalence*** relation which is the unitary equivalence up to multiplicity.

2) The roles of ***sectors as Micro-Macro boundary*** can be seen in ***Micro-Macro duality*** [3, 4] as a mathematical version of "***quantum-classical correpsondence***" between the inside of microscopic ***sectors*** and the macroscopic ***inter-sectorial*** level described by geometrical structures on the central spectrum $Sp(\mathfrak{Z}) := Spec(\mathfrak{Z}_\pi(\mathcal{X}))$:

1.3 Micro-Macro Duality and Emergence of Macro-level

This corresponds mathematically to a Hilbert bimodule ${}_{\pi(\mathcal{X})''}\widetilde{\mathcal{X}}_{L^\infty(E_\mathcal{X})} := \pi(\mathcal{X})'' \otimes L^\infty(E_\mathcal{X})$ with left $\pi(\mathcal{X})''$ and right $L^\infty(E_\mathcal{X}, \mu)$ actions (where $E_\mathcal{X}$ denotes the state space of $\mathcal{X}$ equipped with a central measure μ), controlled by Tomita decomposition theorem:

$\longleftarrow$	Visible ***Macro***	of	***independent objects***	$\cdots$	$\longrightarrow$	***Inter-sectorial***
$\cdots$	γ_N		***sectors*** γ	γ_2	γ_1	$Sp(3)$
	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\uparrow$ ***Intra-sectorial***
$\cdots$	π_{γ_N}		π_γ	π_{γ_2}	π_{γ_1}	$\Vert$
	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\downarrow$ Invisible ***Micro***

Now, Micro-Macro Duality is formulated as a categorical adjunction consisting of an adjoint pair of functors E, F in combination with a unit $\eta : I_{\mathcal{X}} \dot{\rightarrow} T$ intertwining from $\mathcal{X}$ to the monad $T = EF$ and with a counit $\varepsilon : S \dot{\rightarrow} I_{\mathcal{A}}$ intertwining from the comonad $S = FE$ to $\mathcal{A}$:

emergence $\nearrow$	$\underrightarrow{Lim}$(frames): Spec $S = FE$ comonad $\searrow$	***kinematical relativity***
counit $\varepsilon \swarrow$	Arveson spec : $V \uparrow\downarrow I$: Spec subsp	$\searrow$ $\downarrow$: local net
States $\mathcal{A} \leftrightarrows$	bimodule of adjoint pair $\underset{E}{\overset{F}{\leftrightarrows}}$	$\mathcal{X}$ Algebra
dynamical relativity	$\uparrow\downarrow$: Galois	$\swarrow$ η: unit
	Dyn: monad $T = EF$ $\underleftarrow{Lim}$(dynamics)	$\nearrow$ co-emergence

Here the left adjoint functor F intertwines $FT = FEF = SF$ from T to S and the right one E intertwines $ES = EFE = TE$ from S to T. The adjunction as natural isomorphisms $\mathcal{A}(a \leftarrow Fx) \underset{E(-)\eta_x}{\overset{\varepsilon_a F(-)}{\leftrightarrows}} \mathcal{X}(Ea \leftarrow x)$ is characterized by the two sets of identities $\begin{pmatrix} & FEF & \\ \varepsilon F \swarrow & \circlearrowleft & \nwarrow F\eta \\ F & = & F \end{pmatrix}$ and $\begin{pmatrix} E & = & E \\ E\varepsilon \nwarrow & \circlearrowright & \swarrow \eta E \\ & EFE & \end{pmatrix}$, as a homotopical extension of Fierz duality $E = F^{-1} \leftrightarrows F = E^{-1}$ between the orthgonality $FE = I_{\mathcal{A}}$ and the completeness $EF = I_{\mathcal{X}}$ of Fourier & inverse-Fourier transforms.

2 Galois-like Functors in *-categories

If the microscopic dynamics and the internal symmetry of the system are known from the outset, the principle of kinematical relativity tells us that observable quantities observed in reality are essentially the invariants under the transformations of dynamics and symmetry. Since we do not live in the microscopic world, however, all what we can do is just to *guess* the *invisible* microscoipic dynamics and the internal symmetry on the basis of *visible* macroscopic data, essentially consisting of invariants under the transformations.

Therefore, the most essential tools in our scientific activities should be found in the methods to determine unknown quantities by solving such equations that the known coefficients are given in terms of observable invariants and that unobservable non-invariants are the unknown variables to be solved. For this reason, we need the basic concepts pertaining to the Galois theory of equations, among which the most important one is the Galois group. In the usual definition, a Galois group $G = Gal(\mathcal{X}/\mathcal{A}) =: G(\mathcal{X}, \mathcal{A})$ is defined by a pair of an algebra $\mathcal{X}$ containing knowns and unknowns, the former of which constitutes a subalgebra $\mathcal{A}$ of $\mathcal{X}$ providing coefficients of the equations, while the "quotient" $\mathcal{X}/\mathcal{A}$ has no actual meaning. If we interpret the symbol $/\mathcal{A}$ as $\mathcal{A}$ to be reduced to scalars, however, we can regard $\mathcal{X}/\mathcal{A}$ as a G-module whose inverse Fourier transform becomes $Gal(\mathcal{X}/\mathcal{A})$. With the aid of natural transformations, this re-interpretation can be extended categorically, according to which we obtain functors to extract groups or algebras from *-categories of modules as follows:

a) $G := End_{\otimes}(V : \mathcal{T}_{DR} \hookrightarrow FHilb)$: in Doplicher-Roberts sector theory [5], the group G of unbroken internal symmetry is recovered from the Doplicher-Roberts category $\mathcal{T}_{DR}(\subset End(\mathcal{A}))$ consisting of modules describing local excitations via the formula $G := End_{\otimes}(V : \mathcal{T}_{DR} \hookrightarrow FHilb)$ as the group of unitary $\otimes$-natural transformations u from the embedding functor V of $\mathcal{T}_{DR}$ into the category $FHilb$ of finite-dimensional Hilbert spaces

to V: $\begin{array}{ccc} V_{\gamma_1} & \overset{u(\gamma_1)=\gamma_1(u)}{\longleftarrow} & V_{\gamma_1} \\ T\downarrow & \circlearrowleft & \downarrow T \\ V_{\gamma_2} & \underset{u(\gamma_2)=\gamma_2(u)}{\longleftarrow} & V_{\gamma_2} \end{array}$ for $\gamma_i \in \mathcal{T}_{DR}$ and $T \in \mathcal{T}_{DR}(\gamma_2 \leftarrow \gamma_1)$ and

$$\gamma_1(u) \otimes \gamma_2(u) = u(\gamma_1) \otimes u(\gamma_2) = u(\gamma_1 \otimes \gamma_2) = (\gamma_1 \otimes \gamma_2)(u).$$

b) $Nat(I : Mod_B \hookrightarrow Hilb) = B''$: Rieffel's device to extract the universal enveloping von Neumann algebra B'' from the category Mod_B of B-modules, in terms of natrual transformations from the embedding functor I to itself.

b') Takesaki-Bichteler's admissible family of operator fields on $Rep(B \to \mathfrak{H})$ in a sufficiently big Hilbert space $\mathfrak{H}$ to reproduce a von Neumann algebra B (: the example focused up in Dr. Okamura's PhD thesis as a non-commutative extension of Gel'fand-Naimark theorem).

With the aid of this machinery, such a perspective (which has long been advocated by Dr.Saigo and also emphasized recently by Dr.Okamura) can now be envisaged that all the contents of Quantum Field Theory can be unified into a C*-tensor category of physical quantities (joint work in progress).

3 Symmetry Breaking and Emergence of Sector-classifying Space

To understand the third item, 3) emergence processes via "forcing" [Macro $\Longleftarrow$ Micro], at the beginning, it is important to realize the sector-classifying space typically emerging from spontaneous breakdown of symmetry of a dynamical system $\mathcal{X} \curvearrowleft G$ with action of a group G (without changing dynamics of the system= "spontaneous"). For this purpose, we need

Criterion for Symmetry Breaking given by non-triviality of ***central*** dynamical system $\mathfrak{Z}_\pi(\mathcal{X}) \curvearrowleft G$ arising from the original one $\mathcal{X} \curvearrowleft G$.

Namely, symmetry G is ***broken in sectors*** $\in Sp(\mathfrak{Z})$ ***shifted non-trivially by central action*** of G. In the infinitesimal version, the Lie algebra $\mathfrak{g}$ of the group G is decomposed into unbroken $\mathfrak{h}$ and broken $\mathfrak{m} := \mathfrak{g}/\mathfrak{h}$, the former of which is vertical to $Sp(\mathfrak{Z})$ and the latter parallel.

The G-transitivity assumption with ***unbroken*** subgroup H in broken G leads to such a specific form of sector-classifying space as $Sp(\mathfrak{Z}) = G/H =: M$. Then, ***classical geometric*** structure on G/H can be seen to arise physically from an ***emergence*** process via ***condensation*** of a family of ***degenerate vacua***, each of which is mutually distinguished by condensed values $\in Sp(\mathfrak{Z})$. In this way, infinite number of low-energy quanta are condensed into geometry of classical Macro objects $\in Sp(\mathfrak{Z}) = G/H$.

In combination with sector structure $\hat{H}$ of unbroken symmetry H, the total sector structure due to this symmetry breaking is described by a "sector bundle" $G \underset{H}{\times} \hat{H}$ with $\hat{H}$ as a standard fiber over a base space G/H consisting of "***degenerate vacua***" [3, 6].

When this geometric structure is established, all the physical quantities are ***parametrized by condensed values*** $\in G/H$. Then, by means of "***logical extension***" of ***constants*** into ***sector-dependent variables***, we find the origin of local gauge structures. On these bases, the duality emerges between ***kinematical*** & ***dynamical*** sorts of "relativity principles" owing to the duality between converging & diverging families of functors between Macro & Micro: [Kinematics in $\underset{\rightarrow}{Lim} \overset{\swarrow}{\underset{\nwarrow}{\longleftarrow}}$] $\overset{\text{duality}}{\leftrightarrows}$ [$\overset{\nwarrow}{\underset{\swarrow}{\longleftarrow}} \underset{\leftarrow}{Lim}$: Dyn in projective limit].

3.1 Symmetric Space Structure of G/H

We see here that this homogeneous space G/H is a ***symmetric space*** equipped with Cartan involution as follows (IO, in preparation). Assuming Lie structures on $G, H, G/H = M$ we denote by $\mathfrak{g}, \mathfrak{h}, \mathfrak{m}$ the corresponding Lie algebraic quantities satisfying $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{m}] = \mathfrak{m}$. Then the validity of $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ provides the homogeneous space M (at least, locally) with a Cartan involution $\mathcal{I}$ to characterize a symmetric space whose eigenvalues are $\mathcal{I} = +1$ on $\mathfrak{h}$ and $\mathcal{I} = -1$ on $\mathfrak{m}$, respectively. Note that $[\mathfrak{m}, \mathfrak{m}]$ is the ***holonomy*** term corresponding to an infinitesimal loop along the ***broken direction*** $G/H = M = Sp(3)$ as ***inter-sectorial space***. Namely, $[\mathfrak{m}, \mathfrak{m}]$ describes the effect of ***broken*** G transformation along an infinitesimal loop on M starting from a point in M and going back to the same point. According to the above Criterion for Symmetry Breaking in terms of ***non-trivial shift under central action*** of G, the absence of $\mathfrak{m}$-components in $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$, follows from the identity of initial and final points of the loop. Thus, $M = G/H = Sp(3)$ is a symmetric space.

3.2 Example 1: Relativity controlled by Lorentz group

Typical example of the above sort can be found in the case of Lorentz group $\mathcal{L}_+^\uparrow =: G$ with an unbroken subgroup of the rotation group $SO(3) =: H$: here, $G/H = M \cong \mathbb{R}^3$ is a symmetric space of Lorentz frames mutually connected by Lorentz boosts.

With $\mathfrak{h} := \{M_{ij}; i, j = 1, 2, 3, i < j\}$, $\mathfrak{m} := \{M_{0i}; i = 1, 2, 3\}$, the validity of $[\mathfrak{h}, \mathfrak{h}] = \mathfrak{h}$, $[\mathfrak{h}, \mathfrak{m}] = \mathfrak{m}, [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ is evident from the basic Lie algebra structure:

$$[iM_{\mu\nu}, iM_{\rho\sigma}] = -(\eta_{\nu\rho} iM_{\mu\sigma} - \eta_{\nu\sigma} iM_{\mu\rho} - \eta_{\mu\rho} iM_{\nu\sigma} + \eta_{\mu\sigma} iM_{\nu\rho}).$$

While both $\mathfrak{h}$ and $\mathfrak{m}$ are taken as unbroken in the standard physics, such results as Borcher-Arveson theorem (: affiliation of Poincaré generators to the algebra of global observables in vacuum situation) and the spontaneous breakdown of Lorentz boosts at $T \neq 0K$ [7] indicate the ***speciality of the vacuum situation with*** $\mathfrak{m}$ ***unbroken***. In this sense, the symmetric space of Lorentz frames $M \cong \mathbb{R}^3$ with [boosts, boosts] = rotations, gives a typical example of symmetric space structure emerging from symmetry breaking (inevitable in non-vacuum situations).

Along this line, typical examples are provided by the chiral symmetry with the current algebra structure $[V, V] = V, [V, A] = A, [A, A] = V$ with vector currents V and axial vector ones A, and also by the conformal symmetry. In the latter case consisting of translations P_μ, Lorentz transformations $M_{\mu\nu}$, scale transformation S and of special conformal transformations K_μ the unbroken $\mathfrak{h}$ part corresponds to $M_{\mu\nu}$ and S, and the broken $\mathfrak{m}$ to P_μ and K_μ, where $\mathfrak{m}$ is the infinitesimal non-compact form of the self-dual Grassmannian manifold acted by the conformal group.

3.3 Example 2: Second law of thermodynamics

Physically most interesting example can be found in thermodynamics: corresponding to $\mathfrak{h} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{m} = \mathfrak{g}/\mathfrak{h}$, we find here an exact sequence $\Delta' Q \hookrightarrow \Delta E = \Delta' Q + \Delta' W \twoheadrightarrow \Delta' W$ due to the first law of thermodynamics, whose precise form can be found in Caratheodory's formulation. With respect to Cartan involution with + assigned to the heat production $\Delta' Q$ and − to the macroscopic work $\Delta' W$, the holonomy $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$ corresponding to a loop in the space M of thermodynamic variables becomes just

Kelvin's version of second law of thermodynamics,

namely, holonomy $[\mathfrak{m}, \mathfrak{m}]$ in the cyclic process with $\Delta E = \Delta' Q + \Delta' W = 0$, describes the heat production $\Delta' Q \geq 0$: $-\Delta' W = -[\mathfrak{m}, \mathfrak{m}] = \Delta' Q > 0$ (from the system to the outside).

Thus, the essence of the second law of thermodynamics is closely related with the geometry of the symmetric space structure of thermodynamic space M consisting of paths of thermodynamic state-changes caused by works $\Delta' W$. Actually, this symmetric space structure can be seen to correspond to its *causal structure* due to state changes via adiabatic processes, which can be interpreted as the mathematical basis of Lieb-Yngvason axiomatics of thermodynamic entropy.

4 Convergent Kinematics at Macro End

In terms of symmetric space structure with Cartan involution, we find, in the basic structures of both relativity and thermodynamcs, essential common features of convergence [Kinematics in $\varinjlim \overset{\swarrow}{\underset{\nwarrow}{\longleftarrow}}$], which seem to be characteristic to Macro side. Because of this convergence, phenomenological diversity due to the presence of ***many reference frames*** is successfully controlled by the relativity principle with the aid of Lorentz-type transformations. What plays crucial roles here is, however, the ***implicitly postulated unicity of the "true physical system"*** entitled by the ***unicity of microscopic law of dynamics***, in sharp contrast to its phenomenological diversity. But who guarantees this assumption?

4.1 One-sidedness inherent in Relativity

From the duality viewpoint between Micro and Macro: [Macro: $\varinjlim \overset{\swarrow}{\underset{\nwarrow}{\longleftarrow}}$] $\overset{\text{duality}}{\leftrightarrows}$ [$\overset{\nwarrow}{\underset{\swarrow}{\longleftarrow}}$ $\varprojlim$: Micro], mentioned at the beginning, we should notice the one-sidedness inherent in the standard picture of relativity: [Kinematics in $\varinjlim \overset{\swarrow}{\underset{\nwarrow}{\longleftarrow}}$], in contrast to the situations on the Micro side: [$\overset{\nwarrow}{\underset{\swarrow}{\longleftarrow}}$ $\varprojlim$: Dyn in projective limit]. This shows the one-sidedness inherent in the idea of relativity, in sharp contrast to the universal validity of thermodynamic consequences applicable to variety of different systems independently of minor details.

4.2 To relativize dynamics of a system

Generalizing the excellent idea of relativity, we can naturally and legitimately ***relativize and pluralize dynamics*** of a physical system whose ***mutual relations are controlled, deformed and compared;*** the ***freedom*** attained by this extension is expected to liberate us from the stereotyped spell of "a physical system *with a fixed law of dynamics*". Then, it will enable us to develop a theoretical framework for describing a physical systems with ***a family of dynamical laws*** exhibited in an array, where their mutual relations are systematically examined from the viewpoints of deformation and evolutionary theories: in [8], the essence of this line of thought has been proposed under the name of "***Theory Bundle***", bundles of theories patched together by the "***method of variation of natural constants***".

5 Framework for Multiple Laws of Dynamics

For the purpose of materializing the essential ideas explained above, a mathematical framework has been proposed by the present author in [1] by connecting the essence of multiple laws of dynamics based on the "***groupoid dynamical systems***" with the concepts of "***sectors***" and of "***sector space***" in the framework of "***quadrality scheme***" based on "***Micro-Macro duality***" [3, 4]. In a word, a groupoid Γ is a family of invertible transformations from an initial point to a final one, which can be thought of as a family of groups scattered over spacetime. Here, the space $\Gamma^{(0)}$ of units carries many interesting physical contexts, in a similar to the concept of base spaces of fiber bundles. In this sense, it provides not only a generalization of the notion of groups in close relation with the basic ideas of local gauge invariance and of general relativity, but also an algebraic and generalized formulation of "equivalence relations" ubiquitously found at the basis of any kind of mathematical descriptions.

5.1 Definition of a groupoid

A groupoid Γ is defined on a set $\Gamma^{(0)}$ (called unit space) in combination with two maps $s, t : \Gamma \to \Gamma^{(0)}$ characterized by the following three properites R1), R2), R3). When $t(\gamma) = x \in \Gamma^{(0)}, s(\gamma) = y \in \Gamma^{(0)}$, we write $x \overset{\gamma}{\leftarrow} y$ or $\gamma : x \leftarrow y$, calling x and y, respectively, the target and source of an arrow $\gamma \in \Gamma$ from y to x:

R1) For any $x \in \Gamma^{(0)}$, there is an arrow $x \overset{1_x}{\leftarrow} x$ from x to x called a unit arrow :

R2) when $x \overset{\gamma_1}{\leftarrow} y$ and $y \overset{\gamma_2}{\leftarrow} z$, there exists a composition $x \overset{\gamma_1\gamma_2}{\leftarrow} z$ of arrows γ_1 and γ_2 from z to x.

R3) when γ is an arrow $x \overset{\gamma}{\leftarrow} y$ from y to x, there exists the inverse $\gamma^{-1} \in \Gamma$ from x to y in the sense of $\gamma\gamma^{-1} = 1_x : x \leftarrow x$ and of $\gamma^{-1}\gamma = 1_y : y \leftarrow y$. the sense of $\gamma\gamma^{-1} = 1_x : x \leftarrow x$ and of $\gamma^{-1}\gamma = 1_y : y \leftarrow y$.

If we define a relation R on $\Gamma^{(0)}$ by $R(x, y) = (\exists\gamma \in \Gamma$ such that $x \overset{\gamma}{\leftarrow} y)$, then R1), R2), R3) are equivalent to the laws of symmetry, transitivity, and reflexivity, respectively. In this way, a groupoid is an algebraic generalization of an equivalence relation. While the equivalence relation $R(x, y)$ is symmetric in x, y owing to R3), we retain the direction of arrows $x \overset{\gamma}{\leftarrow} y$ for the purpose of unified treatment of such relations with preferred directions as order relations or arrows of time. The totality of the arrows γ is called a groupoid Γ and the set $\Gamma^{(0)}$ of x, y, etc., connected by the arrows $\gamma \in \Gamma$ in such a way as $x \overset{\gamma}{\leftarrow} y$ is called the "unit space" of the groupoid Γ. The element $y \in \Gamma^{(0)}$ in $x \overset{\gamma}{\leftarrow} y$ is called the source of γ and denoted by $s(\gamma) = y$, and, in this situation, $x \in \Gamma^{(0)}$ is called the target of γ and denoted by $t(\gamma) = y$.

In this context, a groupoid Γ can be viewed as a special sort of categories, all of the arrows of which are invertible. Then, the unit space $\Gamma^{(0)}$ is nothing but the set of objects of the category Γ, where

R1) means the assignment of the identity arrow 1_x corresponding to an object $x \in \Gamma^{(0)}$,

R2) explains the relation among the source, target and the composition of arrows in the category Γ,

R3) means the invertibility of all the arrows in Γ.

It can be easily understood that a groupoid is a generalization of the concept of a group and that a group is a special case of a groupoid: for this purpose, we equip a group G with a (virtual) object $*$ which is regarded as connected by any group element $g \in G$ to itself: $* \overset{g}{\leftarrow} *$. In this way, a group G can be viewed as a groupoid G whose unit space is given by $\{*\}$.

The important difference between a general groupoid and a group can be found in that any pair $(g_1, g_2) \in G \times G$ of group elements can be composable: $(g_1, g_2) \longmapsto g_1 g_2 \in G$, whereas the product $\gamma_1\gamma_2$ of a pair $(\gamma_1, \gamma_2) \in \Gamma \times \Gamma$ can be defined only when the condition $s(\gamma_1) = t(\gamma_2)$ is satisfied: $\gamma_1\gamma_2 = [t(\gamma_1) \leftarrow s(\gamma_1) = t(\gamma_2) \leftarrow s(\gamma_2)] = [t(\gamma_1) \overset{\gamma_1\gamma_2}{\leftarrow} s(\gamma_2)]$.

The set of all the composable pairs (γ_1, γ_2) is denoted by $\Gamma^{(2)}$, which can be identified with the fiber product : $\Gamma^{(2)} := \{(\gamma_1, \gamma_2) \in \Gamma \times \Gamma; s(\gamma_1) = t(\gamma_2)\} = \Gamma \underset{\Gamma^{(0)}}{\times} \Gamma$ characterized by a commutative diagram:

$$\begin{tikzcd} \Gamma \arrow[d, "s"'] & \Gamma \underset{\Gamma^{(0)}}{\times} \Gamma \arrow[l, "pr_1"'] \arrow[d, "pr_2"] \arrow[dl, phantom, "\circlearrowleft"] \\ \Gamma^{(0)} & \Gamma \arrow[l, "t"] \end{tikzcd}.$$

For $x, y \in \Gamma^{(0)}$ we denote $\Gamma^x_y := \{\gamma \in \Gamma; t(\gamma) = x, s(\gamma) = y\} = \Gamma(x \leftarrow y)$. Since $\Gamma^x_x \subset \Gamma^{(2)}$, any pair of elements in the subgroupoid Γ^x_x are composable, and hence it is a group $\Gamma^x_x \subset \Gamma$ for any $x \in \Gamma^{(0)}$.

Among many useful and important examples of groupoids, we should mention the concept of transformation groupoid $\Gamma = M \times G$ based on an action of G on M, where the product in the groupoid Γ is defined by $(x_1, g_1) \cdot (x_2, g_2) := (x_1, g_1 g_2)$ under the condition of $s(x_1, g_1) = g_1^{-1} x_1 = x_2 = t(x_2, g_2)$.

5.2 Symmetry breaking patterns classified by unit space $\Gamma^{(0)}$

Going back to the context of symmetry breaking in terms of a theory bundle on the sector-classifying space $Spec(\mathfrak{Z}_\pi(\mathcal{X})) \subset F_{\mathcal{X}}$, we see that the action of broken G-symmetry on it can be identified, under the assumption of transitivity, with a transformation groupoid $\Gamma = \Gamma^{(0)} \times G$ with the G-transitive unit space $\Gamma^{(0)} := \mathrm{supp}\ \mu_\omega = G/H$.

Some remarks on transitivity

$\Gamma^{(0)} = G/H$: transitivity + symmetric space

$\subset$ ergodicity = measure-theoretical transitivity

$\Gamma^{(0)} = \amalg(G/H_i)$: orbit decomposition, ergodic decomposition

Then, in terms of the unit space $\Gamma^{(0)} \subset F_{\mathcal{X}}$, breaking patterns of the symmetry described by $G \underset{\tau}{\curvearrowright} \mathcal{X}$ can be classified into ***unbroken***, ***spontaneously broken***, ***explicitly broken*** ones as follows:

(i) ***unbroken***: $\Gamma^{(0)}_{unbroken}$ = one-point set

(ii) ***spontanesously broken***: $\Gamma^{(0)}_{SSB}$ = sector bundle $G \underset{H}{\times} \hat{H}$ of a theory with a ***fixed dynamics***, whose base space G/H consists of degenerate vacua and whose fibers consist of sectors $\hat{H}$ of unbroken symmetry H

(iii) ***explicitly broken***: $\Gamma^{(0)}_{\text{explicit br.}}$ = double-layer bundle of sectors, whose base space consists of physical constants to parametrize ***different dynamics***, upon each point of which we have a sector bundle $\Gamma^{(0)}_{SSB}$ of SSB corresponding to a fixed dynamics.

6 Applications to Local Gauge Invariance and Renormalization

In the systematic use of the machinery developed above, we can reformulate theories of local gauge invariance and of renormalization from novel physical viewpoints. Because of the limitation of the available space, however, we refere the detailed account of the results to the following two proceeding articles

1) local gauge invariance: see [1],

2) renormalization: in this case, duality between "cutoffs" (or, regularizations of ultra-violet divergences) will play improtant roles (see [9]).

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