Truncated determinants and the refined enumeration of Alternating Sign Matrices and Descending Plane Partitions

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1 Introduction

In these notes, we will be mainly focussing on the proof of the so-called ASM-DPP conjecture of Mills, Robbins and Rumsey [22] which relates refined enumerations of Alternating Sign Matrices (ASM) and Descending Plane Partitions (DPP).

ASMs were introduced by Mills, Robbins and Rumsey [24] in their study of Dodgsons condensation algorithm for the evaluation of determinants. DPPs were introduced by Andrews [1] while attempting to prove a conjectured formula for the generating function of cyclically symmetric plane partitions.

1.1 ASMs from lambda-determinant

The definition of the so-called lambda-determinant of Mills, Robbins and Rumsey [24] is based on the famous Dodgson condensation algorithm [12] for computing determinants, itself based on the Desnanot-Jacobi equation, a particular Plücker relation, relating minors of any square $k + 1 \times k + 1$ matrix M:

$$|M| \times |M^{1,k+1}_{1,k+1}| = |M^{k+1}_{k+1}| \times |M^1_1| - |M^{k+1}_1| \times |M^1_{k+1}| \qquad (1.1)$$

where $|M^{j_1,j_2,...,j_r}_{i_1,i_2,...,i_r}|$ stands for the determinant of the matrix obtained from M by deleting rows $i_1, ..., i_r$ and columns $j_1, ..., j_r$. The relation (1.1) may be used as a recursion relation on the size of the matrix, allowing for efficiently compute its determinant.

More formally, we may recast the algorithm using the so-called A_∞ T-system (also known as discrete Hirota) relation:

$$T_{i,j,k+1}T_{i,j,k-1} = T_{i,j+1,k}T_{i,j-1,k} - T_{i+1,j,k}T_{i-1,j,k} \qquad (1.2)$$

for any $i, j, k \in \mathbb{Z}$ with fixed parity of $i + j + k$. Now let $A = (a_{i,j})_{i,j\in\{1,2,...,n\}}$ be a fixed $n \times n$ matrix. Together with the initial data:

$$\begin{aligned}
T_{\ell,m,0} &= 1 & (\ell, m \in \mathbb{Z}; \ell + m = n \bmod 2) \\
T_{i,j,1} &= a_{\frac{j-i+n+1}{2},\frac{i+j+n+1}{2}} & (i, j \in \mathbb{Z}; i + j = n + 1 \bmod 2; |i| + |j| \leq n - 1),
\end{aligned} \qquad (1.3)$$

the solution of the T-system (1.2) satisfies:

$$T_{0,0,n} = \det(A) \tag{1.4}$$

Given a fixed formal parameter λ, the lambda-determinant of the matrix A, denoted by $|A|_\lambda$ is simply defined as the solution $T_{0,0,n} = |A|_\lambda$ of the deformed T-system

$$T_{i,j,k+1}T_{i,j,k-1} = T_{i,j+1,k}T_{i,j-1,k} + \lambda\, T_{i+1,j,k}T_{i-1,j,k} \tag{1.5}$$

subject to the initial condition (1.3).

The discovery of Mills, Robbins and Rumsey is that the lambda-determinant is a homogeneous Laurent polynomial of the matrix entries of degree n, and that moreover the monomials in the expression are coded by $n \times n$ matrices B with entries $b_{i,j} \in \{0, 1, -1\}$, characterized by the fact that their row and column sums are 1 and that the partial row and column sums are non-negative, namely

$$\sum_{i=1}^{k} b_{i,j} \geq 0 \qquad \sum_{i=1}^{k} b_{j,i} \geq 0 \quad (k = 1, 2, ..., n-1; j = 1, 2, ..., n)$$

$$\sum_{i=1}^{n} b_{i,j} = 1 \qquad \sum_{i=1}^{n} b_{j,i} = 1 \quad (j = 1, 2, ..., n)$$

Such matrices B are called alternating sign matrices (ASMs). These include the permutation matrices (the ASMs with no -1 entry). Here are the 7 ASMs of size 3:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

There is an explicit formula for the lambda-determinant[24]:

$$|A|_\lambda = \sum_{n \times n\, ASM\, B} \lambda^{\text{Inv}(B)-N(B)}(1+\lambda)^{N(B)} \prod_{i,j} a_{i,j}^{b_{i,j}} \tag{1.6}$$

where $\text{Inv}(B)$ and $N(B)$ denote respectively the inversion number and the number of entries -1 in B, with

$$\text{Inv}(B) = \sum_{\substack{1 \leq i < j \leq n \\ 1 \leq k < \ell \leq n}} b_{i,\ell}b_{j,k}$$

$$N(B) = \frac{1}{2}\left(-n + \sum_{1 \leq i,j \leq n} |b_{i,j}|\right)$$

Note that for $\lambda = -1$, only the ASMs with $N(B) = 0$ contribute, i.e. the permutation matrices, for which $\text{Inv}(B)$ coincides with the usual inversion number of the corresponding permutation, and therefore (1.6) reduces to the usual formula for the determinant.

Mills, Robbins and Rumsey [22] noticed that apart from the quantities $\text{Inv}(B)$ and $N(B)$, another "observable" of interest is the position of the unique 1 in the top row of

any ASM. We denote by $t(B)$ the number of 0 entries to the left of the 1 in the top row of B.

Associating a weight

$$W(B) = z^{t(B)} y^{\text{Inv}(B) - N(B)} x^{N(B)} \tag{1.7}$$

to each ASM B, we may form the partition function

$$Z_{ASM}^{(n)}(x, y, z) = \sum_{n \times n\, ASM\, B} W(B) \tag{1.8}$$

In the case $n = 3$ listed above, the ASMs receive respective weights: $1, zy, zxy, y, zy^2, z^2y^2, z^2y^3$, leading to $Z_{ASM}^{(3)}(x, y, z) = 1 + zy + zxy + y + zy^2 + z^2y^2 + z^2y^3$.

1.2 DPPs

Descending plane partitions are arrays of positive integers of the form:

$$
\begin{array}{cccccccc}
a_{1,1} & a_{1,2} & a_{1,3} & \cdots & & \cdots & a_{1,\mu_1-2} & a_{1,\mu_1-1} & a_{1,\mu_1} \\
& a_{2,2} & a_{2,3} & \cdots & & \cdots & a_{2,\mu_2} \\
& & a_{3,3} & \cdots & & a_{3,\mu_3} \\
& & & \ddots & & \cdots \\
& & a_{r,r} & \cdots & a_{r,\mu_r}
\end{array}
$$

such that the sequece μ_i is strictly decreasing $\mu_{i+1} < \mu_i$, and that for $\lambda_i = \mu_i - i + 1$, $\lambda_0 = \infty$:

$$a_{i,j} \ge a_{i,j+1} \qquad a_{i,j} > a_{i+1,j} \qquad \lambda_i < a_{i,i} \le \lambda_{i-1}$$

for all i, j. By convention, the empty partition is a DPP. Here are the 7 DPPs of order 3:

$$\emptyset, \qquad 2, \qquad 3, \qquad 3\ 1, \qquad 3\ 2, \qquad 3\ 3, \qquad \begin{array}{cc} 3 & 3 \\ & 2 \end{array}.$$

The integers $a_{i,j}$ are called parts. A DPP A is said to be of order n if $a_{i,j} \le n$ for all i, j. A part $a_{i,j}$ is said to be special if $a_{i,j} \le j - i$. We denote by $S(A)$ and $NS(A)$ respectively the total number of special parts and the total number of non-special parts of any DPP A. Another observable of interest among the DPPs A of order n is the number of parts in A equal to the order, which we denote by $M(A)$. To each DPP A of given order, we associate a weight

$$W(A) = x^{S(A)} y^{NS(A)} z^{M(A)} \tag{1.9}$$

and define the partition function for DPPs of order n to be:

$$Z_{DPP}^{(n)}(x, y, z) = \sum_{DPP\, A\, of\, order\, n} W(A) \tag{1.10}$$

The 7 DPPs of order 3 listed above have respective weights: $1, y, zy, zxy, zy^2, z^2y^2, z^2y^3$, as $M(A)$ is the number of occurrences of the part 3, and the only special part is the entry 1 in the fourth DPP. This leads to the partition function $Z_{DPP}^{(3)} = 1 + y + zy + zxy + zy^2 + z^2y^2 + z^2y^3$.

1.3 The ASM-DPP conjecture

The ASM-DPP conjecture as stated by Mills, Robbins and Rumsey [22] amounts to the identity between the partition functions of ASMs and DPPs as defined in the previous sections. This is the following:

Theorem 1.1. *The partition functions for the refined enumeration of ASMs and DPPs coincide, namely*

$$Z_{ASM}(x, y, z) = Z_{DPP}(x, y, z)$$

This was finally proved in all its generality in [4], and then generalized so as to include yet another observable in [5]. In the present note, we explain the rationale behind these proofs which strongly rely on manipulations of finite truncations of infinite matrices.

For simplicity of exposition we shall start with the identity between the doubly refined partition functions $Z_{ASM}^{(n)}(x, y, 1)$ and $Z_{DPP}^{(n)}(x, y, 1)$. Each will be expressed as the determinant of a finite truncation (of size $n \times n$) of an infinite matrix, and the identity between determinants will be derived from general principles relating the two "infinite" matrices. One key ingredient is the use of the double generating series for the matrix entries (see Appendix A for definitions and properties).

1.4 Outline

The use of infinite matrices is somewhat non-standard in this context, and we would like to stress the power and beauty of the method. The infinite matrices occurring here actually involve some fundamental object that came up in the study of so-called Lorentzian triangulations [8], giving rise to one of the simplest examples of quantum integrable system. More precisely, random configurations of this particular class of triangulations may be generated by iterated powers of a transfer matrix T of infinite size. The problem was solved exactly by diagonalization of T in [8]. A drastic simplification of the problem comes from the existence of an infinite parametric family of such transfer matrices, which all commute with each other.

The notes are organized as follows.

In Section 2, we recall a number of facts about the transfer matrix of $1 + 1$-dimensional Lorentzian triangulations, including other applications to trees and lattice path enumeration.

In Section 3, we compute $Z_{ASM}^{(n)}(x, y, 1)$ by use of the Izergin-Korepin (IK) [15, 17] determinant formulation of the partition function of the bijectively equivalent configurations of the 6 Vertex (6V) model with Domain-Wall Boundary Conditions (DWBC). The difficulty here is to extract a homogeneous limit out of the IK determinant, and to put it in the form of the determinant of a finite truncation to size $n \times n$ of an infinite matrix which is independent of n.

In Section 4, we compute $Z_{DPP}^{(n)}(x, y, 1)$ by use of the lattice path formulation of the problem [19], and by use of the Lindström Gessel-Viennot (LGV) determinant formula for the partition function of non-intersecting families of lattice paths. This expresses $Z_{DPP}^{(n)}(x, y, 1)$ as the determinant of the finite truncation to size $n \times n$ of an infinite matrix independent of n.

In Section 5, we show how the relation between the double generating functions of the two infinite matrices above implies the identity between the determinants of any finite truncation thereof. This is the key to the proof of the ASM-DPP conjecture. We then show how this has to be adapted to include more refinements.

In the conclusive Section 6, the ASM-DPP correspondence is placed in the wider context of the myriad of combinatorial objects and physical systems connected to ASMs. We also compare the two very different forms of quantum integrability underlying the ASM-DPP correspondence, one coming from the Lorentzian triangulations, the other from the 6V model.

We collect the useful formulas and definitions for generating functions and truncated determinants of infinite matrices in Appendix A.

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2 The main actor: the transfer matrix of Lorentzian gravity

2.1 1+1D Lorentzian gravity

Discrete models for 1+1D Lorentzian gravity are defined as follows. They are statistical models whose configurations are discrete space-times, in the form of random triangulations with a regular discrete time direction (an integer segment $[t_1, t_2] \subset \mathbb{N}$) and a random space direction, modeled by random triangulations of the unit time strips $[t, t + 1]$, $t \in \mathbb{N}$, by arbitrary but finite numbers of triangles with one edge along the time line t (resp. $t + 1$) and the opposite vertex on the time line $t + 1$ (resp. t). All other edges are then glued to their neighbors so as to form a triangulation. Each "horizontal edge" along a time line t is shared by two triangles, one in each time slice $[t - 1, t]$ and $t, t + 1]$. The boundary conditions along the time lines may be taken free, periodic or staircase-like [8]. A typical free boundary Lorentzian triangulation Θ in 1+1D reads as follows:
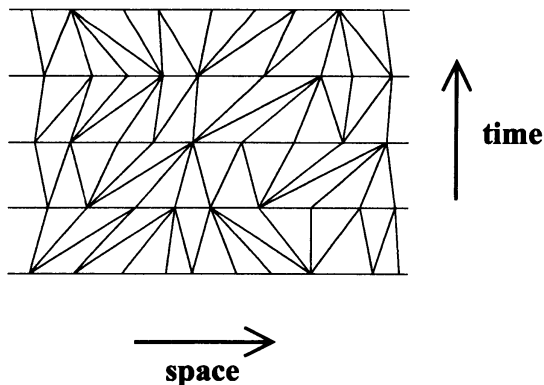
These triangulations are best described in the dual picture by considering triangles as vertical half-edges and pairs of triangles that share a time-like (horizontal) edge as vertical edges between two consecutive time-slices. We may now concentrate on the transition

between two consecutive time-slices which typically reads as follows:

$$
\begin{array}{c}\text{(diagram)}\end{array}
\tag{2.1}
$$

with say i half-edges on the bottom and j on the top (here for instance we have $i = 9$ and $j = 10$). Denoting by $|i\rangle$ and $|j\rangle$, with[1] $i, j \in \mathbb{Z}_+$, the bottom and top Hilbert space state bases, we may describe the generation of a triangulation by the iterated action of a transfer operator $\mathcal{T}$ with matrix elements $T_{i,j} = \binom{i+j}{i}$ between states $|i\rangle$ and $|j\rangle$. Note that the corresponding matrix $T = (T_{i,j})_{i,j \in \mathbb{Z}_+}$ is infinite. We shall deal with such matrices in the following. As detailed in Appendix A, a compact characterization of the infinite matrix T is via its double generating function:

$$
f_T(u, v) = \sum_{i,j \geq 0} T_{i,j} u^i v^j = \frac{1}{1 - u - v} \,.
$$

2.2 Integrability

To make the model more realistic, we may include both area and curvature-dependent terms, by introducing Botlzmann weights $w(\Theta)$ equal to the product of local weights of the form g per triangle (area term) and a per pair of consecutive triangles in a time-slice pointing in the same direction, either both up or both down (curvature term). The rules in the dual picture are as follows:

$$
\begin{array}{cccc}\text{(diagrams)}\\ \mathbf{g} & \mathbf{g} & \mathbf{a} & \mathbf{a}\end{array}
$$

For instance, in the example (2.1) above with $i = 9$ and $j = 10$, the product of local weights is $g^{19}a^9$. For the staircase-like boundary conditions of [8], namely assuming that each state-to-state transition as in (2.1) has at least one leftmost half-edge on the bottom and one rightmost half-edge on top (and not counting the leftmost and rightmost half-edge weights g^2), it is easy to compute the new transfer operator $\mathcal{T}(g, a)$ with matrix elements

$$
T(g, a)_{i,j} = (ag)^{i+j} \sum_{k=0}^{\text{Min}(i,j)} \binom{i}{k}\binom{j}{k} a^{-2k}
\tag{2.2}
$$

for $i, j \geq 0$ which expresses the transition between states $|i + 1\rangle$ and $|j + 1\rangle$. Equivalently, the double generating function reads:

$$
f_{T(g,a)}(u, v) = \sum_{i,j \geq 0} T(g, a)_{i,j} u^i v^j = \frac{1}{1 - ga(u + v) - g^2(1 - a^2)uv}
\tag{2.3}
$$

and we have $T(g, a) = T(ga, ga, g^2(1 - a^2))$ in the notations of Appendix A.

---

[1]Here and throughout the paper, we use the notation $\mathbb{Z}_+ = \{0, 1, 2, ...\}$.

This model turns out to provide one of the simplest examples of quantum integrable system, with an infinite family of commuting transfer matrices. Indeed, we have:

Theorem 2.1. *[8] The transfer matrices $T(g,a)$ and $T(g',a')$ commute if and only if the parameters (g,a,g',a') are such that $\varphi(g,a) = \varphi(g',a')$ where:*

$$\varphi(g,a) = \frac{1 - g^2(1 - a^2)}{ag} \tag{2.4}$$

This is easily proved by using the generating functions. This case corresponds to the family of matrices $T_{s,t}(\alpha)$ of Appendix A, with $s = 1$, $t = \varphi(g,a)$ and $\alpha = ga$.

2.3 Diagonalization

To diagonalize T, we may first consider finite size truncations $T^{[0,k]} = \left(T_{i,j}(g,a)\right)_{i,j\in[0,k]}$. It is easy to see that the $(k+1) \times (k+1)$ symmetric matrix $T^{[0,k]}$ is diagonalizable, with eigenvalues $\lambda_i^{[0,k]}(g,a) = g^{2i}(1 + O(g^2))$, $i \in [0,k]$, all with formal series expansions in powers of g with coefficients in $\mathbb{Z}[a]$. As k increases, we get more and more eigenvalues, with power series expansions that stabilize. In this sense, the limiting infinite matrix has an infinite set of eigenvalues $\lambda_i = g^{2i}(1 + O(g^2))$, $i \in \mathbb{Z}_+$, with well-defined formal power series expansions in g.

Setting $\varphi(g,a) = q + q^{-1}$ for some $q \in \mathbb{C}^*$, and introducing a new variable

$$\lambda = \frac{1 - q^{-1}ga}{1 - qga},$$

we may rewrite the double generating function of $T(g,a)$ as:

$$
\begin{aligned}
f_{T(g,a)}(u,v) &= \frac{1 - \lambda q^2}{(1 - qu)(1 - qv) - \lambda(u - q)(v - q)} \\
&= \sum_{m=0}^{\infty} \frac{\sqrt{1 - q^2}(q - u)^m}{(1 - qu)^{m+1}} \left(\frac{1 - \lambda q^2}{1 - q^2}\lambda^m\right) \frac{\sqrt{1 - q^2}(q - v)^m}{(1 - qv)^{m+1}}
\end{aligned} \tag{2.5}
$$

where we identify

$$\Lambda^{(m)} = \frac{1 - \lambda q^2}{1 - q^2}\lambda^m$$

as the m-th eigenvalue of $T(g,a)$, $m \in \mathbb{Z}_+$ and

$$f_{v^{(m)}}(u) = \sum_{i=0}^{\infty} v_i^{(m)}u^i = \frac{\sqrt{1 - q^2}(q - u)^m}{(1 - qu)^{m+1}}$$

as the generating function for the corresponding eigenvector $v^{(m)}$. Note that $(v^{(m)})_{m\in\mathbb{Z}_+}$ form an orthonormal basis of the Hilbert space of states w.r.t. the standard scalar product $u \cdot v = \sum_{i\in\mathbb{Z}_+} u_i v_i$. It is easy to show that $\Lambda^{(m)} = g^{2m}(1 + O(g^2))$ as a formal power series of g, thereby proving that these are the limits of the eigenvalues of the truncated matrices as the size $k \to \infty$.

This was extensively used in [8] to compute correlation functions of top/bottom boundary loops in random Lorentzian triangulations. We want to stress here the very simple form of the generating function (2.3), which will reappear later in these notes.

2.4 Trees

For staircase boundary conditions, the dual random Lorentzian triangulations introduced above may be viewed as random plane trees. This is easily realized by gluing all the bottom vertices of parallel vertical edges whose both top and bottom halves contribute to the curvature term (no interlacing with the neighboring time slices). A typical such example reads:
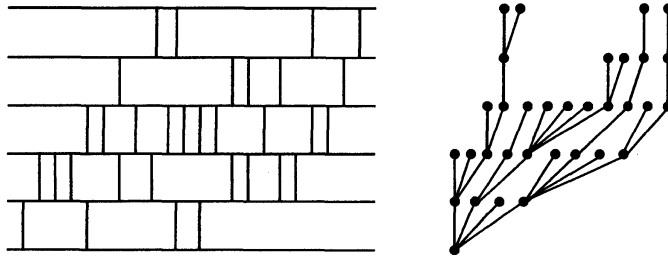
Note that the tree is naturally rooted at its bottom vertex.

To summarize, we have unearthed some integrable structure attached naturally to certain plane trees. Note that in tree language the weights are respectively g^2 per edge (except the left/right boundary ones), and a per pair of consecutive descendent half-edges (from left to right) and per leaf.

2.5 Paths

There is yet another interpretation of the transfer matrix $T(g, a)$ of $1 + 1D$ Lorentzian gravity, in terms of lattice paths. First notice that $T(g, a) = V(g, a)V^t(g, a)$ for some (infinite) lower triangular matrix $V(g, a)$ with entries

$$V(g, a)_{i,k} = g^i a^{i-k} \binom{i}{k} \qquad (i, k \in \mathbb{Z}_+) \qquad (2.6)$$

and double generating function:

$$f_{V(g,a)}(u, v) = \frac{1}{1 - agu - guv} \qquad (2.7)$$

In the notations of Appendix A.2, we have $V(g, a) = L(a^{-1}, ga)$.

Consider paths on the positive quadrant of the two-dimensional square lattice $\mathbb{Z}_+^2$, with steps $(-1, 0)$ (horizontal) and $(0, 1)$ (vertical), as illustrated in Fig.1(a). Then the total number of paths from the point $(i, 0)$ to the point (k, k) on the diagonal is $\binom{i}{k}$. Moreover if we attach a weight ga per horizontal step and g per vertical one, we get a total contribution of $(ga)^{k-i} g^k \binom{i}{k} = V(g, a)_{i,k}$, which we interpret as the partition function for weighted paths from $(i, 0)$ to (k, k). This quantity will also reappear later.

Note that in this language $T(g, a)_{i,j}$ is the partition function of lattice paths from $(i, 0)$ to $(0, j)$ in the positive quadrant, with weights ga (resp. g) per horizontal step below (resp. above) the diagonal and g (resp. ga) per step above (resp. below) the diagonal (see Fig.1(b) for an illustration).
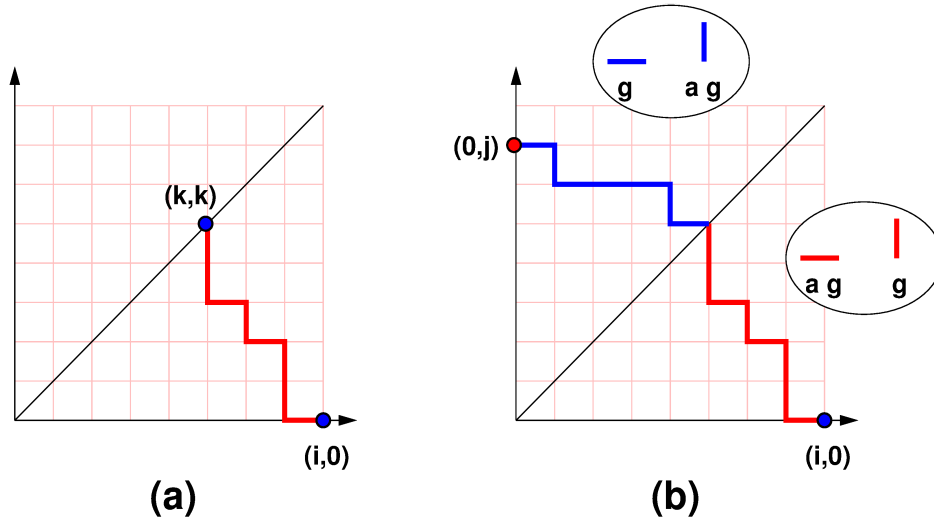
Figure 1: A typical path contributing to the matrix element $V(g,a)_{i,k}$ (a), and to $T(g,a)_{i,j}$ (b). These paths are taken on $\mathbb{Z}_+^2$, with steps $(-1,0)$ and $(0,1)$. In the latter case, we have indicated the weights of the steps below and above the diagonal.

This formulation allows to visualize immediately the truncated transfer matrix $T^{[0,k]}(g,a)$ as corresponding to paths within the square $[0,k] \times [0,k] \subset \mathbb{Z}_+^2$. For such paths, both the portion below the diagonal and that above are within the same square, so that we may write $T^{[0,k]}(g,a) = V^{[0,k]}(g,a)V^{[0,k]}(g,a)^t$, wich immediately yields the determinant of $T^{[0,k]}$, as $V^{[0,k]}(g,a)$ is lower triangular (see also Appendix A.4):

$$\det\left(T^{[0,k]}(g,a)\right) = \det\left(V^{[0,k]}(g,a)\right)^2 = g^{k(k+1)}$$

This is compatible with eigenvalues $\lambda_i^{[0,k]}(g,a) = g^{2i}(1 + O(g^2))$ for $i = 0, 1, ..., k$.

3 Enumerating ASMs

3.1 ASMs, 6V model and the IK deteminant

As discovered by Kuperberg [18], ASMs of size $n \times n$ are in bijection with the so-called Domain-Wall Boundary Condition Six Vertex (6V-DWBC) model on a square grid of size $n \times n$. The latter configurations are choices of orientations of the edges of a $n \times n$ grid of the (two-dimensional) square lattice, in such a way that at each vertex exactly two edges point to- and two point from- the vertex. Moreover oriented external horizontal (resp. vertical) edges are attached to the boundary vertices, in such a way that external horizontal edges point towards the grid and vertical ones from the grid. We display below the 6 possible vertex configurations $a_1, a_2, b_1, b_2, c_1, c_2$ obeying the above rules, as well as

a sample grid showing the external edge boundary condition:
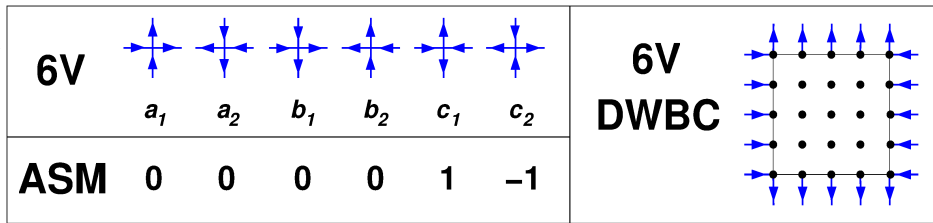
We have also indicated the dictionary between the vertex configurations and the ASM entries. It is easy to understand the bijection as follows. The ± 1 entries correspond to vertices where both the horizontal and vertical flows (indicated by the direction of the edges) are *reflected*. The alternation of 1 and -1 entries corresponds to odd and even order flips along each row or column of the grid. The DWBC boundary condition ensures that the first and last encountered non-zero elements in the ASM along rows and columns must be 1, as one needs an odd total number of flips to reverse the external edge orientation, when going along a row or a column.

The 6V model had been extensively studied in the physics literature. With a suitable parameterization of the Boltzmann weights a_i, b_i, c_i, the model forms the archetypical example of an integrable lattice model, as it admits an infinite family of commuting transfer matrices, that can be diagonalized for various types of bopundary conditions using the Bethe Ansatz techniques. These integrable weights are defined as follows. Each row (resp. column) of the grid carries a complex number z_i, $i = 1, 2, ...n$ (resp. w_i, $i = 1, 2, ...n$) called spectral parameter. Moreover the weights depend on a "quantum" parameter $q \in \mathbb{C}^*$. We have the following parametrization of the weights:

$$a(z, w) = qz - q^{-1}w \qquad b(z, w) = q^{-1}z - qw \qquad c(z, w) = (q^2 - q^{-2})\sqrt{zw}$$

where $a(z, w)$ is the weight for a vertex of type a_1 or a_2 at the intersection of a line with parameter z and column with parameter w, etc. With this parameterization, the model has an infinite family of commuting row-to-row transfer matrices, and can be exactly solved by Bethe Ansatz techniques. Using recursion relations of Korepin [17], Izergin [15] obtained a compact determinantal formula for the partition function of this 6V-DWBC model, defined as the sum over edge configurations of the product of local vertex weights, divided by the normalization factor $\prod_{i=1}^n c(z_i, w_i)$ (to make the answer polynomial in the z's and w's). It reads:

$$Z_{6V}^{(n)}(q; \{z_i\}, \{w_j\}) = \frac{\prod_{i,j} a(z_i, w_j)b(z_i, w_j)}{\Delta(z)\Delta(w)} \det_{1 \leq i,j \leq n} \left(\frac{1}{a(z_i, w_j)b(z_i, w_j)} \right) \qquad (3.1)$$

where $\Delta(z) = \prod_{1 \leq i < j \leq n}(z_i - z_j)$ stands for the Vandermonde determinant of the z's.

3.2 Homogeneous limit and computation of $Z_{ASM}^{(n)}(x, y, 1)$

In the above bijection between 6V configurations and ASMs, it is easy to track both quantities $N(B)$ and $\text{Inv}(B)$ in terms of 6V weights. We find that

$$N(B) = N_{c_2} = \frac{N_c - n}{2} \qquad \text{Inv}(B) - N(B) = N_{a_1} = N_{a_2} = \frac{N_a}{2}$$

where N_{a_i}, N_{b_i}, N_{c_i} stand for the total numbers of vertex configurations of each type. The determinant result above can therefore be used to compute the refined partition function $Z_{ASM}^{(n)}(x, y, 1)$ for ASMs, which counts ASMs with a weight x/y per entry -1 and a weight y for each inversion. Setting

$$x = \left(\frac{c}{b}\right)^2 \qquad y = \left(\frac{a}{b}\right)^2 \tag{3.2}$$

we have:

$$Z_{ASM}^{(n)}(x, y, 1) = \sum_{ASM\, B} x^{N(B)} y^{\text{Inv}(B) - N(b)} = b^{-n(n-1)} Z_{6V}^{(n)}(a, b, c) \tag{3.3}$$

where $Z_{6V}^{(n)}(a, b, c)$ refers to the homogeneous limit of the partition function (3.1) of the 6V model in which all $a(z_i, w_j)$ tend to a, etc. This is obtained by letting all $z_i \to r$ and all $w_i \to r^{-1}$, with $a = a(r, r^{-1})$, $b = b(r, r^{-1})$ and $c = c(r, r^{-1})$. This and more refinements were worked out in [4]. We have the following remarkable result:

Theorem 3.1. *[4] The partition function for refined ASMs reads:*

$$Z_{ASM}^{(n)}(x, y, 1) = \det_{0 \leq i,j \leq n-1} \left((1 - \nu)\mathbb{I} + \nu G \right) \tag{3.4}$$

where ν is any solution to the equation

$$x\nu(1 - \nu) = \nu + y(1 - \nu) \tag{3.5}$$

and the $n \times n$ determinant is the principal minor for the n first rows and columns of the infinite matrix $M_{ASM} = (1 - \nu)\mathbb{I} + \nu G$ whose entries are generated by

$$f_{M_{ASM}}(u, v) = \frac{1 - \nu}{1 - uv} + \frac{\nu}{1 - xu - v - (y - x)uv} \tag{3.6}$$

Note the remarkable similarity between the generating function for the matrix elements of G and the that of the transfer matrix for $1 + 1D$ Lorentzian triangulations (2.3). The two actually match up to a rescaling $u \to u/\sqrt{x}$ and $v \to v\sqrt{x}$ (which amounts to a conjugation by the diagonal matrix $\sqrt{x}\,\mathbb{I}$) and upon identifying $x = g^2 a^2$ and $y = g^2$. Note also that $G = T(x, 1, y - x)$ in the notations of Appendix A.2.

Let us now give a sketch of the proof of Theorem 3.1. The determinant (3.1) is singular in the homogeneous limit, but we may Taylor-expand the matrix entries within the determinant around the homogeneous point. For $a = a(r, r^{-1})$, $b = b(r, r^{-1})$ and $c = c(r, r^{-1})$ this reads:

$$Z_{6V}^{(n)}(a, b, c) = \frac{(ab)^{n^2}}{c^n} \det_{0 \leq i,j \leq n-1} \left(\left\{ \left(\frac{1}{i!}\frac{d^i}{du^i}\right) \left(\frac{1}{j!}\frac{d^j}{dv^j}\right) \frac{c(u^{-1}, v)}{a(u^{-1}, v)b(u^{-1}, v)} \right\} \Big|_{u=v=r^{-1}} \right)$$

Noting further that

$$\frac{c(u^{-1}, v)}{a(u^{-1}, v)b(u^{-1}, v)} = \frac{1}{uv - q^{-2}} - \frac{1}{uv - q^2}$$

and introducing the infinite matrices $A_\pm$ with elements:

$$(A_\pm)_{i,j} = \left\{ \left(\frac{1}{i!}\frac{d^i}{du^i}\right) \left(\frac{1}{j!}\frac{d^j}{dv^j}\right) \frac{1}{uv - q^{\pm 2}} \right\} \Big|_{u=v=r^{-1}} \qquad (i, j \in \mathbb{Z}_+) \tag{3.7}$$

we have the following straightforward:

Lemma 3.2. *We have*

$$A_\pm = \frac{1}{r^{-2} - q^{\pm 2}} \left(U(\alpha_\pm, \beta_\pm)^t \right)^{-1} U(\alpha'_\pm, \beta'_\pm) \tag{3.8}$$

for $U(\alpha, \beta)$ the infinite upper triangular matrix with entries $U(\alpha, \beta)_{i,j} = \binom{j}{i}\alpha^i \beta^j$ (see Appendix A.2), and where the parameters read:

$$\alpha_+ = \frac{1 - q^2 r^2}{r}, \quad \beta_+ = \frac{q^2 - q^{-2}}{r^2 - q^2}, \quad \alpha'_+ = -q^2 r^2 \beta_+, \quad \beta'_+ = -\frac{1}{\alpha +}$$

and the parameters with $-$ index are obtained form those with $+$ by the substitution $q \to q^{-1}$.

Proof. The statement of the lemma is an immediate consequence of the fact that the Taylor-expansion expression (3.7) around $(u, v) = (r^{-1}, r^{-1})$ turns into the following double generating functions for the matrix elements of $A_\pm$:

$$f_{A_\pm}(u, v) = \sum_{i,j \in \mathbb{Z}_+} (A_\pm)_{i,j} u^i v^j = \frac{1}{(r^{-1} + u)(r^{-1} + v) - q^{\pm 2}}$$

Moreover, using the generating function for the matrix elements of $U(\alpha, \beta)$ of Appendix A.2:

$$f_{U(\alpha,\beta)}(u, v) = \frac{1}{1 - \beta v(1 + \alpha u)},$$

and for $\alpha\beta \neq 0$, $U(\alpha, \beta)^{-1} = U(-1/\beta, -1/\alpha)$, we finally compute by convolution product:

$$
\begin{aligned}
f_{U^t(\alpha,\beta)^{-1}U(\alpha',\beta')}(u, v) &= f_{U(-1/\beta,-1/\alpha)}(v, u) * f_{U(\alpha',\beta')}(u, v) \\
&= \oint_{\mathbb{C}} \frac{dt}{2i\pi t} \frac{1}{1 + \frac{1}{\alpha}u(1 - \frac{1}{\beta}t^{-1})} \frac{1}{1 - \beta' v(1 + \alpha' t)} \\
&= \frac{1}{(1 + \frac{1}{\alpha}u)(1 - \beta' v) - uv\frac{\alpha'\beta'}{\alpha\beta}}
\end{aligned}
$$

The lemma follows from comparing this with the expressions for $f_{A_\pm}(u, v)$. $\qquad\square$

The Lemma is easily extended to finite truncations of the infinite matrices $A_\pm$ and U, as U is upper triangular and $(U^t)^{-1}$ is lower triangular. Therefore we may write $A_\pm^{[0,n-1]} = \frac{1}{r^{-2}-q^{\pm 2}}(U_\pm^{[0,n-1]\,t})^{-1} U_\pm^{[0,n-1]'}$ with the obvious shorthand notations. Going back to our original determinant, we find that

$$
\begin{aligned}
Z_{6V}^{(n)}(a, b, c) &= \frac{(ab)^{n^2}}{c^n} \det\left(A_-^{[0,n-1]} - A_+^{[0,n-1]} \right) \\
&= \frac{(ab)^{n^2}}{c^n} \det(A_-^{[0,n-1]}) \det\left(\mathbb{I} - \frac{r^{-2} - q^{-2}}{r^{-2} - q^2} U_-^{[0,n-1]\,t}(U_+^{[0,n-1]\,t})^{-1} U_+^{[0,n-1]'}(U_-^{[0,n-1]'})^{-1} \right)
\end{aligned}
$$

where $\mathbb{I}$ stands for the $n \times n$ identity matrix. The last product of 4 matrices is finally identified with the matrix G of Theorem 3.1 and computed by use of generating functions,
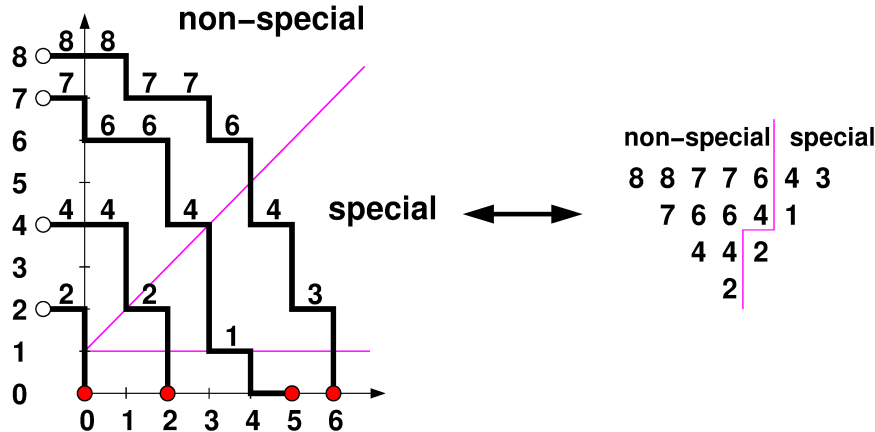
Figure 2: Non-intersecting lattice path configuration for a sample DPP of order $n \geq 8$. We have indicated the domains in which horizontal steps correspond to special/non-special parts.

whereas the determinant of $A_{-}^{[0,n-1]}$ follows from its expression as a product of triangular matrices. Collecting all the factors finally yields (3.4-3.6), with the following identification of parameters:

$$x = \left(\frac{q^2 - q^{-2}}{q^{-1}r - qr^{-1}}\right)^2, \quad y = \left(\frac{qr - q^{-1}r^{-1}}{q^{-1}r - qr^{-1}}\right)^2, \quad \nu = \frac{r^{-2} - q^{-2}}{q^2 - q^{-2}}, \quad 1 - \nu = \frac{q^2 - r^{-2}}{q^2 - q^{-2}}.$$

4 Enumerating DPPs

4.1 Lattice path formulation of DPPs

The DPPs are in bijection with configurations of non-intersecting lattice paths illustrated in Fig.2 and defined as follows. Like in Sect. 2.5, the paths take place in the positive quadrant $(\mathbb{Z}_+)^2$, with the same steps but different weights and boundary conditions. The paths start along the x axis at positions of the form $(s_i, 0)$ $(i = 1, 2, ..., r$ recorded from right to left) and end along the y axis at positions $(0, s_i + 2)$ $(i = 1, 2, ..., r$ recorded from top to bottom). We add a final horizontal left step at the end of each path. Reading paths from left to right and top to bottom, we record the vertical positions $y = a_{i,j}$ of the j-th horizontal step from the left taken on the i-th path from top (steps with $y = 0$ are not recorded). These form a DPP with r rows, of order any $n \geq s_1 + 2$. Conversely to each DPP with r rows we may associate such a path configuration. Note that the starting points are such that $s_i = \lambda_i - 1$, where $\lambda_i = \mu_i - i + 1$ the total number of parts in the row i.

The special parts correspond to horizontal steps taken in the strict upper octant $y \geq x + 1$ of the plane, and the remaining parts correspond to the horizontal steps in the domain $1 \leq y \leq x + 1$, while horizontal steps along the x axis do not count (weight 1).

4.2 Computation of $Z_{DPP}^{(n)(}(x, y, 1)$

The computation of $Z_{DPP}^{(n)}(x, y, 1)$ uses the Lindström-Gessel-Viennot [21, 14] determinant formula expressing the partition function for non-intersecting lattice paths with fixed atarting points and endpoints Z as a determinant $\det(Z_{i,j})$ where $Z_{i,j}$ is the partition function for a single path from the i-th starting point to the j-th endpoint. This leads to the following:

Theorem 4.1. *[4] The partition function $Z_{DPP}^{(n)}(x, y)$ for DPPs of order n with weight x per special part and y per other part reads:*

$$Z_{DPP}^{(n)}(x, y) = \det\left(\mathbb{I} + H^{[0,n-1]}\right) \tag{4.1}$$

where the determinant is that of the finite truncation to the n first rows and columns of the infinite matrix $(M_{DPP})_{i,j} = \delta i, j + H_{i,j}$, $i, j \in \mathbb{Z}_+$, with generating function:

$$f_{M_{DPP}}(u, v) = \sum_{i,j\in\mathbb{Z}_+} (M_{DPP})_{i,j} u^i v^j = \frac{1}{1-uv} + \frac{1}{1-u} \frac{yu}{1-xu-v-(y-x)uv} \tag{4.2}$$

Again, note the close similarity between the matrix H and the transfer matrix T for $1 + 1D$ Lorentzian triangulations. Our proof of the ASM-DPP conjecture will be based on this similarity. Note also that in the notations of Appendix A.2, we have $H = yS(\mathbb{I} - S)^{-1}T(x, 1, y - x)$.

Let us now sketch the proof of Theorem 4.1. The sought after partition function is a sum over all configurations of r non-intersecting paths $(0 \le r \le n - 1)$ with fixed r starting points $(s_i, 0)$, $i = 1, ..., r$ and endpoints $(0, s_i + 2)$, $i = 1, ..., r$. According to the Lindström-Gessel-Viennot theorem, this is the sum over minors $|D|_{s_1,...,s_r}^{s_1,...,s_r}$ of the $n \times n$ matrix D whose entries $D_{i,j}$ is the partition function for a single path from $(i, 0)$ to $(0, j + 2)$. It also has the simple expression:

$$\sum_{r=0}^{n-1} \sum_{0 \le s_1 < ... < s_r \le n-1} |D|_{s_1,...,s_r}^{s_1,...,s_r} = \det(\mathbb{I} + D)$$

Such a path is split into three pieces: (i) between the $x = 0$ axis and the first hit on the $x = 1$ axis (ii) between the $x = 1$ axis and the diagonal line $y = x + 1$ (iii) between the diagonal line $y = x + 1$ and the vertical axis $y = 0$. Each piece receives a specific weight, with total contribution:

$$D_{i,j} = \sum_{k=0}^{i} \sum_{\ell=0}^{Min(k,j+1)} \binom{k}{\ell} x^{k-\ell} \binom{j+1}{\ell} y^{\ell+1} \tag{4.3}$$

where we have first summed over (i) paths from $(i, 0)$ to $(k, 1)$ with $i - k$ horizontal steps along the $x = 0$ axis and one final vertical step (ii) paths from $(k, 1)$ to $(\ell, \ell + 1)$ on the diagonal $y = x + 1$, for which $k - \ell$ horizontal steps must be chosen among a total of k, each weighted by x as these correspond to special parts (iiii) paths from $(\ell, \ell + 1)$ to $(0, j + 2)$ for which ℓ horizontal steps must be chosen among a total $j + 1$, each weighted

by y as they correspond to non-special parts with one extra y factor for the additional final horizontal step.

Note that the matrix elements of D are independent of n. We may therefore consider the extension of D to an infinite matrix $\tilde{D}$ with matrix elements given by $D_{i,j}$ of eqn.(4.3), for $i,j \in \mathbb{Z}_+$. The theorem follows by identifying the infinite matrix H with $\tilde{D}$, and therefore $H^{[0,n-1]}$ with D.

5 Proof of the ASM-DPP conjecture

5.1 Proof of $Z_{ASM}^{(n)}(x,y,1) = Z_{DPP}^{(n)}(x,y,1)$

The expressions (3.4) and (4.1) for respectively the partition functions $Z_{ASM}^{(n)}(x,y,1)$ and $Z_{DPP}^{(n)}(x,y,1)$ are determinants of the principal minor of size n of some infinite matrix, in other words, these are the determinants of a finite truncation to the n first rows and columns of infinite matrices.

There is a very simple relation (independent of n) between the generating functions of the two infinite matrices M_{ASM} and M_{DPP}, namely:

$$(1 - \frac{u}{1-\nu})(1-v)f_{M_{ASM}}(u,v) = (1-u)(1-(1-\nu)v)f_{M_{DPP}}(u,v) \qquad (5.1)$$

as a direct consequence of (3.5).

Let us translate this back into a finite matrix relation upon truncation. First, for any matrix A with generating function $f_A(u,v)$ the function $f_M(u,v) = (1-au)(1-bv)f_A(u,v)$ is actually the generating function of the infinite matrix $M = (\mathbb{I} - aS)A(\mathbb{I} - bS^t)$ where S is the strictly lower triangular shift matrix with elements $S_{i,j} = \delta_{i-j,1}$ for $i,j \in \mathbb{Z}_+$, and S^t its strictly upper triangular transpose. Upon truncation to indices in $[0, n-1]$, we have the obvious relation (see Lemma A.2 in Appendix A): $M^{[0,n-1]} = (\mathbb{I} - aS)^{[0,n-1]}A^{[0,n-1]}(\mathbb{I} - bS^t)^{[0,n-1]}$, due to lower triangularity of $\mathbb{I} - aS$ and upper triangularity of $\mathbb{I} - bS^t$. Note that both matrix truncations are unitriangular, hence have determinant 1 so that $\det(M^{[0,n-1]}) = \det(A^{[0,n-1]})$ for all $n \geq 1$. By the identity (5.1), we therefore conclude that the truncations $M_{ASM}^{[0,n-1]}$ and $M_{DPP}^{[0,n-1]}$ have the same determinant, and the $z = 1$ version of Theorem 1.1 follows.

5.2 Refinement: proof of the MRR conjecture

The observable $t(B)$ for ASMs B may be included by slightly modifying the homogeneous limit of the IK determinant. We simply have to consider vertex weights with homogeneous limits in $\{a(r, r^{-1}), b(r, r^{-1}), c(r, r^{-1})\}$ at points (i,j), $i = 1,2,...,n$ and $j = 1,2,...,n-1$ of the square grid, and different weights $\{a(s, s^{-1}), b(s, s^{-1}), c(s, s^{-1})\}$ for the last column $i = 1,2,...,n$ and $j = n$. Defining further

$$z = \frac{a(s, s^{-1})b(r, r^{-1})}{b(s, s^{-1})a(r, r^{-1})}$$

gives an extra contribution $z^{t(B)}$ to the ASM enumeration.

Adapting the method of enumeration described above, one finds that we simply have to change the definition of the last column of $M_{ASM}^{[0,n-1]}$ to include the z dependence. This in turn is obtained by modifying *all* columns of index $j \geq n - 1$ in the infinite matrix M_{ASM} (we refer the reader to [4] for the technical details). The result is the following:

Theorem 5.1. *The quantity* $(1 + \nu(z - 1))Z_{ASM}(x, y, z)$ *is the determinant of the truncation to the n first rows and columns of the modified infinite matrix M'_{ASM}, with double generating function*

$$f_{M'_{ASM}}(u, v) = \frac{1 - \nu}{1 - uv} + \frac{\nu}{1 - xu - v - (y - x)uv}$$
$$+ \frac{\nu(z - 1)}{1 - (y(z - 1) + x)u}\left(1 + \frac{yu}{1 - xu}\right)^n v^{n-1}\left(1 + \frac{v}{x}\frac{y(\nu - 1) + \nu(xu - 1)}{\nu + (y - \nu x)u}\right)$$

The prefactor $(1 + \nu(z - 1))$ is ad-hoc and comes from a modification of the columns n and higher in the infinite matrix to make it simpler. Note that the new infinite matrix M'_{ASM} has an explicit dependence on n.

Likewise, keeping track of the observable $M(A)$ in a DPP A is easy. The lattice path formulation still holds and yields a LGV-like determinant as well, but for a modified $n \times n$ matrix $D'_{i,j}$, identical to $D_{i,j}$ of (4.3) for $i = 1, 2, ..., n$ and $j = 1, 2, ..., n - 1$ and with a different last column, explicitly depending on z.

The latter dependence is the result of decomposing further the piece (iii) of the DPP (see Sect.4.2), when $j = n$, into $(iii - a)$ and $(iii - b)$ and attaching weights as follows: (iii-a) the piece of the path from $(\ell, \ell+1)$ to its first vertex on the $y = n$ line at (m, n) with a total of $\binom{n-m-1}{\ell-m}$ paths all with weight $y^{\ell-m}$ and (iii-b) the straight path from (m, n) to $(0, n)$ with an extra weight of $(yz)^{m+1}$. This gives:

$$D'_{i,j} = \sum_{k=0}^{i}\sum_{\ell=0}^{k}\binom{k}{\ell}x^{k-\ell}\binom{n - m - 1}{\ell - m}y^{\ell+1}z^{m+1} \tag{5.2}$$

This leads to the following:

Theorem 5.2. *The quantity* $(1 + \nu(z - 1))Z_{DPP}(x, y, z)$ *is the determinant of the truncation to the n first rows and columns of the modified infinite matrix M'_{ASM}, with double generating function:*

$$f_{M'_{DPP}}(u, v) = \frac{1}{1 - zw} + \frac{1}{1 - z}\frac{yz}{1 - xz - w - (y - x)zw}$$
$$+(z - 1)\frac{1 - v}{1 - u}\frac{yu + \nu(1 - xu)}{1 - (y(z - 1) + x)u}\left(1 + \frac{yu}{1 - xu}\right)^n v^{n-1}$$

Like M'_{ASM}, the infinite matrix M'_{DPP} has an explicit dependence on n.

The proof of the complete Mills-Robbins-Rumsey conjecture follows from the following elementary lemma, easily proved by direct computation:

Lemma 5.3. *We have the relation:*

$$(1 + (y - x\nu - 1)u)(1 - v)f_{M'_{ASM}}(u, v) = (1 - u)(1 + (\nu - 1)v)f_{M'_{DPP}}(u, v)$$

As explained above, such a relation between the two infinite matrices M'_{ASM} and M'_{DPP} guarantees that the determinant of their truncation to their first m rows and columns coincide, for any $m \geq 1$, so it holds in particular for $m = n$ and Theorem 1.1 follows.

5.3 More refinements

In [5] a further observable was considered for ASMs and DPPs. For any ASM B let $b(B)$ be the number of 0 entries to the right of the unique 1 in the bottom row of B. For any DPP A of order n, let $P(A)$ be the number of parts equal to $n - 1$ plus the number of rows of length $n - 1$. Defining the two following partition functions:

$$Z_{ASM}^{(n)}(x, y, z, w) = \sum_{n \times n\ ASM\ B} x^{N(B)} y^{Inv(B)-N(B)} z^{t(B)} w^{b(B)}$$

$$Z_{DPP}^{(n)}(x, y, z, w) = \sum_{DPP\ A\ of\ order\ n} x^{S(A)} y^{NS(A)} z^{M(A)} w^{P(A)}$$

for respectively ASMs of size n and DPPs of order n, we have:

Theorem 5.4. *[5] We have the identity:*

$$Z_{ASM}^{(n)}(x, y, z, w) = Z_{DPP}^{(n)}(x, y, z, w)$$

This was proved in [5] by showing that both functions $Z_{ASM}^{(n)}(x, y, z, w)$ and $Z_{DPP}^{(n)}(x, y, z, w)$ obey the following relation as functions of z, w, n:

$$(z-w)Z^{(n)}(z,w)Z^{(n-1)}(1,1) = (z-1)wZ^{(n)}(z,1)Z^{(n-1)}(1,w)-(w-1)zZ^{(n-1)}(z,1)Z^{(n)}(1,w)$$

in both cases as a consequence of the Desnanot-Jacobi identity (1.1).

6 Conclusion

6.1 ASM,DPP,TSSCPP, FPL,DPL, etc.

In these notes, we have detailed the refined enumeration of ASMs and DPPs and established an identity between them.

One could think of further refinements, leading eventually to a bijection between these objects. This is however only the tip of a much larger iceberg (see Fig.3), which on the pure combinatorics side involves other objects: the so-called TSSCPPs (Totally Symmetric Self-Complementary Plane Partitions) which are yet another kind of plane partitions, with a formulation in terms of different configurations of non-intersecting lattice paths (see for instance [6] for a detailed account). There is also a statistical physics side, involving the so-called Fully Packed Loop (FPL) model on a square grid, whose configurations are in bijection with those of the 6V-DWBC model. The latter plays a central role in the so-called Razumov-Stroganov (RS) conjecture [23] proved by Cantini-Sportiello [7], relating its refined enumeration according to link patterns of connections of the loops around the grid to the asymptotic probabilities of connections of the Densely Packed Loop (DPL) model on a semi-infinite cylinder of finite perimeter. The latter is yet another integrable
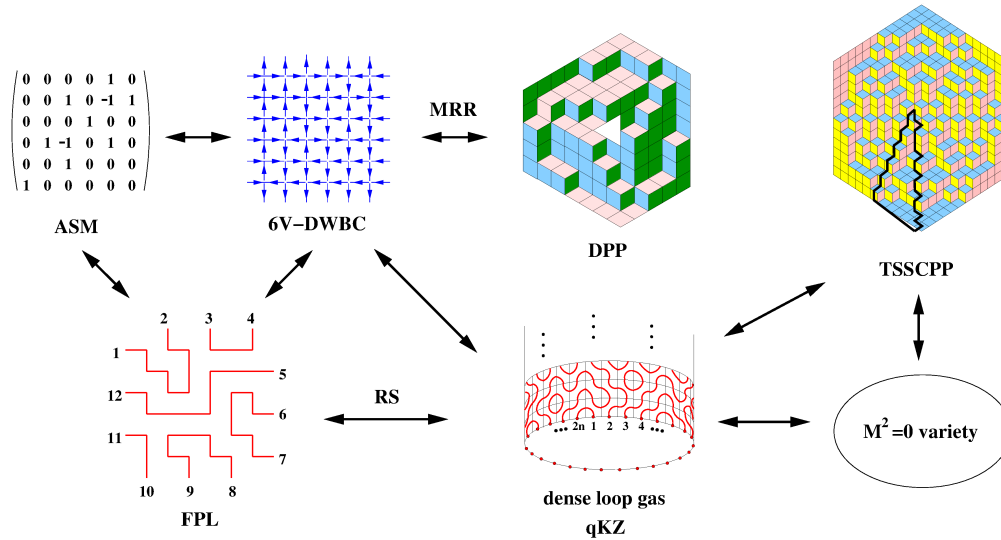
Figure 3: From left to right: ASM, 6V-DWBC and FPL, all in bijection; dense loop gas (DPL): its ground-state/limiting probability vector satisfies the qKZ equation, the components measure FPL correlations (RS conjecture), their sum matches the 6V-DWBC partition function with inhomogeneous spectral parameters z_i, w_j and $q^3 = 1$; DPPs: their refined evaluation matches that of ASMs (MRR conjecture); TSSCPPs: their refined enumeration matches a sum rule for qKZ solutions at generic q and $z_i = 1$; Variety $M^2 = 0$: its degree/multidegree matches solutions of qKZ for $q = 1$.

lattice model based on some pictorial representation of the Temperley-Lieb algebra, whose groundstate vector is a solution to the quantum Knizhnik-Zamolodchikov (qKZ) equation [9, 10]. Finally, there is an algebraic geometry side of the iceberg. For instance, the degree of the variety of upper triangular complex matrices with vanishing square corresponds to a refined enumeration of TSSCPPs, and the (equivariant cohomology) multidegree is obtained via a specialization of the solution to the qKZ equation [11].

Many of the known enumerations of the above objects involve determinants,. In a number of cases, these can be obtained through some application of the Lindström-Gessel-Viennot theorem. This applies to all the "free fermion" cases that are in bijection with non-intersecting lattice path configurations. However, both the 6V model and the DPL model are models of interacting fermions, in which even if there is some kind of lattice path formulation, the latter are no longer just non-intersecting. For instance, in the case of the 6V model, one can define paths going from the left border of the grid to the top border, by going right and up along the oriented edges as much as possible (i.e. when there is a choice, always go up). Such paths are now "osculating" in that two paths can bounce against each other at a vertex (the first going right, then up; the second going up then right; this corresponds to the vertex a_1 of the 6V model), and this configuration receives a different weight, interpreted as the exponential of some interaction energy. Yet, somehow, our formula for the refined enumeration has magically disentangled this interaction, to make it look like a free fermion model, via our determinant evaluation. This mechanism deserves to be better understood.

6.2 Integrabilities

The 6V model is the archetype of 2D integrable lattice model, related to the 1D quantum XXZ spin chain for sl_2. It is know to have an infinite family of transfer matrices $T(a, b, c)$ provided the Boltzmann weights a, b, c satisfy the following relation:

$$\Delta(a, b, c) = \frac{a^2 + b^2 - c^2}{2ab} = \text{const.}$$

This constant is the anisotropy of the associated quantum spin chain. Alternatively, in terms of the x, y variables of (3.2), this turns into the following "6V" variety:

$$\psi(x, y) = \frac{1 + y - x}{\sqrt{y}} = \text{const.} \tag{6.1}$$

as $x, y > 0$.

On the other hand, the infinite matrix M_{ASM}, whose finitely truncated determinant gives the DWBC homogeneous 6V partition function on a grid of same size, also involves a transfer matrix θ of a form analogous to that of 1+1D Lorentzian triangulations, generated by:

$$f_\theta(u, v) = \frac{1}{1 - xu - v - (y - x)uv}$$

The transfer matrices θ commute for different values of the parameters x, y provided they belong to the following "Lorentzian" variety:

$$\varphi(x, y) = \frac{1 + x - y}{\sqrt{x}} = \text{const.} \tag{6.2}$$

obtained by rephrasing (2.4) above.

Comparing (6.1) and (6.2), we see that $\varphi(x, y) = \psi(y, x)$, hence the two varieties are distinct! However, they do intersect. Solving for $\varphi(x, y) = q + q^{-1}$ and $\psi(x, y) = p + p^{-1}$ with say $q, p > 1$, we find that

$$\sqrt{x} = \frac{p(q^2 - 1)}{p^2 q^2 - 1} \qquad \sqrt{y} = \frac{q(p^2 - 1)}{p^2 q^2 - 1}$$

Conversely, any such point for $p, q > 1$ lies at the intersection of two "integrable" varieties of the form (6.1) and (6.2). This intriguing fact deserves a better understanding. In particular, the 6V variety involves commutation of *finite* size transfer matrices, whereas the Lorentzian one concern matrices of infinite size.

A Infinite matrices and truncated determinants

Throughout these notes, we make extensive use of generating functions for infinite matrices. Let us summarize here the main definitions and properties we use.

A.1 Infinite matrices

We consider infinite matrices $A = (a_{i,j})_{i,j\in\mathbb{Z}_+}$. The very concept of an infinite matrix is a bit delicate to work with, for instance the product of two such matrices might not be well defined. This may be repaired by introducing a formal expansion parameter ϵ, and associating to A the matrix $A(\epsilon) = (\epsilon^{i+j}a_{i,j})_{i,j\in\mathbb{Z}_+}$. The product of any two such matrices now makes sense in the sense of formal power series of ϵ. Moreover, even the notion of eigenvector and eigenvalue make sense in this setting, provided one can show that the latter have formal power or Laurent series expansions in ϵ.

Such a construction will always be implicit (when not explicit) throughout these notes. For instance, the parameter g in $T(g,a)$ of (2.2-2.3) plays the role of ϵ. The "diagonalization" of $T(g,a)$ along the integrable variety $\varphi(g,a) = q + q^{-1}$ (see Sect.2.3) is an example of such extended notions of eigenvectors and eigenvalues.

A.2 Generating functions for infinite matrices

For an infinite matrix $A = (a_{i,j})_{i,j\in\mathbb{Z}_+}$ and a vector $w = (w_i)_{i\in\mathbb{Z}_+}$ we define the formal generating functions

$$f_A(u,v) = \sum_{i,j\in\mathbb{Z}_+} a_{i,j}u^i v^j \qquad f_w(u) = \sum_{i\in\mathbb{Z}_+} w_i u^i$$

with the following properties for matrices A, B and a vector w:

$$
\begin{aligned}
f_{\mathbb{I}}(u,v) &= \frac{1}{1-uv} \\
f_{A^t}(u,v) &= f_A(v,u) \\
f_{Aw}(u) &= \oint \frac{dt}{2i\pi t} f_A(u,t^{-1}) f_w(t) \\
f_{AB}(uv) &= (f_A * f_B)(u,v) = \oint \frac{dt}{2i\pi t} f_A(u,t^{-1}) f_B(t,v)
\end{aligned}
$$

where the contour integral picks the constant term in t.

We consider the lower and upper triangular matrices $L(\alpha,\beta)$ and $U(\alpha,\beta)$ with generating functions

$$f_{L(\alpha,\beta)}(u,v) = \frac{1}{1-\beta u(1+\alpha v)} \qquad f_{U(\alpha,\beta)}(u,v) = \frac{1}{1-\beta v(1+\alpha u)} \qquad (\text{A.1})$$

with $U(\alpha,\beta) = L(\alpha,\beta)^t$. Let us also introduce the shift matrix $S = (\delta_{i,j+1})_{i,j\in\mathbb{Z}_+}$ and the transfer matrix $T(\alpha,\beta,\gamma)$ generated respectively by:

$$f_S(u,v) = \frac{u}{1-uv} \qquad f_{T(\alpha,\beta,\gamma)}(u,v) = \frac{1}{1-\alpha u - \beta v - \gamma uv} \qquad (\text{A.2})$$

In particular, we have

$$L(\alpha,\beta) = T(\beta,0,\alpha\beta) \qquad U(\alpha,\beta) = T(0,\beta,\alpha\beta) \qquad \mathbb{I} = T(0,0,1)$$

The case of the infinite transfer matrix $T(g, a)$ for Lorentzian triangulations corresponds to the identification:

$$T(g, a) = T(ga, ga, g^2(1 - a^2))$$

We have the following properties easily derived by contour integrals for the corresponding generating functions:

$$L(\alpha, \beta)L(\alpha', \beta') = L\left(\frac{\alpha\beta'\alpha'}{1 + \alpha\beta'}, \beta(1 + \alpha\beta')\right) \tag{A.3}$$

$$U(\alpha, \beta)U(\alpha', \beta') = U\left(\frac{\alpha\beta\alpha'}{1 + \alpha'\beta}, \beta(1 + \alpha'\beta)\right) \tag{A.4}$$

$$L(\alpha, \beta)^{-1} = L\left(-\frac{1}{\beta}, -\frac{1}{\alpha}\right) \qquad U(\alpha, \beta)^{-1} = U\left(-\frac{1}{\beta}, -\frac{1}{\alpha}\right) \tag{A.5}$$

$$L(\alpha, \beta)U(\alpha', \beta') = T\left(\beta, \beta', \beta\beta'(\alpha\alpha' - 1)\right) \tag{A.6}$$

$$U(\alpha', \beta')L(\alpha, \beta) = \frac{1}{1 - \beta\beta'} T\left(\frac{\alpha'\beta\beta'}{1 - \beta\beta'}, \frac{\alpha\beta\beta'}{1 - \beta\beta'}, \frac{\alpha\alpha'\beta\beta'}{1 - \beta\beta'}\right) \tag{A.7}$$

$$T(\alpha, \beta, \gamma)T(\alpha', \beta', \gamma') = \frac{1}{1 - \beta\alpha'} T\left(\frac{\alpha + \gamma\alpha'}{1 - \beta\alpha'}, \frac{\beta' + \gamma'\beta}{1 - \beta\alpha'}, \frac{\gamma\gamma' - \alpha\beta'}{1 - \beta\alpha'}\right) \tag{A.8}$$

$$T(\alpha, \beta, \gamma)^{-1} = \frac{\gamma}{\alpha\beta + \gamma} T\left(-\frac{\alpha}{\gamma}, -\frac{\beta}{\gamma}, \frac{1}{\gamma}\right) \tag{A.9}$$

These hold whenever the denominators are non-vanishing.

A.3 Commuting families and addition formulas

Using the formulas above, it is easy to derive the following:

Theorem A.1. *The following family* $\{T_{s,t}(\alpha)\}_{a \in \mathbb{C}}$ *of infinite matrices commute among themselves for any fixed values of* s *and* t:

$$T_{s,t}(\alpha) = T(\alpha, s\alpha, 1 - t\alpha)$$

We also have the following "addition" formula:

$$T_{s,t}(\alpha)T_{s,t}(\alpha') = \frac{1}{1 - s\alpha\alpha'} T_{s,t}\left(\frac{\alpha + \alpha' - t\alpha\alpha'}{1 - s\alpha\alpha'}\right)$$

For $s = 0$, we obtain a family of commuting lower triangular matrices

$$L_t(\alpha) = T_{0,t}(\alpha) = L\left(\frac{1}{\alpha} - t, \alpha\right)$$

with the "addition" formula:

$$L_t(\alpha)L_t(\alpha') = L_t(\alpha + \alpha' - t\alpha\alpha')$$

Changing variables from α to a, with $\alpha = \frac{1 - e^{-ta}}{t}$, and writing $\ell_t(a) = L_t(\alpha)$ we finally get the addition formula:

$$\ell_t(a)\ell_t(a') = \ell_t(a + a') \tag{A.10}$$

from which we deduce that $\ell_t(a)$ is an infinite matrix exponential. More precisely, let M_t be the infinite matrix generated by

$$f_{M_t} = \frac{(tv-1)u}{(1-uv)^2} \qquad M_t = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ -1 & t & 0 & \cdots \\ 0 & -2 & 2t & \cdots \\ 0 & 0 & -3 & 3t \\ \vdots & & & \ddots \ddots \end{pmatrix}$$

then we have $\ell_t(a) = \exp(-aM_t)$, which by triangularity holds for any finite truncation as well. A similar analysis holds for $T_{s,t}(\alpha)$, but only for the infinite matrix. Assuming that $t^2 - 4s > 0$, and introducing another parameter $r = \sqrt{1 - 4s/t^2}$, the relevant change of variables is:

$$\alpha = \frac{2(e^{rta} - 1)}{t(te^{rta}(r+1) + r - 1)} \qquad \tau_{r,t}(a) = T_{s,t}(\alpha)$$

in terms of which

$$\tau_{r,t}(a)\tau_{r,t}(a') = \frac{1}{1 - \frac{(1-r^2)(e^{rta}-1)(e^{rta'}-1)}{(te^{rta}(r+1)+r-1)(te^{rta'}(r+1)+r-1)}} \tau_{r,t}(a + a')$$

This reduces to (A.10) when $r = 1$ (corresponding to $s = 0$).

A.4   Truncated determinants

For any infinite matrix $A = (a_{i,j})_{i,j\in\mathbb{Z}_+}$, we denote by $A^{[0,n-1]}$ the finite $n \times n$ truncation of A to its n first rows and columns, namely the matrix with entries: $A^{[0,n-1]} = (a_{i,j})_{i,j\in[0,n-1]}$.

In general, the matrix product does not respect truncation. However, if L, U are respectively lower and upper triangular infinite matrices, then $(LU)^{[0,n-1]} = L^{[0,n-1]}U^{[0,n-1]}$. Note that this does not hold for (UL) (see the example below).

Let us now examine truncated determinants, namely the determinant of such finitely truncated matrices. By triangularity it is immediate to compute:

$$\det\left(L(\alpha,\beta)^{[0,k]}\right) = (\alpha\beta)^{k(k+1)/2} = \det\left(U(\alpha,\beta)^{[0,k]}\right) \qquad (\text{A.11})$$

and by the above property we deduce from (A.6) and (A.11) that:

$$\det\left(T^{[0,k]}(\beta,\beta',\beta\beta'(\alpha\alpha' - 1))\right) = \det\left(L(\alpha,\beta)^{[0,k]}U(\alpha',\beta')^{[0,k]}\right) = (\alpha\beta\alpha'\beta')^{k(k+1)/2}$$

and more generally

$$\det\left(T^{[0,k]}(\alpha,\beta,\gamma)\right) = (\alpha\beta + \gamma)^{k(k+1)/2}$$

whereas

$$\det\left(\left(U(\alpha',\beta')L(\alpha,\beta)\right)^{[0,k]}\right) = \frac{\det\left(T^{[0,k]}\left(\frac{\alpha'\beta\beta'}{1-\beta\beta'}, \frac{\alpha\beta\beta'}{1-\beta\beta'}, \frac{\alpha\alpha'\beta\beta'}{1-\beta\beta'}\right)\right)}{(1-\beta\beta')^{k+1}}$$

$$= \frac{(\alpha\beta\alpha'\beta')^{k(k+1)/2}}{(1-\beta\beta')^{k+1}}$$

by use of (A.7). The discrepancy with the LU result is because the matrix product now involves all the elements of the $(k+1)\times$ infinite and infinite $\times(k+1)$ rectangular matrices of the truncated product.

The main property allowing for proving truncated determinant identities from relations between generating functions is the following:

Lemma A.2. *Let L, U, A be respectively lower triangular, upper triangular and arbitrary infinite matrices, and let $M = LAU$. Then:*

$$M^{[0,k]} = L^{[0,k]} A^{[0,k]} U^{[0,k]}$$

Assuming further that both L and U are unitriangular, we then deduce that $\det(M^{[0,k]}) = \det(A^{[0,k]})$ for all $k \geq 0$. So we will have identity between all truncated determinants of two infinite matrices M and A if there is a relation $M = LAU$ for L, U lower and upper unitriangular infinite matrices.

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