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# Necessary condition for existence of conditional SIC-POVM

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## 1 Introduction

This paper is an announcement of our result and the detailed version will be submitted to somewhere. (See [2].)

POVMs (positive operator valued measure) on quantum systems are considered as a measurement in quantum physics. If POVMs have a good condition, then we can determine a state by results of a measurement. In this case, the POVM is called informationally complete and the process is called quantum state tomography.

First, we introduce SIC-POVMs (symmetric informationally complete POVM). SIC-POVMs is generated by vectors in $\mathbb{C}^n$ whose the absolute values of inner products of each vectors are same. Zauner conjectured that there exist such $n^2$ vectors in $\mathbb{C}^n$ for any $n$. But the existence is only proved when $n \leq 15$ and $n = 19, 24, 35, 48$ [5, 8].

Next, we introduce conditional SIC-POVMs. A state of a quantum system is a density matrix which has several parameters. When a few parameters are known, then SIC-POVM is not the best measurement to determine the state. Hence we need another POVM and it is a conditional SIC-POVM. Conditional SIC-POVMs are also generated by vectors in $\mathbb{C}^n$. But the existence of conditional SIC-POVMs depends on the system. We will discuss the details in Sect. 4.

## 2 Preliminaries

**Definition 2.1** *$\rho \in M_n(\mathbb{C})$ is called a density matrix (or state) if $\rho \geq 0$ and*

$$\mathrm{Tr}(\rho) = 1.$$

For any density matrix $\rho$, we can define a state $\hat{\rho}$ by

$$\hat{\rho}(X) = \mathrm{Tr}(\rho X).$$

Conversely, any state is written by the above form. Therefore, there exists a one-to-one correspondence between density matrices and states.

**Definition 2.2** *A set of positive operators $\{P_i\}_{i=1}^k \subset M_n(\mathbb{C})$ is called a positive operator valued measure (POVM) if $P_i \geq 0$ $(1 \leq i \leq k)$ and*

$$\sum_{i=1}^{k} P_i = I.$$

*If $P_i$ is a projection, then $\{P_i\}_{i=1}^k$ is called a projection valued measure (PVM).*

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For a POVM $\{P_i\}_{i=1}^k$, we can consider a measurement device by using this POVM:

$$\boxed{\rho} \xrightarrow{input} \boxed{\text{Measurement device by } \{P_i\}_{i=1}^k}$$

$$\xrightarrow{output} \begin{cases} 1 & \text{with probability } \mathrm{Tr}(\rho P_1) \\ 2 & \text{with probability } \mathrm{Tr}(\rho P_2) \\ \vdots & \qquad\vdots \\ k & \text{with probability } \mathrm{Tr}(\rho P_k) \end{cases}$$

This means that we can know $\mathrm{Tr}(\rho P_i)$ for all $(1 \leq i \leq k)$, if we have many copies of $\rho$. Therefore, if a POVM $\{P_i\}_{i=1}^k$ has a good condition, then we can determine the quantum state $\rho$ by $\mathrm{Tr}(\rho P_i)$. This is called **quantum state tomography**.

**Definition 2.3** *A POVM $\{P_i\}_{i=1}^k$ is called informationally complete if for density matrices $\rho \neq \sigma$, there exists $P_i$ such that*

$$\mathrm{Tr}(\rho P_i) \neq \mathrm{Tr}(\sigma P_i).$$

If a POVM $\{P_i\}_{i=1}^k$ is informationally complete, then we can determine the quantum state $\rho$ by $\mathrm{Tr}(\rho P_i)$. If $\rho$ is a state in $M_n(\mathbb{C})$, then the following statement holds.

**Theorem 2.4** *A POVM $\{P_i\}_{i=1}^k \subset M_n(\mathbb{C})$ is informationally complete if and only if*

$$\mathrm{span}\{P_i\}_{i=1}^k = M_n(\mathbb{C}).$$

*In particular, we can determine the quantum state $\rho$ by* $\mathrm{Tr}(\rho P_i)$.

**Example 2.5** For small $\varepsilon > 0$. Let

$$\begin{aligned} P_1 &= \frac{1}{2+2\varepsilon^2}\begin{bmatrix} 1 & \varepsilon \\ \varepsilon & \varepsilon^2 \end{bmatrix}, P_2 = \frac{1}{2+2\varepsilon^2}\begin{bmatrix} 1 & -\varepsilon \\ -\varepsilon & \varepsilon^2 \end{bmatrix}, \\ P_3 &= \frac{1}{2+2\varepsilon^2}\begin{bmatrix} \varepsilon^2 & \varepsilon i \\ -\varepsilon i & 1 \end{bmatrix}, P_4 = \frac{1}{2+2\varepsilon^2}\begin{bmatrix} \varepsilon^2 & -i\varepsilon \\ i\varepsilon & 1 \end{bmatrix}. \end{aligned}$$

Then $\{P_i\}_{i=1}^4$ is an informationally complete POVM. For a state $\rho$ in $M_n(\mathbb{C})$, by equations

$$\begin{aligned} (2+2\varepsilon^2)\mathrm{Tr}(\rho P_1) &= \rho_{11} + \varepsilon\rho_{12} + \varepsilon\rho_{21} + \varepsilon^2\rho_{22} \\ (2+2\varepsilon^2)\mathrm{Tr}(\rho P_2) &= \rho_{11} - \varepsilon\rho_{12} - \varepsilon\rho_{21} + \varepsilon^2\rho_{22} \\ (2+2\varepsilon^2)\mathrm{Tr}(\rho P_3) &= \varepsilon^2\rho_{11} - i\varepsilon\rho_{12} + i\varepsilon\rho_{21} + \rho_{22} \\ (2+2\varepsilon^2)\mathrm{Tr}(\rho P_4) &= \varepsilon^2\rho_{11} + i\varepsilon\rho_{12} - i\varepsilon\rho_{21} + \rho_{22}, \end{aligned}$$

we have

$$\rho = \mathrm{Tr}(\rho P_1)\begin{bmatrix} \dfrac{1}{1-\varepsilon^2} & \dfrac{1}{4\varepsilon(1+\varepsilon^2)} \\ \dfrac{1}{4\varepsilon(1+\varepsilon^2)} & \dfrac{-\varepsilon^2}{1-\varepsilon^2} \end{bmatrix} + \mathrm{Tr}(\rho P_2)\begin{bmatrix} \dfrac{1}{1-\varepsilon^2} & \dfrac{-1}{4\varepsilon(1+\varepsilon^2)} \\ \dfrac{-1}{4\varepsilon(1+\varepsilon^2)} & \dfrac{-\varepsilon^2}{1-\varepsilon^2} \end{bmatrix}$$
$$+ \mathrm{Tr}(\rho P_3)\begin{bmatrix} \dfrac{-\varepsilon^2}{1-\varepsilon^2} & \dfrac{i}{4\varepsilon(1+\varepsilon^2)} \\ \dfrac{-i}{4\varepsilon(1+\varepsilon^2)} & \dfrac{1}{1-\varepsilon^2} \end{bmatrix} + \mathrm{Tr}(\rho P_4)\begin{bmatrix} \dfrac{-\varepsilon^2}{1-\varepsilon^2} & \dfrac{-i}{4\varepsilon(1+\varepsilon^2)} \\ \dfrac{i}{4\varepsilon(1+\varepsilon^2)} & \dfrac{1}{1-\varepsilon^2} \end{bmatrix}.$$

Hence we can determine a state $\rho$. But this is not a good POVM to detect $\rho$. Since

$$\rho_{12} = \frac{1}{4\varepsilon(1+\varepsilon^2)}\left(\mathrm{Tr}(P_1\rho) - \mathrm{Tr}(P_2\rho) - i\left(\mathrm{Tr}(P_3\rho) - \mathrm{Tr}(P_4\rho)\right)\right),$$

a small error causes a big difference.

If a POVM is informationally complete, then we can determine a state $\rho$ by $\{\mathrm{Tr}(\rho P_i)\}_{i=1}^k$. But by $\ell$ experiments, we can only obtain approximate values of $\{\mathrm{Tr}(\rho P_i)\}_{i=1}^k$. Let the candidate generated by these approximate values be $\hat{\rho}$. A POVM is called optimal, if the expected value of

$$\|\rho - \hat{\rho}\|_2$$

is the minimum among all candidates generated by any POVM and $\ell$ experiments. If a POVM is optimal, then it satisfies the following condition.

**Theorem 2.6** [4] *A POVM in $M_n(\mathbb{C})$ with rank one positive operators $\{\frac{n}{k}P_i\}_{i=1}^k$ are optimal POVM if and only if*

$$\sum_{i=1}^{k} \frac{n}{k}|P_i\rangle\langle P_i| = \frac{1}{n+1}\left(\mathrm{id}_{M_n(\mathbb{C})} + |I\rangle\langle I|\right),$$

*where $|P_i\rangle\langle P_i|$ is a superoperator $M_n(\mathbb{C}) \to M_n(\mathbb{C})$ with $A \mapsto \mathrm{Tr}(AP_i)P_i$.*

# 3 SIC-POVM

In this section, we introduce a SIC-POVM (symmetric informationally complete positive operator valued measure) which is an optimal POVM.

**Definition 3.1** *A set of vectors $\{\xi_i\}_{i=1}^{n^2} \subset \mathbb{C}^n$ is called symmetric informationally complete POVM (SIC-POVM) if*

$$|\langle \xi_i, \xi_j\rangle| = \frac{1}{\sqrt{n+1}}.$$

*A POVM generated by the above vectors*

$$\left\{\frac{1}{n}|\xi_i\rangle\langle\xi_i|\right\}_{i=1}^{n^2}$$

*is also called a SIC-POVM, where* $|x\rangle\langle y|z = \langle y, z\rangle x$ *for all* $x, y, z \in \mathbb{C}^n$.

A SIC-POVM is informationally complete. Indeed, if we assume

$$\sum_{i=1}^{n^2} a_i|\xi_i\rangle\langle\xi_i| = 0,$$

then for all $1 \leq j \leq n^2$ we have

$$0 = \mathrm{Tr}\left(\sum_{i=1}^{n^2} a_i|\xi_i\rangle\langle\xi_i| \cdot |\xi_j\rangle\langle\xi_j|\right) = a_j + \frac{1}{n+1}\sum_{i\neq j} a_i.$$

So it is easy to see that $\{|\xi_i\rangle\langle\xi_i|\}_{i=1}^{n^2}$ is linearly independent. Moreover, for all $1 \leq j \leq n^2$,

$$\mathrm{Tr}\left(\sum_{i=1}^{n^2} \frac{1}{n}|\xi_i\rangle\langle\xi_i| \cdot |\xi_j\rangle\langle\xi_j|\right) = \frac{1}{n} + \sum_{i\neq j}\frac{1}{n(n+1)} = 1.$$

Hence $\sum_{i=1}^{n^2} \frac{1}{n}|\xi_i\rangle\langle\xi_i| = I$.

**Example 3.2** In $\mathbb{C}^2$,

$$\xi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \xi_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}, \xi_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ \lambda\sqrt{2} \end{bmatrix}, \xi_4 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ \bar{\lambda}\sqrt{2} \end{bmatrix}$$

is a SIC-POVM, where $\lambda = e^{2\pi i/3} = \dfrac{-1+\sqrt{3}i}{2}$.

In $\mathbb{C}^3$,

$$\begin{aligned}
\xi_1 &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \xi_2 = \frac{1}{2}\begin{bmatrix} 1 \\ \sqrt{3} \\ 0 \end{bmatrix}, \xi_3 = \frac{1}{2}\begin{bmatrix} 1 \\ -\sqrt{3} \\ 0 \end{bmatrix}, \\
\xi_4 &= \frac{1}{2}\begin{bmatrix} 1 \\ i \\ \sqrt{2} \end{bmatrix}, \xi_5 = \frac{1}{2}\begin{bmatrix} 1 \\ i \\ \sqrt{2}\lambda \end{bmatrix}, \xi_6 = \frac{1}{2}\begin{bmatrix} 1 \\ i \\ \sqrt{2}\bar{\lambda} \end{bmatrix}, \\
\xi_7 &= \frac{1}{2}\begin{bmatrix} 1 \\ -i \\ \sqrt{2} \end{bmatrix}, \xi_8 = \frac{1}{2}\begin{bmatrix} 1 \\ -i \\ \sqrt{2}\lambda \end{bmatrix}, \xi_9 = \frac{1}{2}\begin{bmatrix} 1 \\ -i \\ \sqrt{2}\bar{\lambda} \end{bmatrix},
\end{aligned}$$

is a SIC-POVM, where $\lambda = e^{2\pi i/3}$.

It is known that a SIC-POVM exists if $n \leq 15$ or $n = 19, 24, 35, 48$. Numerical solutions have been found when $n \leq 67$. But for other cases, the existence is an open problem. The following is conjectured by G. Zauner in 1999 [7].

Let

$$W = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \lambda & 0 & \dots & 0 \\ 0 & 0 & \lambda^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda^{n-1} \end{bmatrix}, S = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{bmatrix},$$

where $\lambda = \exp \frac{2\pi i}{n}$. $\{W^k S^\ell\}_{k,\ell=1}$ are called generalized Pauli matrices.

**Definition 3.3** *A unit vector $\xi$ is called a fiducial vector if*

$$\{W^k S^\ell \xi\}_{k,\ell=1}^n$$

*is a SIC-POVM.*

**Conjecture** (Zauner's conjecture [7]) A fiducial vector exists in $\mathbb{C}^n$ for all $n \geq 2$. In particular, a SIC-POVM exists in $\mathbb{C}^n$.

# 4 Conditional SIC-POVM

Recently, SIC-POVMs in arbitrary subspace of $M_n(\mathbb{C})$ are also considered. Let $M_n(\mathbb{C})$ be decomposed as

$$M_n(\mathbb{C}) = \mathbb{C}I \oplus \mathcal{A} \oplus \mathcal{B}$$

and $\rho \in M_n(\mathbb{C})$ be a density matrix. Let $P_\mathcal{A}$ and $P_\mathcal{B}$ be projections onto $\mathcal{A}$ and $\mathcal{B}$. Assume we know $P_\mathcal{A}\rho$. Then we want to know an optimal POVM which determines $\rho$. Since we already know $P_\mathcal{A}\rho$, SIC-POVM is not suitable.

Let $\dim \mathcal{A} = m$ then $\dim \mathcal{B} = n^2 - m - 1$ and let $N = n^2 - m$. For rank one informationally complete POVM $\{\frac{n}{k}P_i\}_{i=1}^k$, let

$$\mathcal{F} = \frac{n}{k}|P_i\rangle\langle P_i|.$$

Then the following theorem holds.

**Theorem 4.1** [3] *Rank one informationally complete POVM $\{\frac{n}{N}P_i\}_{i=1}^N$ is optimal if and only if*

$$\mathcal{F} = |I\rangle\langle I| + \frac{n-1}{N-1}P_\mathcal{B}.$$

*In this case,*

$$\sum_{i=1}^k P_i = \frac{N}{n}I, \qquad \mathrm{Tr}(P_i P_j) = \frac{N-n}{n(N-1)}.$$

Such POVM is called a conditional SIC-POVM. Examples of conditional SIC-POVMs are following.

**Example 4.2** If we do not have any information a priory about the state ($m = 0, N = n^2$), then

$$\mathrm{Tr} P_i P_j = \frac{1}{n+1} \quad (i \neq j)$$

so the optimal POVM is the well-known SIC-POVM (if it exists).

**Example 4.3** If we know the off-diagonal elements of the state, and we want to estimate the diagonal entries ($m = n^2 - n, N = n$), then from Theorem 4.1 it follows that the optimal POVM has the properties

$$\mathrm{Tr} P_i P_j = 0 \quad (i \neq j), \quad \sum_{i=1}^{n} P_i = I, \quad \text{and} \quad P_i \text{ is diagonal.}$$

So the diagonal matrix units form an optimal POVM.

**Example 4.4** If we know the diagonal elements of the state, and we want to estimate the off-diagonal entries ($m = n-1, N = n^2-n+1$), then from Theorem 4.1 it follows that the optimal POVM has the properties

$$\mathrm{Tr} P_i P_j = \frac{n-1}{n^2} \quad (i \neq j), \quad \sum_{i=1}^{n} P_i = \frac{n^2-n+1}{n} I$$

and $P_i$ has a constant diagonal.

The existence is not clear generally, but if $n-1$ is a prime power then it can be constructed. Details are written in [3].

Next, we present a necessary condition for existence of a conditional SIC-POVM.

**Lemma 4.5** *Let $\{P_i\}_{i=1}^N$ be a conditional SIC-POVM in $A \oplus C$ and let*

$$Q_i = \sqrt{\frac{n(N-1)}{N(n-1)}} \left( P_i - \frac{1}{n} \left( 1 + \sqrt{\frac{n-1}{N-1}} \right) I \right). \tag{1}$$

*Then $\{Q_i\}_{i=1}^N$ is an orthonormal basis of $A \oplus C$.*

Proof. For any $1 \le i \le N$, we have

$$\begin{aligned}
&\mathrm{Tr}\left(\left(P_i - \frac{1}{n}\left(1+\sqrt{\frac{n-1}{N-1}}\right)I\right)^2\right)\\
&= \mathrm{Tr}\left(P_i - \frac{2}{n}\left(1+\sqrt{\frac{n-1}{N-1}}\right)P_i + \frac{1}{n^2}\left(1+\sqrt{\frac{n-1}{N-1}}\right)^2 I\right)\\
&= 1 - \frac{2}{n}\left(1+\sqrt{\frac{n-1}{N-1}}\right) + \frac{1}{n}\left(1+2\sqrt{\frac{n-1}{N-1}}+\frac{n-1}{N-1}\right)\\
&= 1 - \frac{1}{n} + \frac{n-1}{n(N-1)}\\
&= \frac{N(n-1)}{n(N-1)}.
\end{aligned}$$

Moreover, for any $1 \le i < j \le N$,

$$\begin{aligned}
&\mathrm{Tr}\left(\left(P_i - \frac{1}{n}\left(1+\sqrt{\frac{n-1}{N-1}}\right)I\right)\left(P_j - \frac{1}{n}\left(1+\sqrt{\frac{n-1}{N-1}}\right)I\right)\right)\\
&= \frac{N-n}{n(N-1)} - \frac{2}{n}\left(1+\sqrt{\frac{n-1}{N-1}}\right) + \frac{1}{n}\left(1+2\sqrt{\frac{n-1}{N-1}}+\frac{n-1}{N-1}\right)\\
&= \frac{N-n}{n(N-1)} - \frac{1}{n} + \frac{n-1}{n(N-1)} = 0.
\end{aligned}$$

These equations imply $\langle Q_i, Q_j\rangle = \mathrm{Tr}(Q_i^* Q_j) = \delta_{ij}$ so that $\{Q_i\}_{i=1}^N$ is an orthonormal basis of $A \oplus C$. □

**Theorem 4.6** *If there exists a conditional SIC-POVM in $A \oplus C$, then for any $X \in B$ and any orthonormal basis $\{R_i\}_{i=1}^m$ of $B$,*

$$\sum_{i=1}^m R_i^* X R_i = \frac{N-n}{n(n-1)} X.$$

Proof. Let $\{P_i\}_{i=1}^N$ be a conditional SIC-POVM in $A \oplus C$ and define $\{Q_i\}_{i=1}^N$ by (1). Then from the previous lemma, $\{Q_1, \ldots, Q_N, R_1, \ldots R_m\}$ is an orthonormal basis of $M_n(\mathbb{C})$. It is well known that

$$\sum_{i=1}^N Q_i^* X Q_i + \sum_{i=1}^m R_i^* X R_i = \mathrm{Tr}(X).$$

$B$ is orthogonal to $A = \mathbb{C}I$ so that $\mathrm{Tr}(X) = 0$. Hence we will calculate $\sum_{i=1}^N Q_i^* X Q_i$. Since $P_i$ is a rank one projection, $P_i X P_i = tP_i$ for some $t \in \mathbb{C}$. But $\mathrm{Tr}(P_i X P_i) =$

$\langle P_i, X\rangle = 0$ implies $t = 0$. Therefore $P_i X P_i = 0$. From the equation

$$\sum_{i=1}^{N} P_i = \frac{N}{n} I,$$

we have

$$\begin{aligned}
&\frac{N(n-1)}{n(N-1)} \sum_{i=1}^{N} Q_i^* X Q_i \\
=\ & \sum_{i=1}^{N} \left(P_i - \frac{1}{n}\left(1+\sqrt{\frac{n-1}{N-1}}\right)\right) X \left(P_i - \frac{1}{n}\left(1+\sqrt{\frac{n-1}{N-1}}\right)\right) \\
=\ & \sum_{i=1}^{N} \left(P_i X P_i - \frac{1}{n}\left(1+\sqrt{\frac{n-1}{N-1}}\right)(XP_i + P_i X) + \frac{1}{n^2}\left(1+\sqrt{\frac{n-1}{N-1}}\right)^2 X\right) \\
=\ & \left(-\frac{1}{n}\left(1+\sqrt{\frac{n-1}{N-1}}\right)(X\sum_{i=1}^{N} P_i + \sum_{i=1}^{N} P_i X) + \frac{N}{n^2}\left(1+\sqrt{\frac{n-1}{N-1}}\right)^2 X\right) \\
=\ & \left(-\frac{2N}{n^2}\left(1+\sqrt{\frac{n-1}{N-1}}\right) + \frac{N}{n^2}\left(1+2\sqrt{\frac{n-1}{N-1}} + \frac{n-1}{N-1}\right)\right) X \\
=\ & \frac{N}{n^2}\left(-1+\frac{n-1}{N-1}\right) X = \frac{N(n-N)}{n^2(N-1)} X.
\end{aligned}$$

This implies the assertion. □

**Example 4.7** Now we consider $M_4(\mathbb{C}) = M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$. A density matrix

$$\rho = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

has reduced densities:

$$\rho_1 = \begin{bmatrix} a_{11}+a_{22} & a_{13}+a_{24} \\ a_{31}+a_{42} & a_{33}+a_{44} \end{bmatrix}, \qquad \rho_2 = \begin{bmatrix} a_{11}+a_{33} & a_{12}+a_{34} \\ a_{21}+a_{43} & a_{22}+a_{44} \end{bmatrix}.$$

The condition $\rho_1 = \rho_2$ implies

$$a_{22} = a_{33} \quad \text{and} \quad a_{13}+a_{24} = a_{12}+a_{34}.$$

Let

$$R_1 = \frac{1}{\sqrt{2}}(e_{22}-e_{33}), \quad R_2 = \frac{1}{2}(e_{12}-e_{13}-e_{24}+e_{34}), \quad R_3 = \frac{1}{2}(e_{21}-e_{31}-e_{42}+e_{43}),$$

and $B = \text{span}\{R_1, R_2, R_3\}$, then

$$\rho = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12}^* & b & a_{23} & c - a_{13} \\ a_{13}^* & a_{23}^* & b & c - a_{12} \\ a_{14}^* & c^* - a_{13}^* & c^* - a_{12}^* & a_{44} \end{bmatrix}$$

is orthogonal to $B$. But a conditional SIC-POVM for this $\rho$ does not exist. Indeed, the equations

$$R_1^* R_1 R_1 = \frac{1}{2} R_1, \quad R_2^* R_1 R_2 = 0, \quad R_3^* R_1 R_3 = 0$$

imply $\sum_{i=1}^3 R_i^* R_1 R_i = \frac{1}{2} R_1$ and this is in contradict to the condition in Theorem 4.6.

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