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# On the arc index of knots and links

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## 1 Introduction

A knot is a one-dimensional circle embedded in three-dimensional space and a link is a disjoint union of knots which may be tangled up together. An *arc presentation* of a knot or a link $L$ is a special way of presenting $L$. It is an ambient isotopic image of $L$ contained in the union of finitely many half planes, called *pages*, with a common boundary line, called *binding axis*, in such a way that each half plane contains a properly embedded single simple arc as in Figure 1. It is an essential condition that $L$ meets each page in a single arc. If the requirement is removed then $L$ can be embedded in at most three pages [9]. It is known that every knot or link has an arc presentation [5, 6]. So for a link $L$ we can define an invariant, called *arc index* and denoted by $\alpha(L)$, as the minimum number of pages among all arc presentations of $L$.

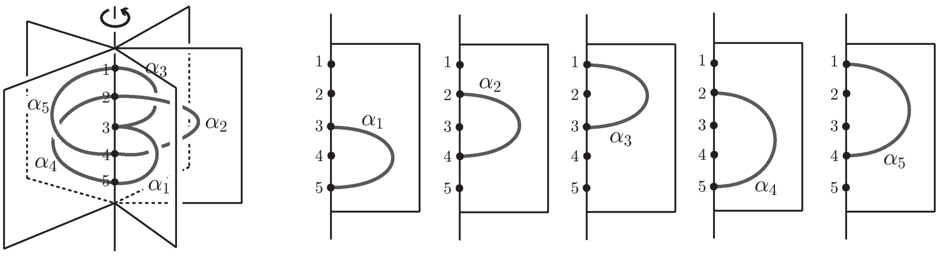


Figure 1: An arc presentation of the trefoil knot

In this paper, we present a small survey and introduce recent results on arc index. Arc presentations were originally described by Brunn [5] more than 100 years ago, when he proved that any link has a diagram with only one multiple point. Birman and Menasco used arc-presentations of companion knots to find braid presentations for some satellites [4]. Cromwell adapted Birman-Menasco's method and used the term "arc index" as an invariant and established some of its basic properties.

**Proposition 1.1 (Cromwell [6])** *Every link has an arc-presentation.*

**Theorem 1.2 (Cromwell [6])** *For two nonalternating links $L_1$ and $L_2$, we have*

$$\alpha(L_1 \sqcup L_2) = \alpha(L_1) + \alpha(L_2),$$
$$\alpha(L_1 \sharp L_2) = \alpha(L_1) + \alpha(L_2) - 2.$$

The theory of arc-presentations was developed by Dynnikov [8]. He proved that any arc presentation of the unknot admits a monotonic simplification by elementary moves. He also showed that the problem of recognizing split links and of factorizing a composite link can be solved in a similar manner.

**Theorem 1.3 (Dynnikov [8])** *The decomposition problem of arc-presentations is solvable by monotonic simplification.*

## 2 Arc index and other knot invariants

Arc presentations can be represented in various ways. Figure 2 depicts different ways to describe an arc presentation. All arcs named and all integers correspond to each other in the figure.

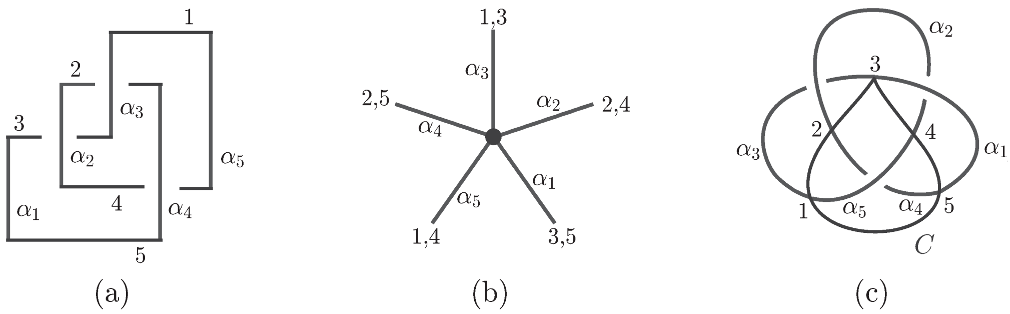


Figure 2: Different ways to describe the arc presentation of Figure 1

Cromwell and Dynnikov used the arc presentation called *grid diagram* to prove Proposition 1.1 and Theorem 1.3, respectively. A *grid diagram* is a finite union of vertical line segments and the same number of horizontal line segments with the properties that at every crossing the vertical strand crosses over the horizontal strand and no two horizontal segments are collinear and no two vertical segments are collinear as in Figure 2(a). A grid diagram can be converted easily to an arc presentation with the number of arcs which is equal to the number of vertical line segments and vice versa. If we consider an oriented grid diagram of a link, we can get a braid form of the link by cutting open horizontal arcs of same orientation. The fact yields the followings:

**Proposition 2.1 (Cromwell [6])** *Let $\beta(L)$ denote the braid index of a link $L$. Then*

$$\alpha(L) \geq 2\beta(L).$$

It is well-known that grid diagrams are closely related to front projections of its Legendrian imbedding in contact geometry. Grid diagrams are also used to compute Heegaard Floer homology and Khovanov homology. Due to the connections and other nice properties, arc presentations became very popular in recent years. Matsuda described a relation between arc index $\alpha(K)$ and the maximal Thurston-Bennequinn numbers of a knot $K$ and its mirror $K^*$, denoted by $\overline{\mathrm{tb}}(K)$ and $\overline{\mathrm{tb}}(K^*)$.

**Theorem 2.2 (Matsuda [24])**

$$-\alpha(K) \leq \overline{\mathrm{tb}}(K) + \overline{\mathrm{tb}}(K^*).$$

In [1] Bae and Park presented an algorithm for constructing arc presentations of a link which is given by edge contractions on a link diagram. The resulting diagram is called *wheel diagram*. The projection of an arc presentation of a knot or link into the plane perpendicular to the binding axis is of this shape. See Figure 2(b). Unordered pairs of integers in the figure indicate $z$-levels of the end point of the corresponding arcs in Figure 1. They showed that the algorithm leads an upper bound on the arc index in terms of the crossing number, $c(L)$, of a nonsplit link $L$.

**Theorem 2.3 (Bae-Park [1])** *Let $L$ be any prime non-split link. Then*

$$\alpha(L) \leq c(L) + 2.$$

By refining Bae-Park's algorithm, Beltrami constructed minimal arc presentations of $n$-semi-alternating links and Jin and Park obtained an inequality sharper than the one in Theorem 2.3 for non-alternating prime links.

**Theorem 2.4 (Beltrami [2])** *Let $L$ be an $n$-semi-alternating link. Then*

$$\alpha(L) = c(L) - 2(n-2).$$

**Theorem 2.5 (Jin-Park [15])** *A prime link $L$ is non-alternating if and only if*

$$\alpha(L) \leq c(L).$$

In [25] Morton and Beltrami gave an explicit lower bound for the arc index of a link $L$ in terms of the Laurent degree of the Kauffman polynomial $F_L(a, z)$. In [18] the reader will find details of the Kauffman polynomial.

**Theorem 2.6 (Morton-Beltrami [25])** *For every link $L$ we have*

$$\mathrm{spread}_a(F_L) + 2 \leq \alpha(L).$$

Combing Theorem 2.3, Theorem 2.6 and an observation of Thistlethwaite [29] on the Kauffman polynomial of alternating links, we have the following equality:

**Corollary 2.7** *For a non-split alternating link $L$,*

$$\alpha(L) = c(L) + 2.$$

By Thistlethwaite's work [30], if a link $L$ admits an adequate diagram, the lower bound of $\mathrm{spread}_a(F_L)$ can be calculated from a graph theoretical viewpoint. Since $n$-semi-alternating links are adequate, from Corollary 1.1 in [30], Beltrami got $\mathrm{spread}_a(F_L) \geq c(L) - 2(n-1)$ for an $n$-semi-alternating link $L$. These permit that the equality of arc index holds for $n$-semi-alternating links.

## 3 The arc index of some knots and links

No one can doubt that the arc index of the unknot is 2. Table 1 gives all list of links with arc index up to 5. Beltrami [2] and Ng [26] determined arc index for prime knots up to 10 and 11 crossings, respectively. Nutt [27], Jin *et al.* [11] and Jin and Park [14] identified all prime knots up to arc index 9, 10 and 11, respectively. In [13] the author with Jin showed that the existence of certain local diagrams indicates that the arc index is strictly less than the crossing number. They also determined arc index for new 364 knots with 13 crossings and 15 knots with 14 crossings. Recently, Jin and Kim [12] identified all prime knots with arc index 12 up to 16 crossings.

Table 1: All links with arc index up to 5

| $\alpha(L)$ | 2 | 3 | 4 | 5 |
|---|---|---|---|---|
| $L$ | unknot | none | 2-component unlink, Hopf link | trefoil |

Matsuda determined the arc index for torus knots.

**Theorem 3.1 (Matsuda [24])** *Let $T_{(p,q)}$ be a torus knot of type $(p,q)$. Then*

$$\alpha(T_{(p,q)}) = |p| + |q|.$$

The author determined the arc index of some of Pretzel knots of type $(-p,q,r)$ (with Jin) and Montesinos links of type $(-r_1,r_2,r_3)$, denoted by $P(-p,q,r)$ and $M(-r_1,r_2,r_3)$, respectively. $P(p,q,r)$ particularly satisfies the following properties for nonzero integers $p$,$q$, and $r$:

- The link type of $P(p,q,r)$ is independent of the order of $p,q,r$,
- $P(p,q,r)$ is a knot if and only if at most one of $p,q,r$ is an even number.

Since we consider Pretzel knots of type $(-p,q,r)$[1], we may assume that $p,q,r \geq 2$ and $r \geq q$.

**Theorem 3.2 (Lee-Jin [20])** *Let $p,q,r$ be integers with $p,q \geq 2$ and $r \geq q$.*

(1) *If $K = P(-2,q,r)$ is a knot with $q \geq 3$, then $\alpha(K) \leq c(K) - 1$.*

(2) *If $K = P(-p,2,r)$ is a knot with $p \geq 3$, then $\alpha(K) = c(K)$.*

(3) *If $K = P(-p,3,r)$ is a knot with $p \geq 3$, then $\alpha(K) = c(K) - 1$.*

(4) *If $K = P(-p,4,r)$ is a knot with $p \geq 5$, then $\alpha(K) = c(K) - 2$.*

(5) *If $K = P(-3,4,r)$ is a knot with $r \geq 7$, then $c(K) - 4 \leq \alpha(K) \leq c(K) - 2$.*

[1] By Lickorish Thistlethwaite's work [23], it is known that $c(P(-p,q,r)) = p+q+r$. We also know reduced Montesinos links admit minimal crossing diagrams.

**Theorem 3.3 (Lee [19])**

(1) *Let $L$ be a reduced Montesinos link $M(-r_1, r_2, r_3)$ for all positive irreducible rational numbers $r_i$. If $r_1 > 1$, $r_2 > 2$ and $r_3 > 2$, then*

$$\alpha(L) \leq c(L) - 1.$$

(2) *Let $n$ be a positive integer greater than 1 and $r_2, r_3$ be all positive irreducible rational numbers. Let $L$ be a reduced Montesinos link $M(-n, r_2, r_3)$. If $r_2 > 3$ and $r_3$ has a continued fraction $(a_1, a_2, \ldots, a_m)$ with $a_1 \geq 3$ and $a_2 \geq 2$, then*

$$\alpha(L) \leq c(L) - 2.$$

(3) *Let $n, m$ be positive integers and let $L$ be a reduced Montesinos link $M(-n, \frac{m}{2}, \frac{m}{2})$.*

  (a) *If $n > 1$ and $m = 3$, then $\alpha(L) = c(L)$.*

  (b) *If $n > 2$ and $m = 5$, then $\alpha(L) = c(L) - 1$.*

  (c) *If $n > 3$ and $m = 7$, then $\alpha(L) = c(L) - 2$.*

(4) *Let $n, m$ be positive integers and let $L$ be a reduced Montesinos link $M(-n, m, \frac{17}{5})$.*

  (a) *If $m = 2$, then $\alpha(L) = c(L)$.*

  (b) *If $m = 3$, then $\alpha(L) = c(L) - 1$.*

  (c) *If $m = 4$, then $\alpha(L) = c(L) - 2$.*

In [16, 17] Kanenobu introduced an infinite family of knots, denoted by $K(p, q)$, that is composed of infinite classes of knots which have the same HOMFLY-PT and Jones polynomials which are hyperbolic, fibered, ribbon, of genus 2 and 3-bridge, but with distinct Alexander module structures. Since $K(p, q) \approx K(q, p)$, $K(p, q)^* \approx K(-p, -q) \approx K(-q, -p)$ and $\alpha(L) = \alpha(L^*)$ for a link $L$, it is sufficient to consider $K(p, q)$ with $|p| \leq q$ in order to determine the arc index of $K(p, q)$.

**Theorem 3.4 (Lee-Takioka [21])**

(1) *Let $1 \leq p \leq q$ and $pq \geq 3$. Then $\alpha(K(p, q)) = p + q + 6$.*

(2) *Let $q \geq 3$. Then $q + 6 \leq \alpha(K(0, q)) \leq q + 7$.*

(3) *Let $q \geq 3$. Then $q + 5 \leq \alpha(K(-1, q)) \leq q + 7$.*

(4) *Let $q \geq 3$. Then $q + 4 \leq \alpha(K(-2, q)) \leq q + 7$.*

The author with Takioka found some examples to show that the bounds of (2) and (3) in Theorem 3.4 are best possible.

To prove Theorem 3.2, 3.3, 3.4, the author with Jin and Takioka in the proper paper used the way of finding arc presentations on knot or link diagrams as depicted in Figure 2(c). The idea was introduced by Cromwell and Nutt [7] first. The definition is as follows:

Let $D$ be a diagram of a knot or a link $L$. Suppose that there is a simple closed curve $C$ meeting $D$ in $k$ distinct points which divide $D$ into $k$ arcs $\alpha_1, \alpha_2, \ldots, \alpha_k$ with the following properties:

1. Each $\alpha_i$ has no self-crossing.
2. If $\alpha_i$ crosses over $\alpha_j$ at a crossing in $R_I$(resp. $R_O$), then $i > j$(resp. $i < j$) and it crosses over $\alpha_j$ at any other crossings with $\alpha_j$, respectively. Here, $R_I$ and $R_O$ is the inner and the outer region divided by $C$, respectively.[2]
3. For each $i$, there exists an embedded disk $d_i$ such that $\partial d_i = C$ and $\alpha_i \subset d_i$.
4. $d_i \cap d_j = C$, for distinct $i$ and $j$.

Then the pair $(D, C)$ is called an *arc presentation* of $L$ with $k$ arcs, and $C$ is called the *binding circle* of the arc presentation. Figure 2(c) shows an arc presentation of the trefoil knot.

Finally, we consider satellite knots. The class of satellite knots contains basic families of composite knots, cable knots and Whitehead doubles. The arc index of composite knots was determined by Cromwell as stated in Theorem 1.2. The others were dealt with in [22].

Let $p, q$ and $t$ be integers with $p > 1$. Given a knot $K$, let $K^{(p,q)}$, $K^{(+,t)}$ and $K^{(-,t)}$ be the $(p,q)$-*cable link*, the $t$-*twisted positive Whitehead double* and the $t$-*twisted negative Whitehead double* of $K$, respectively. Let $\mathrm{ne}(G)$ and $\mathrm{se}(G)$ denote the number of north-east corners and south-east corners for a grid diagram $G$ of a knot, respectively.

**Theorem 3.5 (Lee-Takioka [22])** [3] *Let $G$ be a grid diagram of a knot $K$ and $p, q$ be integers with $p > 1$. Suppose that $n(G) = q - pw(G)$.*

(1) *If $n(G) \geq 0$, $\exists!\ m(G)$ s.t. $pm(G) \leq n(G) < p(m(G)+1)$. Then,*

$$\alpha(K^{(p,q)}) \leq \begin{cases} p\alpha(G) & \text{if } \mathrm{ne}(G) > m(G) \\ p(\alpha(G) + \mathrm{tb}(G^*)) + q & \text{if } \mathrm{ne}(G) \leq m(G) \end{cases}$$

(2) *If $n(G) < 0$, $\exists!\ m'(G)$ s.t. $p(m'(G)-1) < n(G) \leq pm'(G)$. Then,*

$$\alpha(K^{(p,q)}) \leq \begin{cases} p\alpha(G) & \text{if } \mathrm{se}(G) > -m'(G) \\ p(\alpha(G) + \mathrm{tb}(G)) - q & \text{if } \mathrm{se}(G) \leq -m'(G) \end{cases}$$

**Theorem 3.6 (Lee-Takioka [22])** *Let $G$ be a grid diagram of a knot $K$ and $t$ be an integer. Suppose that $n(G) = 2t - 2w(G)$.*

(1) *If $n(G) \geq 0$, then*

$$\alpha(K^{(+,t)}) \leq \begin{cases} 2\alpha(G) + 1 & \text{if } 2\mathrm{ne}(G) > n(G) \\ 2(\alpha(G) + \mathrm{tb}(G^*) + t + 1) & \text{if } 2\mathrm{ne}(G) \leq n(G) \end{cases}$$

[2] For example, in Figure 2(c) $\alpha_4$ and $\alpha_5$ are only in $R_I$.

[3] $w(G)$ is the writhe of $G$ and $\mathrm{tb}(G) = w(G) - se(G)$.

$$\alpha(K^{(-,t)}) \leq \begin{cases} 2\alpha(G)+1 & \textit{if } 2\mathrm{ne}(G) \geq n(G) \\ 2(\alpha(G)+\mathrm{tb}(G^*)+t+1) & \textit{if } 2\mathrm{ne}(G) < n(G) \end{cases}$$

(2) *If* $n(G) < 0$, *then*

$$\alpha(K^{(+,t)}) \leq \begin{cases} 2\alpha(G)+1 & \textit{if } 2\mathrm{se}(G) \geq -n(G) \\ 2(\alpha(G)+\mathrm{tb}(G)-t)+1 & \textit{if } 2\mathrm{se}(G) < -n(G) \end{cases}$$

$$\alpha(K^{(-,t)}) \leq \begin{cases} 2\alpha(G)+1 & \textit{if } 2se(G) > -n(G) \\ 2(\alpha(G)+\mathrm{tb}(G)-t)+1 & \textit{if } 2\mathrm{se}(G) \leq -n(G) \end{cases}$$

Using Theorem 3.5 and 3.6, the author and Takioka exactly determined the arc index of infinite families of the $(2,q)$-cable link, the $t$-twisted positive Whitehead double and the $t$-negative Whitehead double of all knots with up to 8 crossings.

**Example.** The table below gives the arc index of the $(2,q)$-cable link, the $t$-twisted positive Whitehead double and the $t$-negative Whitehead double of $3_1^*$. Here, $3_1^*$ is the mirror image of the diagram of $3_1$ in Rolfsen's tables [28].

| $K$ | $\alpha(K^{(2,q)})$ | $\alpha(K^{(+,t)})$ | $\alpha(K^{(-,t)})$ |
|---|---|---|---|
| $3_1^*$ | $-q+12$ *if* $q \leq 1$ | $-2t+13$ *if* $t \leq 0$ | $-2t+14$ *if* $t \leq 1$ |
| | $10$ *if* $2 \leq q \leq 12$ | $11$ *if* $1 \leq t \leq 5$ | $11$ *if* $2 \leq t \leq 6$ |
| | $q-2$ *if* $q \geq 13$ | $2t$ *if* $t \geq 6$ | $2t-1$ *if* $t \geq 7$ |

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### References

[1] Y. Bae and C. Y. Park, *An upper bound of arc index of links*, Math. Proc. Camb. Phil. Soc. **129** (2000) 491–500.

[2] E. Beltrami, *Arc index of non-alternating links*, J. Knot Theory Ramifications **11**(3) (2002) 431–444.

[3] E. Beltrami and P. R. Cromwell, *Minimal arc-presentations of some nonalternatning knots*, Topology and its Applications **81** (1997) 137–145.

[4] J. S. Birman and W. W. Menasco, *Special positions for essectial tori in link complements*, Topology **33** (1994) 525–556.

[5] H. Brunn, *Uber verknotete Kurven*, Mathematiker-Kongresses Zurich 1897 (Leipzig, 1898) 256–259.

[6] P. R. Cromwell, *Embedding knots and links in an open book I: Basic properties*, Topology and its Applications **64** (1995) 37–58.

[7] P. R. Cromwell and Ian J. Nutt, *Embedding knots and links in an open book II. Bounds on arc index*, Math. Proc. Camb. Phil. Soc. **119** (1996), 309–319.

[8] I. A. Dynnikov, *Arc-presentations of links: Monotonic simplification*, Fundamenta Mathematicae **190** (2006), 29–76.

[9] I. A. Dynnikov, *Three-page representation of links*, Uspekhi Mat. Nauk, **53** (5) (1998), 237–238.

[10] J. B. Etnyre and K. Honda, *Knots and contact geometry I*, Symplectic Geom. **1** (2001), 63–120.

[11] G. T. Jin et al., *Prime knots with arc index up to 10*, Series on Knots and Everything Book vol. 40, World Scientific Publishing Co., 6574, 2006.

[12] G. T. Jin and H. Kim, *Prime knots with arc index 12 up to 16 crossings*, preprint.

[13] G. T. Jin and H. J. Lee, *Prime knots whose arc index is smaller than the crossing number*, J. Knot Theory Ramifications. **21**(10) (2012) 1655–1672.

[14] G. T. Jin and W. K. Park, *A tabulation of prime knots up to arc index 11*, J. Knot Theory Ramifications **20**(11) (2010) 1537–1635.

[15] G. T. Jin and W. K. Park, *Prime knots with arc index up to 11 and an upper bound of arc index for nonalternating knots*, J. Knot Theory Ramifications **19**(12) (2010) 1655–1672.

[16] T. Kanenobu, *Infinitely many knots with the same polynomial invariant*, Proc. Amer. Math. Soc. 97 (1986) 158–162.

[17] T. Kanenobu, *Examples on polynomial invariants of knots and links*, Math. Ann. **275** (1986) 555–572.

[18] L. H. Kauffman, *An invariant of regular isotopy*, Trans. Amer. MAth. Soc., **318** (1990), 417–471.

[19] H. J. Lee, *Arc index of Montesinos links of type of* $(-r_1, r_2, r_3)$, preprint.

[20] H. J. Lee and G. T. Jin, *Arc index of pretzel knots type of* $(-p, q, r)$, arXiv:1204.0597.

[21] H. J. Lee and T. Hideo, *Arc index of Kanenobu knots*, preprint.

[22] H. J. Lee and T. Hideo, *On the arc index of cable links*, preprint.

[23] W. B. R. Lickorish and M. B. Thistlethwaite, *Some links with non-trivial polynomials and their crossing-numbers*, Comment. Math. Helvetici **63** (1988) 527–539.

[24] H. Matsuda, *Links in an open book decomposition and in the standerd contact structure*, Proc. Amer. Math. Soc. **134**(12) (2006) 3697–3702

[25] H. R. Morton and E. Beltrami, *Arc index and the Kauffman polynomial*, Math. Proc. Camb. Phil. Soc. **123** (1998), 41-48.

[26] L. Ng, *On the arc index and maximal Thurston-Bennequin number*, J. Knot Theory Ramifications **121**(04) (2012).

[27] I. J. Nutt, *Embedding knots and links in an open book III : On the braid index of satellite links*, Math. Proc. Camb. Phil. Soc. **126** (1999), 77–98.

[28] D. Rolfsen, *Knots and links* (AMS Chelsea Publishing, 2003).

[29] M. B. Thistlethwaite, *Kauffman polynomial and alternating links*, Topology **27** (1987), 297–309.

[30] M. B. Thistlethwaite, *On the Kauffman polynomail of an adequate link*, Invent. Math. **93** (1988), 285–296.

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