$(q, t)$-hook formula for Birds

Masao ISHIKAWA

Department of Mathematics, Faculty of Education, University of the Ryukyus, Nishihara, Okinawa 901-0213, Japan, ishikawa@edu.u-ryukyu.ac.jp

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Abstract

We study Okada's conjecture on $(q, t)$-hook formula of general $d$-complete posets. Proctor classified $d$-complete posets into 15 irreducible ones. We try to give a case-by-case proof of Okada's $(q, t)$-hook formula conjecture using the symmetric functions. Here we give a proof of the conjecture for birds, in which we use Gasper's identity for VWP-series $13W_{11}$.

1 Introduction and the main results

Let $\mathbb{N}$ (resp. $\mathbb{Z}$) be the set of nonnegative integers (resp. integers). Throughout this paper we use the standard notation for $q$-series (see [1, 3, 4, 5]):

$$(a; q)_{\infty} = \prod_{k=0}^{\infty} (1 - aq^{k}), \quad (a; q)_{n} = \frac{(a; q)_{\infty}}{(aq^{n}; q)_{\infty}}$$

for any integer $n$. Usually $(a; q)_{n}$ is called the $q$-shifted factorial, and we frequently use the compact notation:

$$(a_{1}, a_{2}, \ldots, a_{r}; q)_{n} = (a_{1}; q)_{n}(a_{2}; q)_{n}\cdots(a_{r}; q)_{n}.$$ 

The $r+1 \phi_{r}$ basic hypergeometric series is defined by

$$r+1 \phi_{r} \left( \begin{array}{c} a_{1}, a_{2}, \ldots, a_{r+1} \\ b_{1}, \ldots, b_{r} \end{array}; q, z \right) = \sum_{n=0}^{\infty} \frac{(a_{1}, a_{2}, \ldots, a_{r+1}; q)_{n}}{(q, b_{1}, \ldots, b_{r}; q)_{n}} z^{n}.$$ (1.1)

A basic hypergeometric series $r+1 \phi_{r}$ is said to be balanced if it satisfies $a_{1}\cdots a_{r+1} = b_{1}\cdots b_{r}$ and $z = q$, well-poised if it satisfies $a_{1} = a_{2}b_{1} = \cdots = a_{r+1}b_{r}$, very well-poised if it is well-poised and satisfies $b_{1} = a_{1}^{\frac{1}{12}}$ and $b_{2} = -a_{1}^{\frac{1}{12}}$ (see [3, §2.1]). If $r+1 \phi_{r}$ is very well-poised series, we use the notation

$$r+1W_{r}(a_{1}; a_{4}, \ldots, a_{r+1}; q, z) = r+1 \phi_{r} \left[ \begin{array}{c} a_{1}, a_{2}, \ldots, a_{r+1} \\ a_{1}^{\frac{1}{12}}, -a_{1}^{\frac{1}{12}}, a_{4}, \ldots, a_{r+1} \end{array}; q, z \right].$$

Proposition 1.1. Gasper's formula ([2, p.1065, (3.2)], [3, pp.250, Ex.8.15]) reads as follows:

$$q_{b}^{3} \left[ a, b, c, d \right] = \frac{(a/d, bq/d, cq/d, abc/d; q)_{\infty}}{(q/d, ab/d, ac/d, bcq/d; q)_{\infty}}$$

$$\times 12W_{11} \left( \frac{bc}{d}; \frac{bc}{ad}; \frac{bc}{ad}, \frac{bc}{ad}, -q \left( \frac{bc}{d} \right)^{\frac{1}{2}}, -q \left( \frac{bc}{d} \right)^{\frac{1}{2}}, \frac{ab}{d}; a, b, c; q, \frac{q}{a} \right),$$ (1.2)

where at least one of $a, b, c$ is of the form $q^{-n} (n = 0, 1, \ldots)$. 

We use the notation in [8]. For nonnegative integers $n$ and $m$ we write
\[
f(n; m) = f_{q,t}(n; m) = \frac{(tm+1; q)_n}{(tm; q)_n},
\]
and
\[
F(x) = F(x; q, t) = \frac{(tx; q)_\infty}{(x; q)_\infty},
\]
where $q$ and $t$ are parameters and $x$ is a variable (see [8, (5)(6)]). Hereafter we use the convention that $f_{q,t}(n; m) = 0$ for a negative integer $n < 0$.

We use the notation in [7, 12] for partitions. Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition, i.e., $\lambda_1 \geq \lambda_2 \geq \cdots$ with finitely many $\lambda_i$ unequal to zero. The length and weight of $\lambda$, denoted by $\ell(\lambda)$ and $|\lambda|$, are the number and sum of the non-zero $\lambda_i$ respectively. When $|\lambda| = N$ we say that $\lambda$ is a partition of $N$, and the unique partition of zero is denoted by $\emptyset$. The multiplicity of the part $i$ in the partition $\lambda$ is denoted by $m_i(\lambda)$. We identify a partition with its diagram (Ferrers graph)
\[
D(\lambda) = \{(i, j) \in \mathbb{Z}^2 : 1 \leq j \leq \lambda_i\}.
\]
(1.3)
The conjugate $\lambda'$ of $\lambda$ is the partition obtained by reflecting the diagram of $\lambda$ in the main diagonal. A partition is said to be strict if we have strict inequalities $\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0$ with $r = \ell(\lambda)$. If $\lambda$ is a strict partition, then its shifted diagram is defined by
\[
S(\lambda) = \{(i, j) \in \mathbb{Z}^2 : i \leq j \leq \lambda_i + i - 1\}.
\]
(1.4)
Hereafter we may use the same symbol $\lambda$ to represent its diagram (or shifted diagram).

We use standard notation and terminology of [12, Chapter 3] related to posets. We write $x < y$ if $x$ is covered by $y$, i.e., $x < y$ and there is no $z \in P$ such that $x < z < y$. A Hasse diagram is a diagram in which one represents each element of $P$ as a vertex in the plane and draws an edge that goes upward from $x$ to $y$ whenever $y$ covers $x$.

**Definition 1.2.** ([11], [12, §3.15]) Let $P$ be a poset. A $P$-partition is a map $\pi : P \to \mathbb{N}$ satisfying
\[
x \leq y \in P \implies \pi(x) \geq \pi(y) \text{ in } \mathbb{N}.
\]
(1.5)
Let $\mathcal{A}(P)$ denote the set of $P$-partitions.

First, we review the definition and some properties of d-complete posets. (See [9, 10].) For $k \geq 3$, we denote by $d_k(1)$ the poset consisting of $2k - 2$ elements, called double-tailed diamond poset, with the Hasse diagram depicted in Figure 1. The two incomparable elements are called the sides, the $k - 2$ elements above them are called neck elements, and the maximum and minimum elements are called top and bottom respectively. If $k = 3$ then we call $d_3(1)$ a diamond. Let $P$ be a poset. An interval $[w, v] = \{x \in P : w \leq x \leq v\}$ is called a $d_k$-interval if it is isomorphic to $d_k(1)$. A $d_k^{-}$-interval ($k \geq 4$) is an interval isomorphic to $d_k(1) - \{\text{top}\}$. A $d_k^{-}$-interval consists of three elements $x, y$ and $w$ such that $w$ is covered by both $x$ and $y$. A poset $P$ is $d$-complete if it satisfies the following three conditions for every $k \geq 3$:

(D1) If $I$ is a $d_k^{-}$-interval, then there exists an element $v$ such that $v$ covers the maximal elements of $I$ and $I \cup \{v\}$ is a $d_k$-interval.

(D2) If $I = [w, v]$ is a $d_k$-interval and the top $v$ covers $u$ in $P$, then $u \in I$.

(D3) There are no $d_k^{-}$-intervals which differ only in the minimal elements.

We quote a proposition due to Proctor [9, Proposition in §3] (also see [8, Proposition 4.1]):

**Proposition 1.3.** ([9, Proposition in §3]) Let $P$ be a $d$-complete poset. Suppose that $P$ is connected, i.e., the Hasse diagram of $P$ is connected. Then we have
(a) $P$ has a unique maximal element $v_0$.

(b) For each $v \in P$, every saturated chain from $v$ to the maximum element $v_0$ has the same length.

Hence $P$ admits a rank function $r : P \rightarrow \mathbb{N}$ such that $r(x) = r(y) + 1$ if $x$ covers $y$.

A rooted tree is a poset which has a unique maximal element, and is such that each non-maximal element is covered by exactly one other element. Let $P$ be a poset with a unique maximal element. The top tree $T$ of $P$ is the filter (i.e., $x \in T$ and $y \geq x$ implies $y \in T$) of $P$, whose vertex set consists of all elements $x \in P$ such that every $y \geq x$ is covered by at most one other element of $P$. $T$ is clearly a rooted tree and an element of $T$ is called top tree element. Afterwards we use a particular kind of rooted tree. Let $f \geq 0$ and $h \geq g \geq 0$ be integers. The rooted tree $Y(f; g, h)$ consists of one branch element above which a chain of $f$ elements has been adjoined and below which two non-adjacent chains with $g$ and $h$ elements, respectively.

Let $P$ be a connected $d$-complete poset with top tree $T$. An element $x \in P$ is said to be acyclic if $x \in T$ and it is not in the neck of any $d_k$-interval for any $k \geq 3$. An element of $P$ is said to be cyclic if it is not acyclic. Let $Q$ be a $d$-complete poset containing an acyclic element $y$. Let $P$ be a connected $d$-complete poset. By Proposition 1.3 (a), let $x$ denote the unique maximal element of $P$. Then the slant sum of $Q$ with $P$ at $y$, denoted $Q \uparrow_y P$, is the poset formed by creating a covering relation $x < y$. A $d$-complete poset $P$ is slant irreducible if it is connected and it cannot be expressed as a slant sum of two non-empty $d$-complete posets. Suppose that $P$ is a connected $d$-complete poset with top tree $T$. An edge $x < y$ of $P$ is a slant edge if $x, y \in T$ and $y$ is acyclic. In [9] Proctor proves $P$ is slant irreducible if and only if it contains no slant edges. Also, $P$ is slant irreducible if and only if every acyclic element is a minimal element of its top tree. ([9, Proposition C of §4]) Given any connected $d$-complete poset $P$, first locate all of its slant edges. These may be erased in any order to produce a collection $P_1, P_2, \ldots$ of uniquely determined smaller non-adjacent connected $d$-complete posets. No new slant edges are created, and so each of $P_1, P_2, \ldots$ are slant irreducible. We say that $P_1, P_2, \ldots$ are the slant irreducible components of $P$. If $P$ is an irreducible component, then its top tree $T$ is of the form $Y(f; g, h)$ for some $f \geq 0$ and $h \geq g \geq 1$ ([9, Theorem of §5]). In the paper he establish the following theorem, which describe the structure of any connected $d$-complete poset.

Theorem 1.4. (Proctor [9, Theorem in §4]) Let $P$ be a connected $d$-complete poset. It may be uniquely decomposed into a slant sum of one element posets and irreducible components. The top tree of $P$ is an analogous slant sum of the top trees of the irreducible components.

In §7 of [9] Proctor defines 15 disjoint classes of irreducible components $\mathcal{C}_1, \ldots, \mathcal{C}_{15}$ and have shown that these 15 disjoint classes exhaust the set of all irreducible components. For the list of 15
classes of irreducible $d$-complete posets see [9, Table 1]. The diagram (1.3) of an ordinary partition $\lambda$ or the shifted diagram (1.4) of a shifted partition $\lambda$ is regarded as a poset by defining its order structure as

$$(i_1, j_1) \geq (i_2, j_2) \iff i_1 \leq i_2 \text{ and } j_1 \leq j_2.$$  

(1.6)

By this order the poset represented by a diagram $P = D(\lambda)$ is called a shape with its top tree $T = Y(f; g, h)$ where $f = 0$, $g = \ell(\lambda)$ and $h = \ell(\lambda')$. We use $\mathscr{C}_1$ to express the class of shapes which is a class of irreducible $d$-complete posets defined in [9].

Another important class $\mathscr{C}_2$ is the set of posets $P = S(\alpha)$ for strict partitions $\alpha$, which is called shifted shapes with its top tree $T = Y(f, g, h)$ where $f = g = 1$ and $h = \ell(\alpha)$. Its Hasse diagram is designated by Figure 1 in which the first row has $\alpha_1$ vertices, the second row $\alpha_2$ vertices and so on. When depicting these posets as a Hasse diagram, we use the convention that a northwest vertex is larger than another in southeast. Here the larger dots and the heavier edges indicate the top tree. For later use we denote by $P = P_2(\alpha)$ the shifted shape associated with a strict partition $\alpha$. If $P = P_2(\alpha)$ is the shifted shape associated with a strict partition $\alpha$,

![Figure 2: Shifted shapes $C_2$](image)

then $P$-partition

$$\pi = (\pi_{ij})_{(i,j) \in S(\alpha)}$$  

(1.7)

satisfies

$$\pi_{ij} \leq \pi_{i+1,j}, \quad \pi_{ij} \leq \pi_{i,j+1},$$  

(1.8)

whenever the both sides defined. For example, Figure 1 is a $P$-partition for shifted shape $(8, 5, 2, 1)$.

![Figure 3: $P$-partition for shifted shape $(8, 5, 2, 1)$](image)

In this paper we mainly consider only birds $\mathscr{C}_3$ (Figure 1). Let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ be strict partitions such that $\alpha_1 > \alpha_2 > 0$ and $\beta_1 > \beta_2 > 0$. Define the bird $P = P_3(\alpha, \beta; f)$ by

$$P = P_H \cup P_R \cup P_L \cup P_T$$

where $P_H$, $P_R$, $P_L$, and $P_T$ denote the north, east, west, and south Leg.

...
Figure 4: Birds $C_3$

where

$$P_H = \{(1, j) : -f+1 \leq j \leq 1\},$$
$$P_R = \{(i,j) : i \leq j \leq \alpha_i + i - 1 \ (i = 1, 2)\},$$
$$P_L = \{(i,j) : j \leq i \leq \beta_j + j - 1 \ (j = 1, 2)\},$$
$$P_T = \{(i, i) : 2 \leq i \leq f+2\}$$

as a set and we regard it as a poset by defining its order structure (1.6) if and only if the both of $(i_1, j_1)$ and $(i_2, j_2)$ are in $P_H \cup P_R \cup P_L$ or in $P_T$ (see [9, Table 1 and Figure 5.3]). We call $P_H$ the head, $P_T$ the tail, $P_R$ (resp. $P_L$) the right (resp. left) wing of $P$. The Hasse diagram of a bird is as in Figure 1. Strictly speaking, we have to impose the condition $\alpha_1 = \alpha_2 + 1$ and $\beta_1 = \beta_2 + 1$ to let $P$ be slant irreducible, but here we don’t need this condition. For example, Figure 1 stands for $P = P_3((4,3),(4,2); 2)$. We have the chain $[v, v_2]$ (resp. $[w_2, w]$), which is the head (resp. tail) of $P$. Recall that a $P$-partition $\pi$ satisfies the condition (1.5). When $P = P_3(\alpha, \beta; f)$, we associate the quadruple $(\sigma, \tau; \rho, \theta)$ with $\pi$, where

$$\sigma = (\sigma_{i,j})_{(i,j) \in P_R}, \quad \tau = (\tau_{i,j})_{(j,i) \in P_L}, \quad \rho = (\rho_i)_{i=0, \ldots, f}, \quad \theta = (\theta_i)_{i=0, \ldots, f}$$

with

$$\sigma_{i,j} = \pi(i,j) \quad \text{for} \ (i,j) \in P_R, \quad \tau_{i,j} = \pi(j, i) \quad \text{for} \ (i,j) \in P_L, \quad \rho_{-i+1} = \pi(1, i) \quad \text{for} \ (1, i) \in P_H, \quad \theta_{i-2} = \pi(i,i) \quad \text{for} \ (i, i) \in P_T.$$

Hence we use the convention that $\rho_0 = \sigma_{11} = \tau_{11}$ and $\theta_0 = \sigma_{22} = \tau_{22}$. We write $\pi = (\sigma, \tau; \rho, \theta)$ hereafter. If $P = P_3((4, 3), (4, 2); 2)$ then $\pi$ is as the left picture of Figure 1.

Figure 5: Bird $P = P_3((4, 3), (3, 2); 2)$ and banner $P = P_6((9, 6, 3, 2); 2)$
Figure 6: A $P$-partition

Let $P$ be a connected $d$-complete poset and $T$ its top tree. Let $C$ be a set, called a set of colors, whose cardinality is the same as $T$. A coloring of $P$ a coloring map $c$ of $P$ to the set of colors $C$. $P$ is said to be properly colored if the coloring map $c$ satisfies

(C1) $c(x) \neq c(y)$ if $x$ and $y$ are incomparable,

(C2) $c(x) \neq c(y)$ if $x$ covers $y$.

It is simply colored if, in addition:

(C3) whenever an interval $[w, v]$ is a chain, the colors of the elements $c(x)$ in the interval $[w, v]$ are distinct.

If $P$ is a rooted tree, then it is simply colored by the identity map $P \to P$, i.e. we assign a distinct color to each vertex of $P$.

**Proposition 1.5.** ([10, Proposition 8.6]) Let $P$ be a connected $d$-complete poset and $T$ its top tree. Let $C$ be a set whose cardinality is the same as $T$. Then a bijection $c : T \to C$ can be uniquely extended to a proper coloring $c : P \to C$ satisfying the following condition:

(C4) If $[w, v]$ is a $d_k$-interval then $c(w) = c(v)$.

Such a map $c : P \to I$ is called a $d$-complete coloring.

For example, in the both picture of Figure 1 because $[w_2, v_2]$ (resp. $[w_1, v_1]$) is a $d_5$-interval (resp. $d_4$-interval), $w_2$ (resp. $w_1$, $w$) and $v_2$ (resp. $v_1$, $v$) have the same color. In Figure ?? $v_1$ (resp. $v_2$) and $v_2$ (resp. $v_4$, $v_2$) have the same color since $[v_2, v_1]$ (resp. $[v_4, v_2]$) is a $d_4$-interval, however, the $v_1$ and $v_2$ have distinct colors since the both are in the top tree.

**Proposition 1.6.** (1) If $\alpha$ is a strict partition with length $\geq 2$, then the top tree of the shifted shape $P = P_2(\alpha)$ is given by

$$T = \{(1, j) : 1 \leq j \leq \alpha_1\} \cup \{(2, 2)\} \quad (1.10)$$

and a $d$-complete coloring $c : P \to \{0, 0', 1, 2, \ldots, \alpha_1 - 1\}$ is given by

$$c(i, j) = \begin{cases} j - i & \text{if } i < j, \\ 0 & \text{if } i = j \text{ and } i \text{ is odd,} \\ 0' & \text{if } i = j \text{ and } i \text{ is even.} \end{cases} \quad (1.11)$$

Hence we see that $P$ has the top tree $Y(1; 1, \alpha_1 - 1)$.

(2) If $\alpha$ and $\beta$ are strict partitions with length $= 2$ and $f \geq 1$ then the top tree of the bird $P = P_3(\alpha, \beta; f)$ is given by

$$T = \{(1, j) : -f + 1 \leq j \leq \alpha_1\} \cup \{(i, 1) : 1 \leq i \leq \beta_1\} \quad (1.12)$$
and a \(d\)-complete coloring \(c : P \to \{-f, \ldots, -1, 1, 2, \ldots, \alpha_1 - 1\} \cup \{1', 2', \ldots, (\beta_1 - 1)\}' is given by

\[
c(i, j) = \begin{cases} 
  j - i & \text{if } i < j, \text{ i.e., } (i, j) \in P_H, \\
  (j - i)' & \text{if } 1 \leq j < i, \text{ i.e., } (i, j) \in P_L, \\
  j - 1 & \text{if } i = 1 \text{ and } j \leq 1, \text{ i.e., } (i, j) \in P_H, \\
  -i + 2 & \text{if } i = j \geq 2, \text{ i.e., } (i, j) \in P_T.
\end{cases}
\]

(1.13)

Hence we see that \(P\) has the top tree \(Y(f; \alpha_1 - 1, \beta_1 - 1)\).

Let \(P\) be a connected \(d\)-complete poset and \(c : P \to C\) a \(d\)-complete coloring. Let \(z_i (i \in C)\) be indeterminates. For a \(P\)-partition \(\pi \in \mathcal{A}(P)\), we put

\[
z^\pi = \prod_{v \in P} z_{c(v)}^{\pi(v)}.
\]

As in [8, p.412] we associate a monomial \(z[H_P(v)]\) to each \(v \in P\), called the hook monomial, which is uniquely determined by induction as follows:

(a) If \(v\) is not the top of any \(d_k\)-interval, then we define

\[
z[H_P(v)] = \prod_{w \leq v} z_{c(w)}.
\]

(b) If \(v\) is the top of a \(d_k\)-interval \([w, v]\), then we define

\[
z[H_P(v)] = \frac{z[H_P(x)] \cdot z[H_P(y)]}{z[H_P(w)]},
\]

where \(x\) and \(y\) are the sides of \([w, v]\).

Further we denote \(z[H_P] = \{z[H_P(v)] : v \in P\}\) the set of the hook monomials, and let \(F(z[H_P]; q, t)\) denote the product of \(F(z[H_P(v)]; q, t)\) over \(v \in P\), i.e.,

\[
F(z[H_P]; q, t) = \prod_{v \in P} F(z[H_P(v)]; q, t).
\]

Let \(P\) be a connected \(d\)-complete poset with the maximum element \(v_0\), and the rank function \(r : P \to \mathbb{N}\). Let \(T\) be the top tree of \(P\). Take \(T\) as a set of colors and let \(c : P \to T\) be the \(d\)-complete coloring such that \(c(v) = v\) for all \(v \in T\). Let \(\hat{P} = P \cup \{\hat{1}\}\) be the extended poset, where \(\hat{1}\) is the new maximum element of \(\hat{P}\) which covers \(v_0\). Then \(\hat{P}\) has its top tree \(\hat{T} = T \cup \{\hat{1}\}\), where \(\hat{c} : \hat{P} \to \hat{T}\) with \(\hat{c}(\hat{1}) = \hat{1}\).

**Definition 1.7.** Given a \(P\)-partition \(\pi \in \mathcal{A}(P)\), let \(\hat{\pi} : \hat{P} \to \mathbb{N}\) be the extensions of \(\pi\) defined by \(\hat{\pi}(\hat{1}) = 0\). Define a weight \(W_P(\sigma; q, t)\) by putting

\[
W_P(\pi; q, t) = \prod_{x, y \in P} \frac{f(\pi(x) - \pi(y); d(x, y))}{f(\sigma(x) - \sigma(y); e(x, y)) f(\sigma(x) - \sigma(y); e(x, y) - 1)},
\]

(1.14)

where \(\hat{c}(x) \sim \hat{c}(y)\) means that \(\hat{c}(x)\) and \(\hat{c}(y)\) are adjacent to each other in \(T\), and

\[
d(x, y) = \frac{r(y) - r(x) - 1}{2}, \quad e(x, y) = \frac{r(y) - r(x)}{2}.
\]

Note that if \(c(x) \sim c(y)\) then \(r(y) - r(x)\) is odd, and if \(c(x) = c(y)\) then \(r(y) - r(x)\) is even, hence \(d(x, y)\) and \(e(x, y)\) are nonnegative integers.
Now we quote Okada’s \((q, t)\)-hook formula conjecture.

**Conjecture 1.8.** (Okada [8]) Let \(P\) be a connected \(d\)-complete poset. Using the notations defined above, we have
\[
\sum_{\pi \in \mathcal{A}(P)} W_P(\pi; q, t)z^\pi = F(z[H_p]; q, t). \tag{1.15}
\]

Okada has proven this conjecture for Shapes and Shifted shapes. The purpose of this paper is to prove his conjecture for birds and banners.

**Theorem 1.9.** Okada’s \((q, t)\)-hook formula conjecture is true for birds and banners.

Given a \(P\)-partition \(\pi \in \mathcal{A}(P)\) for the shifed shape \(P = P_2(\alpha)\) for a strict partition \(\alpha\), we write
\[
f_{\alpha}^{ND}(\pi; q, t) = \prod_{(i, j) \in \alpha} \frac{f(\pi_{i,j} - \pi_{i-m,j-m}; m)}{f(\pi_{i,j} - \pi_{i-m,j-m}; m)}.
\]
\[
f_{\alpha}^{D}(\pi; q, t) = \prod_{(i, j) \in \alpha} \frac{f(\pi_{i,j} - \pi_{i-m,j-m}; m)}{f(\pi_{i,j} - \pi_{i-m,j-m}; m)}.
\]

Here we use the convention that \(\pi_{i,j} = 0\) if \(i \leq 0\) or \(j \leq 0\). Further we use the following short notation. Let \(m\) and \(n\) be positive integers such that \(m \leq n\). When \(\rho = (\rho_m, \ldots, \rho_n)\) and \(\theta = (\theta_m, \ldots, \theta_n)\) satisfy
\[
0 \leq \rho_n \leq \cdots \leq \rho_m \leq \theta_m \leq \cdots \leq \theta_n,
\]
we write
\[
\Phi_m^\alpha(\rho, \theta;q,t) = \prod_{i=m+1}^n \frac{f(\pi_{i-1} - \pi_{i-m}; m)}{f(\pi_{i-1} - \pi_{i-m}; m)}.
\]

**Proposition 1.10.** (1) Let \(\alpha\) be a strict partition of length \(r\) and \(P = P_2(\alpha)\) the associated shifted shape. If \(\pi = (\pi_{i,j})_{(i,j) \in \alpha}\) is a \(P\)-partition (1.7) satisfying the condition (1.8), then its weight \(W_P(\pi; q, t)\) is given by
\[
W_P(\pi; q, t) = f_{\alpha}^D(\pi; q, t) f_{\alpha}^{ND}(\pi; q, t).
\]

(2) Let \(\alpha\) and \(\beta\) be strict partitions of length 2. Let \(f > 0\) be a positive integer, and set \(P = P_2(\alpha, \beta; f)\) to be the bird associated with \(\alpha\), \(\beta\) and \(f\). If \(\pi = (\sigma, \tau; \rho, \theta)\) is a \(P\)-partition satisfying the condition (1.9), then its weight \(W_P(\pi; q, t)\) is given by
\[
W_P(\pi; q, t) = \frac{f(\sigma_{22} - \sigma_{21}; 0)f(\tau_{22} - \tau_{21}; 0)f(\rho_{f}; 0)f(\theta_{f}; f + 1)}{f(\sigma_{22} - \sigma_{12}; 0)f(\sigma_{22} - \sigma_{12}; 1)} \times \Phi_m^\alpha(\rho, \theta;q,t) f_{\alpha}^{ND}(\sigma, \tau;q,t) f_{\beta}^{ND}(\tau; q,t).
\]

Here we use the convention that \(\sigma_{11} = \tau_{11} = \rho_0\) and \(\sigma_{22} = \tau_{22} = \theta_0\).

**Proposition 1.11.** (1) Let \(\alpha\) be a strict partition of length \(r\) and \(P = P_2(\alpha)\) the associated shifted shape. Let \(n\) be an integer such that \(n \geq \alpha_1\), and let \(\alpha^c\) be the strict partition formed by the complement of \(\alpha\) in \([n]\), i.e.,
\[
\{\alpha_1, \ldots, \alpha_r\} \cup \{\alpha_1^c, \ldots, \alpha_n^c\} = [n].
\]
We write \(y_0 = z_0\) (see Proposition 1.6 (1)) hereafter. Then we have
\[
F(z[H_p]; q, t) = \prod_{\alpha_1^c < \alpha_1^c} F\left(\bar{z}_{\alpha_i}^{-1} z_{\alpha_j}; q, t\right) \prod_i F(\bar{z}_{\alpha_i}; q, t) \prod_{i<j} F\left(w \bar{z}_{\alpha_i} z_{\alpha_j}; q, t\right), \tag{1.22}
\]
where
\[
\begin{aligned}
w &= y_0/z_0 \quad \text{and} \quad \bar{z}_i = \prod_{k=1}^{i-1} z_k \quad \text{(i = 1, \ldots, n).} \quad \text{if } r \text{ is odd,} \\
w &= z_0/y_0 \quad \text{and} \quad \bar{z}_i = y_0 \prod_{k=1}^{i-1} z_k \quad \text{(i = 1, \ldots, n).} \quad \text{if } r \text{ is even.}
\end{aligned}
\]
(2) Let \( \alpha = (\alpha_1, \alpha_2) \) and \( \beta = (\beta_1, \beta_2) \) be strict partitions of length 2. Let \( f > 0 \) be a positive integer, and set \( P = P_f(\alpha, \beta; f) \) the bird associated with \( f, \alpha \) and \( \beta \). Let \( m, n \) be integers such that \( m \geq \ell(\alpha) \) and \( n \geq \ell(\beta) \), and let \( \alpha^* \) (resp. \( \beta^* \)) be the strict partition formed by the complement of \( \alpha \) (resp. \( \beta \)) in \([m] \) (resp. \([n] \)). We write \( y_i = z_{-i} \) for \( i = 1, \ldots, \beta_1 - 1 \) and \( x_i = z_{-i} \) for \( i = 1, \ldots, f \). Further we may write \( x_0 = y_0 = z_0 \). (See Proposition 1.6 (2)). Then we have

\[
F(z[H_\beta]; q, t) = \prod_{\alpha_j < \alpha_2} F(z_{-\alpha_j}; q, t) \prod_{\beta_j < \beta_1} F(y_{-\beta_j}; q, t) \prod_{i=1}^f F(\tilde{x}_i; q, t)
\]

\[
\times \prod_{i=1}^f \left( \prod_{k, i=1}^2 \frac{\tilde{y}_k \tilde{z}_i}{\tilde{x}_i} ; q, t \right) \prod_{i,j=1}^2 F(\tilde{x}_0 \tilde{y}_j \tilde{z}_i; q, t) (1.23)
\]

where \( \tilde{x}_i = \prod_{k=1}^{f-i} x_i \) for \( i = 0, \ldots, f \), \( \tilde{y}_i = \prod_{k=1}^{i-1} y_k \) for \( i = 1, \ldots, n \), and \( \tilde{z}_i = \prod_{k=1}^{i-1} z_k \) for \( i = 1, \ldots, m \).

### 2 Macdonald polynomials

We follow the notation and terminology of [7] for the symmetric functions. If \( \lambda \) and \( \mu \) are partitions then \( \mu \subseteq \lambda \) if \( \mu \) is contained in \( \lambda \), i.e., \( \mu_i \leq \lambda_i \) for all \( i \geq 1 \). If \( \mu \subseteq \lambda \) then the skew-diagram \( \lambda/\mu \) denotes the set-theoretic difference between \( \lambda \) and \( \mu \), i.e., those squares of \( \lambda \) not contained in \( \mu \). The skew diagram \( \lambda/\mu \) is a vertical \( r \)-strip if \( |\lambda - \mu| = |\lambda| - |\mu| = r \) and if, for all \( i \geq 1 \), \( \lambda_i \geq \mu_i \) is at most one, i.e., each row of \( \lambda - \mu \) contains at most one square. The set of all vertical \( r \)-strips is denoted by \( \mathcal{V}_r \) and the set of all vertical strips by \( \mathcal{V} = \bigcup_{r=0}^\infty \mathcal{V}_r \). The skew diagram \( \lambda/\mu \) is a horizontal \( r \)-strip if \( |\lambda - \mu| = r \) and if, for all \( i \geq 1 \), \( \lambda'_i - \mu'_i \) is at most one, i.e., each column of \( \lambda - \mu \) contains at most one square. For two partitions \( \lambda \) and \( \mu \), we write \( \lambda \succ \mu \) if \( \lambda \supset \mu \) and \( \lambda/\mu \) is a horizontal strip. Note that \( \lambda/\mu \) is a horizontal strip if and only if \( \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \).

The set of all horizontal \( r \)-strips is denoted by \( \mathcal{H}_r \) and the set of all horizontal strips by \( \mathcal{H} \). Let \( s = (i, j) \) be a square in the diagram of \( \lambda \), and let \( a(s) \) and \( l(s) \) be the arm-length and leg-length of \( s \), given by

\[ a(s) = \lambda_i - j, \quad l(s) = \lambda'_j - i \]

Then we define the rational functions let

\[ b_\lambda(s) = b_\lambda(s; q, t) := \begin{cases} \frac{1 - q^{a(s)} t^{l(s) + 1}}{1 - q^{a(s)} t^{l(s)}}, & \text{if } s \in \lambda, \\ 1, & \text{otherwise,} \end{cases} \]

and [6, (3.6)] [7, VI.7 (6.19), VI.7 Ex.4]

\[ b_\lambda(q, t) := \prod_{s \in \lambda} b_\lambda(s; q, t) = \prod_{i \geq 1} \prod_{m \geq 0} \frac{f_{q, t}(\lambda_i - \lambda_{i+m+1}; m)}{f_{q, t}(\lambda_i - \lambda_{i+m}; m)}, \quad (2.1) \]

\[ b_\lambda^i(q, t) := \prod_{s \in \lambda} b_\lambda(s; q, t) = \prod_{i \geq 1} \prod_{m \geq 0} \frac{f_{q, t}(\lambda_i - \lambda_{i+m+1}; m)}{f_{q, t}(\lambda_i - \lambda_{i+m}; m)}, \quad (2.2) \]

\[ b_\lambda^a(q, t) := \prod_{a(s) \text{ even}} b_\lambda(s; q, t). \quad (2.3) \]

If \( x = (x_1, x_2, \ldots) \) and \( y = (y_1, y_2, \ldots) \) are two sequences of independent indeterminates, then we write

\[ \Pi(x; y; q, t) = \prod_{i,j} \frac{(t x_i y_j; q)_{\infty}}{(x_i y_j; q)_{\infty}} = \prod_{i,j} F(x_i y_j; q, t). \quad (2.4) \]
Let $\mathfrak{S}_n$ denote the symmetric group, acting on $x = (x_1, \ldots, x_n)$ by permuting the $x_i$, and let $\Lambda_n = \mathbb{Z}[x_1, \ldots, x_n]^{\mathfrak{S}_n}$ and $\Lambda$ denote the ring of symmetric polynomials in $n$ independent variables and the ring of symmetric polynomials in countably many variables, respectively. For $\lambda = (\lambda_1, \ldots, \lambda_n)$ a partition of at most $n$ parts the monomial symmetric function $m_\lambda$ is defined as

$$m_\lambda(x) = \sum_\alpha x^\alpha$$

where the sum is over all distinct permutations $\alpha$ of $\lambda$, and $x = (x_1, \ldots, x_n)$. For $\ell(\lambda) > n$ we set $m_\lambda(x) = 0$. The monomial symmetric functions $m_\lambda(x)$ for $\ell(\lambda) \leq n$ form a $\mathbb{Z}$-basis of $\Lambda_n$. For $r$ a nonnegative integer the power sums $p_r$ are given by $p_0 = 1$ and $p_r = m_r(\alpha)$ for $r > 1$. More generally the power-sum products are defined as $p_\lambda(x) = p_{\lambda_1}(x)p_{\lambda_2}(x) \cdots$ for an arbitrary partition $\lambda = (\lambda_1, \lambda_2, \ldots)$. Define the Macdonald scalar product $\langle \cdot, \cdot \rangle_{q,t}$ on the ring of symmetric functions by

$$\langle p_\lambda, p_\mu \rangle_{q,t} = \delta_{\lambda \mu} \prod_i \prod_{i=1}^n \frac{1-q^\lambda_i}{1-t^\lambda_i}$$

with $z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!$ and $m_\mu = m_\mu(\lambda)$. If we denote the ring of symmetric functions in $\Lambda_n$ variables over the field $F = \mathbb{Q}(q, t)$ of rational functions in $q$ and $t$ by $\Lambda_{n,F}$, then the Macdonald polynomial $P_\lambda(x) = P_\lambda(x; q, t)$ is the unique symmetric polynomial in $\Lambda_{n,F}$ such that [VI (4.7)]Mac:

$$P_\lambda = \sum_{\mu \leq \lambda} u_{\lambda \mu}(q, t) m_\mu(x)$$

with $u_{\lambda \lambda} = 1$ and

$$\langle P_\lambda, P_\mu \rangle_{q,t} = 0 \quad \text{if} \ \lambda \neq \mu.$$

The Macdonald polynomials $P_\lambda(x; q, t)$ with $\ell(\lambda) \leq n$ form an $F$-basis of $\Lambda_{n,F}$. If $\ell(\lambda) > n$ then $P_\lambda(x; q, t) = 0$. $P_\lambda(x; q, t)$ is called Macdonald’s $P$-function. Since $P_\lambda(x_1, \ldots, x_n; 0; q, t) =$ $P_\lambda(x_1, \ldots, x_n; q, t)$ one can extend the Macdonald polynomials to symmetric functions containing an infinite number of independent variables $x = (x_1, x_2, \ldots)$, to obtain a basis of $F = \Lambda \otimes F$. A second Macdonald symmetric function, called Macdonald’s $Q$-function, is defined as

$$Q_\lambda(x; q, t) = b_\lambda(q, t) P_\lambda(x; q, t). \quad (2.5)$$

The normalization of the Macdonald inner product is then $\langle P_\lambda, Q_\mu \rangle_{q,t} = \delta_{\lambda \mu}$ for all $\lambda, \mu$, which is equivalent to

$$\sum_\lambda P_\lambda(x; q, t) Q_\lambda(y; q, t) = \Pi(x; y; q, t). \quad (2.6)$$

(See [7, VI.4, (4.13)].) Let $g_r(x; q, t) := Q_{(r)}(x; q, t)$, or equivalently, [7, VI.2, (2.8)]

$$\prod_{i=1}^\infty \frac{(tx_i q)_\infty}{(x_i q)_\infty} = \sum_{r=0}^\infty g_r(x; q, t) q^r.$$  

Then the Pieri coefficients $\phi_{\lambda/\mu}$ and $\psi_{\lambda/\mu}$ are given by [7, VI.6, (6.24)]

$$P_\mu(x; q, t) g_r(x; q, t) = \sum_{\lambda \vdash r \times \alpha} \phi_{\lambda/\mu}(q, t) P_\lambda(x; q, t),$$

$$Q_\mu(x; q, t) g_r(x; q, t) = \sum_{\lambda \vdash r \times \alpha} \psi_{\lambda/\mu}(q, t) Q_\lambda(x; q, t).$$

Another direct expressions for $\phi_{\lambda/\mu}$ and $\psi_{\lambda/\mu}$ is given in [7, VI.6, Ex.2] as

$$\phi_{\lambda/\mu}(q, t) = \prod_{1 \leq i < j \leq \ell(\lambda)} \frac{f(\lambda_i - \mu_j, j - i)f(\mu_i - \lambda_{j+1}, j - i)}{f(\lambda_i - \lambda_j, j - i)f(\mu_i - \mu_{j+1}, j - i)}, \quad (2.7)$$

$$\psi_{\lambda/\mu}(q, t) = \prod_{1 \leq i < j \leq \ell(\mu)} \frac{f(\lambda_i - \mu_j, j - i)f(\mu_i - \lambda_{j+1}, j - i)}{f(\mu_i - \mu_j, j - i)f(\lambda_i - \lambda_{j+1}, j - i)}, \quad (2.8)$$
Here we use these expressions to rewrite Okada's \((q, t)\)-hook formula conjectures by the Pieri coefficients. For any three partitions \(\lambda, \mu, \nu\) let \(f_{\mu\nu}^\lambda\) be the coefficient \(P_\lambda\) in the product \(P_\mu P_\nu\): [7, VI (7.1')]:

\[
P_\mu(x; q, t)P_\nu(x; q, t) = \sum_\lambda f_{\mu\nu}^\lambda P_\lambda(x; q, t) \tag{2.9}
\]

Now let \(\lambda, \mu\) be partitions and define \(Q_{\lambda/\mu} \in \Delta_F\) by

\[
Q_{\lambda/\mu}(x; q, t) = \sum_\nu f_{\mu\nu}^\lambda Q_\nu(x; q, t). \tag{2.10}
\]

Then \(Q_{\lambda/\mu}(x; q, t) = 0\) unless \(\lambda \supset \mu\), and \(Q_{\lambda/\mu}\) is homogeneous of degree \(|\lambda| - |\mu|\), which is called Macdonald's skew \(Q\)-function. We define Macdonald's skew \(P\)-function \(P_{\lambda/\mu}\) as

\[
Q_{\lambda/\mu}(x; q, t) = \frac{b_\lambda(q, t)}{b_\lambda(q, t)}P_{\lambda/\mu}(x; q, t). \tag{2.11}
\]

holds. Let \(T\) be a tableau of shape \(\lambda - \mu\) and weight \(\nu\), thought as a sequence of partitions \((\lambda^{(0)}, \ldots, \lambda^{(r)})\) such that

\[
\mu = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(r)} = \lambda
\]

and such that each \(\lambda^{(i)} - \lambda^{(i-1)}\) is a horizontal strip. Let

\[
\phi_T(q, t) = \prod_{i=1}^r \phi_{\lambda^{(i)}/\lambda^{(i-1)}}(q, t),
\]

\[
\psi_T(q, t) = \prod_{i=1}^r \psi_{\lambda^{(i)}/\lambda^{(i-1)}}(q, t).
\]

Then we have [7, VI, (7.13), (7.13')]

\[
Q_{\lambda/\mu}(x; q, t) = \sum_T \phi_T(q, t)x_T,
\]

\[
P_{\lambda/\mu}(x; q, t) = \sum_T \psi_T(q, t)x_T,
\]

summed over tableaux \(T\) of shape \(\lambda - \mu\), where \(x_T = \prod_{i=1}^r x_{\lambda^{(i)} - \lambda^{(i-1)}}\). It also holds [7, VI.7, (7.9) (7.9')]

\[
Q_\lambda(x, z; q, t) = \sum_\mu Q_{\lambda/\mu}(x, z; q, t)Q_\mu(x, z; q, t), \tag{2.12}
\]

\[
P_\lambda(x, z; q, t) = \sum_\mu P_{\lambda/\mu}(x, z; q, t)P_\mu(x, z; q, t), \tag{2.13}
\]

where the sums on the right are over partitions \(\mu \subset \lambda\). The following lemma has appeared in the proof of [13, Proposition 2.2] (also see [7, I.5, Ex.26] and [14, Proposition 5.1]).

**Lemma 2.1.** Let \(\mu\) and \(\nu\) be partitions, and \(x = (x_1, x_2, \ldots)\) and \(y = (y_1, y_2, \ldots)\) are independent indeterminates.

\[
\sum_\lambda Q_{\lambda/\mu}(x; q, t)P_{\lambda/\nu}(y; q, t) = \Pi(x; y; q, t) \sum_\tau Q_{\nu/\tau}(x; q, t)P_{\mu/\tau}(y; q, t) \tag{2.14}
\]

In [13] Vuletić has presented so-called a generalized MacMahon's formula. The following theorem gives a generalized form of [13, Proposition 2.2], which we use in the proof of Okada's conjecture.
Theorem 2.2. Fix a positive integer $T$ and two partitions $\mu^0$ and $\mu^T$. Let $x^0, \ldots, x^{T-1}, y^1, \ldots, y^T$ be sets of variables. Then we have

\begin{equation}
\sum_{(\lambda^1, \mu^1, \lambda^2, \ldots, \lambda^T)} \prod_{i=1}^{T} Q_{\lambda^i/\mu^{i-1}}(x^{i-1}; q, t) P_{\lambda^i/\mu^i}(y^i; q, t) = \prod_{0 \leq i < j \leq T} \Pi(x^i; y^j; q, t) \sum_{\nu} Q_{\mu^T/\nu}(x^0, \ldots, x^{T-1}; q, t) P_{\mu^0/\nu}(y^1, \ldots, y^T; q, t)
\end{equation}

where the sum runs over $(2T-1)$-tuples $(\lambda^1, \mu^1, \lambda^2, \ldots, \mu^{T-1}, \lambda^T)$ of partitions satisfying

\begin{equation}
\mu^0 \subset \lambda^1 \supset \mu^1 \subset \lambda^2 \supset \mu^2 \subset \cdots \supset \mu^{T-1} \subset \lambda^T \supset \mu^T.
\end{equation}

We define $P_{[\lambda, \mu]_\delta}(x; q, t)$ and $Q_{[\lambda, \mu]_\delta}(x; q, t)$ for a pair $(\lambda, \mu)$ of partitions, a set $x = (x_1, x_2, \ldots)$ of independent variables and $\delta = \pm 1$ by

\begin{align*}
P_{[\lambda, \mu]_\delta}(x; q, t) & = \left\{ \begin{array}{ll} P_{\lambda/\mu}(x; q, t) & \text{if } \delta = +1, \\ Q_{\mu/\lambda}(x; q, t) & \text{if } \delta = -1, \end{array} \right. \\
Q_{[\lambda, \mu]_\delta}(x; q, t) & = \left\{ \begin{array}{ll} Q_{\lambda/\mu}(x; q, t) & \text{if } \delta = +1, \\ P_{\mu/\lambda}(x; q, t) & \text{if } \delta = -1. \end{array} \right.
\end{align*}

Here we assume $\lambda \supset \mu$ if $\delta = +1$, and $\lambda \subset \mu$ if $\delta = -1$.

Corollary 2.3. Let $n$ be a positive integer, and $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ a sequence of $\pm 1$. Fix a positive integer $T$ and two partitions $\lambda^0$ and $\lambda^n$. Let $x^1, \ldots, x^n$ be sets of variables. Then we have

\begin{align*}
\sum_{(\lambda^1, \lambda^2, \ldots, \lambda^n)} & \prod_{i=1}^{n} P_{[\lambda^{i-1}, \lambda^i]_{\epsilon_i}}(x^i; q, t) \\
= & \prod_{\{\epsilon_i, \epsilon_j\} = (-1, +1)} \Pi(x^i; x^j; q, t) \sum_{\nu} Q_{\lambda^n/\nu}(\{x^i\}_{\epsilon_i = -1}; q, t) P_{\lambda^0/\nu}(\{x^i\}_{\epsilon_i = +1}; q, t)
\end{align*}

where the sum runs over $(n-1)$-tuples $(\lambda^1, \lambda^2, \ldots, \lambda^{n-1})$ of partitions satisfying

\begin{equation}
\begin{cases}
\lambda^{i-1} \supset \lambda^i & \text{if } \epsilon_i = +1, \\
\lambda^{i-1} \subset \lambda^i & \text{if } \epsilon_i = -1.
\end{cases}
\end{equation}

Theorem 2.4. (Warnaar [15, Proposition 1.3, (1.17)])

\begin{equation}
\sum_{\lambda} w^{r(\lambda)} b_{\lambda}^p(q, t) P_{\lambda}(x; q, t) = \prod_{i \geq 1} \frac{(1 + wx_i)(qtx_i^2; q^2)_{\infty}}{(x_i^2; q^2)_{\infty}} \prod_{i < j} \frac{(tx_i x_j; q)_{\infty}}{(x_i x_j; q)_{\infty}},
\end{equation}

where $r(\lambda)$ is the number of rows of odd length.

Applying $w_{q,t}$ [7, VI.2, (2.14)] to the both sides of (2.20), we obtain

Corollary 2.5.

\begin{equation}
\sum_{\lambda} w^{r(\lambda)} b_{\lambda}^q(q, t) P_{\lambda}(x; q, t) = \prod_{i \geq 1} \frac{(tx_i x_j; q)_{\infty}}{(x_i x_j; q)_{\infty}} \prod_{i < j} \frac{(tx_i x_j; q)_{\infty}}{(x_i x_j; q)_{\infty}}.
\end{equation}
From (2.21), we easily obtain
\[
\sum_{\lambda} w^{\frac{\lambda \cdot \gamma(q,t)}{2}} b_{\lambda}^{q}(q,t) P_{\lambda}(x; q, t) = \prod_{i \geq 1} \left( \frac{t w x_{i}; q}{w x_{i}; q} \right) \prod_{i<j} \left( \frac{t w x_{i} x_{j}; q}{w x_{i} x_{j}; q} \right),
\]
and
\[
\sum_{\lambda} w^{\frac{\lambda \cdot \gamma(q,t)}{2}} b_{\lambda}^{q}(q,t) P_{\lambda}(x; q, t) = \prod_{i \geq 1} \left( \frac{t x_{i}; q}{x_{i}; q} \right) \prod_{i<j} \left( \frac{t w x_{i} x_{j}; q}{w x_{i} x_{j}; q} \right) \prod_{i \geq 1} \frac{(twx_{i}; q)_{\infty}}{(ux_{i}; q)_{\infty}} \prod_{i<j} \frac{(twx_{i}x_{j}; q)_{\infty}}{(wx_{i}x_{j}; q)_{\infty}}.
\]

3 \quad (q, t)\text{-hook formula and Macdonald polynomials}

We define $\phi_{[\lambda, \mu]}^{\delta}(q, t)$ and $\psi_{[\lambda, \mu]}^{\delta}(q, t)$ for a pair $(\lambda, \mu)$ of partitions and $\delta = \pm 1$ by
\[
\phi_{[\lambda, \mu]}^{\delta}(q, t) = \begin{cases} 
\phi_{\lambda/\mu}(q, t) & \text{if } \delta = +1, \\
\psi_{\lambda/\mu}(q, t) & \text{if } \delta = -1,
\end{cases} \quad \psi_{[\lambda, \mu]}^{\delta}(q, t) = \begin{cases} 
\psi_{\lambda/\mu}(q, t) & \text{if } \delta = +1, \\
\phi_{\mu/\lambda}(q, t) & \text{if } \delta = -1.
\end{cases}
\]
Here we assume $\lambda \succ \mu$ if $\delta = +1$, and $\lambda \prec \mu$ if $\delta = -1$. We also write
\[
|\lambda - \mu|_{\delta} = \begin{cases} 
|\lambda - \mu| & \text{if } \delta = +1, \\
|\mu - \lambda| & \text{if } \delta = -1.
\end{cases}
\]

Let $n$ be a positive integer. Let $\epsilon = (\epsilon_{1}, \ldots, \epsilon_{n})$ be a sequence of $\pm 1$. Let $(\lambda^{0}, \lambda^{1}, \ldots, \lambda^{n})$ be an $(n+1)$-tuple of partitions such that $\lambda^{i-1} \succ \lambda^{i}$ if $\epsilon = +1$, and $\lambda^{i-1} \prec \lambda^{i}$ if $\epsilon = -1$. Then we write
\[
\phi_{[\lambda^{0}, \lambda^{1}, \ldots, \lambda^{n}]}^{\epsilon}(q, t) = \prod_{i=1}^{n} \phi_{[\lambda^{i-1}, \lambda^{i}]}^{\epsilon_{i}}(q, t), \quad \psi_{[\lambda^{0}, \lambda^{1}, \ldots, \lambda^{n}]}^{\epsilon}(q, t) = \prod_{i=1}^{n} \psi_{[\lambda^{i-1}, \lambda^{i}]}^{\epsilon_{i}}(q, t).
\]

Let $\alpha$ be a strict partition, and let $n$ be an integer such that $n \geq \alpha_{1}$. Define a sequence $\epsilon = \epsilon_{n}(\alpha) = (\epsilon_{1}, \ldots, \epsilon_{n})$ of $\pm 1$ by putting
\[
\epsilon_{k}(\alpha) = \begin{cases} 
+1 & \text{if } k \text{ is a part of } \alpha, \\
-1 & \text{if } k \text{ is not a part of } \alpha.
\end{cases}
\]

For example, if $\alpha = (8,5,2,1)$ and $n = 10$, then we have $\epsilon = (+ + - - + - - - +)$, and $\epsilon_{n}(\alpha) = (\epsilon_{1}, \ldots, \epsilon_{n})$. Let $\pi \in \mathscr{A}(P)$ be a $P$-partition for the the shifted shape $P = P_{\alpha}(\alpha)$. For each integer $k = 0, \ldots, n$ we define the $k$th trace $\pi[k]$ to be the sequence $(\ldots, \pi_{2,k+2}, \pi_{1,k+1})$ obtained by reading the $k$th diagonal from SE to NW. Here we use the convention that $\pi[k] = \emptyset$ if $k \geq \alpha_{1}$. For example, if $\pi$ is the $P$-partition of shifted shape $\alpha = (8,5,2,1)$ in Figure 1, then we have $\pi[0] = (\pi_{44}, \pi_{33}, \pi_{22}, \pi_{11})$, $\pi[1] = (\pi_{24}, \pi_{23}, \pi_{12})$, $\pi[2] = (\pi_{24}, \pi_{13})$, $\pi[3] = (\pi_{25}, \pi_{14})$, $\pi[4] = (\pi_{26}, \pi_{15})$, $\pi[5] = (\pi_{16})$, $\pi[6] = (\pi_{17})$, $\pi[7] = (\pi_{18})$, $\pi[8] = \pi[9] = \pi[10] = \emptyset$, and
\[
\pi[0] \succ \pi[1] \succ \pi[2] \succ \pi[3] \succ \pi[4] \succ \pi[5] \succ \pi[6] \succ \pi[7] \succ \pi[8] \succ \pi[9] \succ \pi[10].
\]

By direct computation one can easily check
\[
W_{P}(\pi; q, t) = b_{[\pi[0], \ldots, \pi[10]]}^{[\pi]}(q, t) \psi_{[\pi[0], \ldots, \pi[10]]}^{[\pi]}(q, t) = b_{[\pi[0], \ldots, \pi[10]]}^{[\pi]}(q, t) \psi_{[\pi[0], \ldots, \pi[10]]}^{[\pi]}(q, t) \times \phi_{[\pi[0], \ldots, \pi[10]]}^{[\pi]}(q, t) \psi_{[\pi[0], \ldots, \pi[10]]}^{[\pi]}(q, t).
\]

In the following we write
\[
\tilde{\Phi}_{m}^{n}(\tilde{x}; \rho, \theta;q, t) = \Phi_{m}^{n}(\rho; q, t) \prod_{i=m+1}^{n} \tilde{x}_{i}^{\rho_{i}+\theta_{i}+1},
\]
and
\[
\tilde{\Phi}_{m}^{n}(\tilde{x}; \rho, \theta;q, t) = \Phi_{m}^{n}(\rho; q, t) \prod_{i=m+1}^{n} \tilde{x}_{i}^{\rho_{i}+\theta_{i}+1}.
\]
in short, where \( \rho = (\rho_m, \ldots, \rho_n) \) and \( \theta = (\theta_m, \ldots, \theta_n) \) satisfy (1.18), and \( \tilde{x} = (\tilde{x}_m, \ldots, \tilde{x}_n) \) are indeterminates. For example, if \( \pi = (\sigma, \tau; f) \) is the \( P \)-partition of the bird \( P = P_3(\alpha, \beta; f) \) for \( \alpha = (4, 3), \beta = (4, 2) \) and \( f = 2 \) (see Figure 1) and satisfies (1.9), then we have

\[
W_P(\pi; q, t) = \hat{\Phi}_{0}^{f}(\rho, \theta; q, t)\psi_{[\sigma[0], \ldots, \sigma[4]]}(q, t)\phi_{[\tau[0], \ldots, \tau[4]]}(q, t).
\]

**Proposition 3.1.** (1) Let \( P = P_2(\alpha) \) be the shifted shape associated with a strict partition \( \alpha \) such that \( \ell(\alpha) = r \), and let \( n \) be an integer such that \( n \geq \alpha_1 \). If \( \pi \in \mathscr{A}(P) \) is a \( P \)-partition satisfying the condition (1.8), then we have

\[
W_P(\pi; q, t) = b_{\pi[0]}^{e1}(q, t)\psi_{[\pi[0], \ldots, \pi[n]]}(q, t) = b_{\pi[0]}(q, t)^{\phi_{[\pi[0], \ldots, \pi[n]]}(q, t)}\left(3.1\right)
\]

and

\[
z^{\pi} = w^{w[0] - \pi[0] + \phi_{[0], \pi[4]}} \prod_{i=1}^{n} \tilde{z}_i^{\epsilon(\alpha)_{i-1} - \pi[0]_{i_1}}, \quad \left(3.2\right)
\]

where \( w \) and \( \tilde{z}_i \) (\( 1 \leq i \leq n \)) are as in Proposition 1.11 (1).

(2) Let \( \alpha = (\alpha_1, \alpha_2) \) and \( \beta = (\beta_1, \beta_2) \) be strict partitions such that \( \ell(\alpha) = \ell(\beta) = 2 \). Let \( f > 0 \) be a positive integer, and set \( P = P_3(\alpha, \beta; f) \) the bird associated with \( \alpha, \beta \) and \( f \). Let \( m \) (resp. \( n \)) be a positive integer such that \( m \geq \alpha_1 \) (resp. \( n \geq \beta_1 \)). If \( \pi = (\sigma, \tau; \rho, \theta) \) is a \( P \)-partition satisfying the condition (1.9), then we have

\[
W_P(\pi; q, t) = \hat{\Phi}_{0}^{f}(\rho, \theta; q, t)\psi_{[\sigma[0], \ldots, \sigma[m]]}(q, t)\phi_{[\tau[0], \ldots, \tau[n]]}(q, t) \quad \left(3.3\right)
\]

and

\[
z^{\pi} = \overline{x}_0^{\alpha_0 + \theta_0} \prod_{i=1}^{m} \overline{y}_i^{\epsilon(\alpha)_{i-1} - \pi[0]_{i_1}} \prod_{i=1}^{n} \tilde{z}_i^{\epsilon(\beta)_{i-1} - \pi[0]_{i_1}}, \quad \left(3.4\right)
\]

where \( \overline{x}_i \) (\( 0 \leq i \leq f \)), \( \overline{y}_i \) (\( 1 \leq i \leq n \)) and \( \tilde{z}_i \) (\( 1 \leq i \leq m \)) are as in Proposition 1.11 (2).

**Proof.** (1) From (1.16) and (2.8) we have

\[
f_{\alpha}^{ND}(\pi; q, t) = \begin{cases} \prod_{1 \leq i \leq j} f_{[\pi[0], \ldots, \pi[j-1]]}(q, t) \prod_{1 \leq i \leq j} \psi_{[\pi[0], \ldots, \pi[n]]}(q, t) & \text{if } \epsilon_1(\alpha) = +, \\ \prod_{1 \leq i \leq j} f_{[\pi[0], \ldots, \pi[j-1]]}(q, t) \prod_{1 \leq i \leq j} \psi_{[\pi[0], \ldots, \pi[n]]}(q, t) & \text{if } \epsilon_1(\alpha) = -. \end{cases}
\]

Similarly, from (1.17) and (2.2) we have

\[
f_{\alpha}^{D}(\pi; q, t) = \begin{cases} \prod_{1 \leq i \leq j} f_{[\pi[0], \ldots, \pi[j-1]]}(q, t) \prod_{1 \leq i \leq j} \phi_{[\pi[0], \ldots, \pi[n]]}(q, t) & \text{if } \epsilon_1(\alpha) = +, \\ \prod_{1 \leq i \leq j} f_{[\pi[0], \ldots, \pi[j-1]]}(q, t) \prod_{1 \leq i \leq j} \phi_{[\pi[0], \ldots, \pi[n]]}(q, t) & \text{if } \epsilon_1(\alpha) = -. \end{cases}
\]

Hence we obtain (3.1) from (1.20) since

\[
\psi_{[\pi[0], \pi[1]]}(q, t) = \begin{cases} \prod_{1 \leq i \leq j} f_{[\pi[0], \ldots, \pi[j-1]]}(q, t) f_{[\pi[0], \ldots, \pi[n]]}(q, t) & \text{if } \epsilon_1(\alpha) = +, \\ \prod_{1 \leq i \leq j} f_{[\pi[0], \ldots, \pi[j-1]]}(q, t) f_{[\pi[0], \ldots, \pi[n]]}(q, t) & \text{if } \epsilon_1(\alpha) = -. \end{cases}
\]

Meanwhile, (3.2) can be easily obtained from

\[
z^{\pi} = w^{w[0] - \pi[0] + \phi_{[0], \pi[4]}} \prod_{i=1}^{n} \tilde{z}_i^{\epsilon(\alpha)_{i-1} - \pi[0]_{i_1}}.
\]
As in (1) we have
\[
\begin{align*}
f_{\alpha}^{\text{ND}}(\sigma; q, t) &= \begin{cases} 
\frac{f(\sigma_{12} - \sigma_{11}; 0) \prod_{i=2}^{n} \psi_{\sigma_{i-1}, \sigma_{i}}^{\epsilon_{i}(\alpha)}(q, t)}{f(\sigma_{23} - \sigma_{11}; 1) / (\sigma_{23} - \sigma_{11}; 0)} & \text{if } \epsilon_{1}(\alpha) = +, \\
\frac{f(\sigma_{12} - \sigma_{11}; 0) \prod_{i=2}^{n} \psi_{\sigma_{i-1}, \sigma_{i}}^{\epsilon_{i}(\alpha)}(q, t)}{f(\sigma_{23} - \sigma_{11}; 1) / (\sigma_{23} - \sigma_{11}; 0)} & \text{if } \epsilon_{1}(\alpha) = -.
\end{cases}
\end{align*}
\]

From (1.16) and (2.7) we have
\[
\begin{align*}
f_{\beta}^{\text{ND}}(\tau; q, t) &= \begin{cases} 
\frac{f(\tau_{12} - \tau_{11}; 0) \prod_{i=2}^{m} \phi_{\tau_{i-1}, \tau_{i}}^{\epsilon_{i}(\beta)}(q, t)}{f(\tau_{23} - \tau_{11}; 0) / (\tau_{23} - \tau_{11}; 0)} & \text{if } \epsilon_{1}(\beta) = +, \\
\frac{f(\tau_{12} - \tau_{11}; 0) \prod_{i=2}^{m} \phi_{\tau_{i-1}, \tau_{i}}^{\epsilon_{i}(\beta)}(q, t)}{f(\tau_{23} - \tau_{11}; 0) / (\tau_{23} - \tau_{11}; 0)} & \text{if } \epsilon_{1}(\beta) = -.
\end{cases}
\end{align*}
\]

Hence, if we use (2.7) or (2.8), then we obtain (3.3) from (1.21). On the other hand, (3.4) is easily obtained from
\[
z^{\pi} = z_{0}^{\sigma_{11} + \sigma_{22}} \prod_{i=1}^{f} x_{i}^{\rho_{i} + \theta_{i}} = (z_{0} \tilde{x}_{1})^{\rho_{0} + \theta_{0}} \prod_{i=1}^{f} \tilde{x}_{i}^{\rho_{i} + \theta_{i} - \rho_{i-1} - \theta_{i-1}},
\]
where we use the convention $\sigma_{11} = \rho_{0}$ and $\sigma_{22} = \theta_{0}$.

**Theorem 3.2.** (1) Let $P = P_{2}(\alpha)$ be the shifted shape associated with a strict partition $\alpha$ of length $r$. Let $n$ be an integer such that $n \geq \alpha_{1}$, and let $\alpha^{c}$ be the strict partition formed by the complement of $\alpha$ in $[n]$. Then we have
\[
\sum_{\pi \in \mathcal{A}(P)} W_{P}(\pi; q, t) z^{\pi} = \prod_{\sigma_{i}^{c} < \sigma_{j}} F(\tilde{z}_{\sigma_{i}^{c}}^{-1} \tilde{z}_{\sigma_{j}}; q, t) \prod_{\beta_{i}^{c} < \beta_{j}} F(\tilde{y}_{\beta_{i}^{c}}^{-1} \tilde{y}_{\beta_{j}}; q, t) \sum_{(\rho, \theta)} \Phi_{0}^{f}(\tilde{x}; \rho, \theta; q, t) P_{(\theta_{O_{r}} \rho \ldots t) Q_{(\theta_{0}) \rho 0)}(\tilde{y}_{\beta_{1}}, \tilde{z}_{\beta_{2}}; q, t),
\]
where the sum on the right-hand side is taken over all pairs $(\rho, \theta)$ with $\rho = (\rho_{0}, \ldots, \rho_{f})$ and $\theta = (\theta_{0}, \ldots, \theta_{f})$ satisfying
\[
0 \leq \rho_{f} \leq \cdots \leq \rho_{0} \leq \theta_{0} \leq \cdots \leq \theta_{f}.
\]

Here $\tilde{x}_{i} (0 \leq i \leq f)$, $y_{i} (1 \leq i \leq m)$ and $z_{i} (1 \leq i \leq m)$ are as in Proposition 1.11 (2).

**Proof.** (1) Since
\[
\begin{align*}
\psi_{\pi[i-1]/\pi[i]}(q, t) &\tilde{z}_{i}^{-|\pi[i-1] - \pi[i]|} = P_{\pi[i-1]/\pi[i]}(\tilde{z}_{i}; q, t), \\
\phi_{\pi[i]/\pi[i-1]}(q, t) &\tilde{x}_{i}^{-|\pi[i-1] - \pi[i]|} = Q_{\pi[i]/\pi[i-1]}(\tilde{x}_{i}; q, t)
\end{align*}
\]
(see [7, VI.7, (7.14)(7.14')]), we can use (2.17) to take the sum of the product of (3.1) and (3.2), then we obtain

\[
\sum_{\pi} W_{P}(\pi; q, t) z^{\pi} = \prod_{\alpha_{k}^{c} < \alpha_{1}} F\left(\tilde{z}_{\alpha_{k}^{c}}^{-1} \tilde{z}_{\alpha_{1}}\right) \sum_{\pi[0]} b_{\pi[0]}^{e1}(q_{\backslash} t) w^{(|\pi[0]|-r(\pi[0]'))/2} P_{\pi[0]}(\tilde{z}_{\alpha_{1}}, \ldots, \tilde{z}_{\alpha_{r}}, q, t),
\]

where the sum on the right-hand side runs over all partitions \(\pi[0]\).

(2) Again, using (2.17) to take the sum of the product of (3.3) and (3.4), we obtain

\[
\sum_{\pi} W_{P}(\pi; q, t) z^{\pi} = \prod_{\alpha_{k}^{c} < \alpha_{1}} F\left(\tilde{z}_{\alpha_{k}^{c}}^{-1} \tilde{z}_{\alpha_{1}}\right) \prod_{\beta_{k}^{c} < \beta_{1}} F\left(\overline{y}_{\beta_{k}^{c}}^{-1} \overline{y}_{\beta_{1}}\right) \tilde{x}_{0}^{\rho_{0}+\theta_{0}} P_{(\theta_{0}, \rho_{0})}(\tilde{z}_{\alpha_{1}}, \tilde{z}_{\alpha_{2}}; q, t) Q_{(\rho_{0}, \theta_{0})}(\tilde{y}_{\beta_{1}}, \tilde{y}_{\beta_{2}}; q, t),
\]

where the sum on the right-hand side runs over all pairs \((\rho, \theta)\) satisfying (3.7) with \(\sigma[0] = (\theta_{0}, \rho_{0})\). Finally we use \(\tilde{x}_{0}^{\rho_{0}+\theta_{0}} P_{(\theta_{0}, \rho_{0})}(\tilde{z}_{\alpha_{1}}, \tilde{z}_{\alpha_{2}}; q, t) = P_{(\theta_{0}, \rho_{0})}(\tilde{x}_{0}\tilde{z}_{\alpha_{1}}, \tilde{x}_{0}\tilde{z}_{\alpha_{2}}; q, t)\).

If we apply Warner's formula (2.23) to (3.5) we can obtain the \((q, t)\)-hook formula (1.22) for shifted shapes. This gives another proof of [8, Proposition 4.5 (b)]. Now we look at the right-hand side of the conjectured identities in the cases of birds. From Proposition 1.11 we can derive the following theorem.

**Theorem 3.3.** Let \(\alpha = (\alpha_{1}, \alpha_{2})\) and \(\beta = (\beta_{1}, \beta_{2})\) be strict partitions of length 2. Let \(f > 0\) be a positive integer, and set \(P_{f} = P_{3}(\alpha, \beta; f)\) the bird associated with \(f, \alpha\) and \(\beta\). Let \(m, n\) be integers such that \(m \geq \ell(\alpha)\) and \(n \geq \ell(\beta)\), and let \(\alpha^{c}\) (resp. \(\beta^{c}\)) be the strict partition formed by the complement of \(\alpha\) (resp. \(\beta\)) in \([m]\) (resp. \([n]\)). Then we have

\[
F\left(z[H_{p}]; q, t\right) = \prod_{\alpha_{i}^{c} < \alpha_{j}} F\left(\tilde{z}_{\alpha_{i}^{c}}^{-1} \tilde{z}_{\alpha_{j}}\right) \prod_{\beta_{i}^{c} < \beta_{j}} F\left(\overline{y}_{\beta_{i}^{c}}^{-1} \overline{y}_{\beta_{j}}\right) \prod_{i=1}^{f} F\left(\frac{\tilde{x}_{1}^{2}}{\tilde{x}_{i}} \prod_{k,l=1}^{2} \overline{y}_{l} \overline{z}_{k} ; q, t\right)
\]

\[
\times \sum_{l_{1}, \ldots, l_{f} \geq 0} \sum_{\lambda_{1}, \ldots, \lambda_{f} \geq 0} \prod_{i=1}^{f} f(k_{i}; 0) f(l_{i}; 0) \tilde{x}_{i}^{k_{i}-l_{i}} \overline{x}_{i}^{l_{i}} P_{\lambda}(\tilde{x}_{1} \tilde{z}_{\alpha_{1}}, \tilde{x}_{1} \tilde{z}_{\alpha_{2}}; q, t) Q_{\lambda}(\overline{y}_{\beta_{1}}, \overline{y}_{\beta_{2}}; q, t)
\]

\[
(3.8)
\]

where \(\tilde{x}_{i} (1 \leq i \leq f), \overline{y}_{i} (1 \leq i \leq n)\) and \(\tilde{z}_{i} (1 \leq i \leq m)\) are as in Proposition 1.11 (2).

**Proof.** From (2.6) we have

\[
\prod_{i,j=1}^{2} F\left(\tilde{x}_{1} \overline{y}_{\beta_{j}} \tilde{z}_{\alpha_{i}}; q, t\right) = \sum_{\mu} P_{\mu}(\tilde{x}_{1} \tilde{z}_{\alpha_{1}}, \tilde{x}_{1} \tilde{z}_{\alpha_{2}}) Q_{\mu}(\overline{y}_{\beta_{1}}, \overline{y}_{\beta_{2}}).
\]

By the binomials theorem we have

\[
\prod_{i=1}^{f} F\left(\tilde{x}_{i}; q, t\right) = \sum_{k_{1}, \ldots, k_{f} \geq 0} \prod_{i=1}^{f} f(k_{i}; 0) \tilde{x}_{i}^{k_{i}};
\]

\[
\prod_{i=1}^{f} F\left(\frac{\tilde{x}_{1}^{2}}{\tilde{x}_{i}} \prod_{k,l=1}^{2} \overline{y}_{l} \overline{z}_{k}; q, t\right) = \sum_{l_{1}, \ldots, l_{f} \geq 0} \prod_{i=1}^{f} f(l_{i}; 0) \tilde{x}_{i}^{l_{i}} \overline{x}_{i}^{2} \prod_{k,l=1}^{2} \overline{y}_{l} \overline{z}_{k}.
\]
By \( [7, \text{VI.4}, (4.17)] \) and (2.5) we obtain
\[
(x_1^2 x_2)^l P_{\mu} (x_1 z_{\alpha_1}, x_1 z_{\alpha_2}) = P_{\mu + l \cdot 1^2} (x_1 z_{\alpha_1}, x_1 z_{\alpha_2})
\]
\[
(y_1 y_2)^l Q_{\mu} (y_{\beta_1}, y_{\beta_2}) = \frac{b_{\mu}(q, t)}{b_{\mu + l \cdot 1^2}(q, t)} Q_{\mu + l \cdot 1^2} (y_{\beta_1}, y_{\beta_2}).
\]

From (1.23) we obtain
\[
F(z[H_p]; q, t) = \prod_{\alpha_i^c < \alpha_j} F(z_{\alpha_i^c}^{-1} z_{\alpha_j}; q, t) \prod_{\beta_i^c < \beta_j} F(y_{\beta_i^c}^{-1} y_{\beta_j}; q, t)
\]
\[
\times \sum_{0} \sum_{\rho, \theta} \sum_{k_i \geq 0} f(l_i; 0) f(l_i + r_i; 0)
\]
\[
\times \frac{b_{\rho}(q, t)}{b_{\rho + l \cdot 1^2}(q, t)} P_{\rho + l \cdot 1^2} (x_1 z_{\alpha_1}, x_1 z_{\alpha_2}; q, t) Q_{\rho + l \cdot 1^2} (y_{\beta_1}, y_{\beta_2}; q, t).
\]

This immediately implies (3.8).

4 Proof by Gasper's formula

Now we are in position to prove Okada's conjecture for Birds and Banners, i.e., Theorem 1.9. We use the fact that Macdonald's polynomials are the basis of \( \Lambda_{\mathbb{F}} \) (cf. [6]). To prove the birds case, we fix integers \( \rho_0 \) and \( \theta_0 \) such that \( \theta_0 \geq \rho_0 \geq 0 \), and nonnegative integers \( r_1, \ldots, r_f \). If we compare the coefficients of \( \prod_{i=1}^f x_i^{r_i} \). \( P_{\lambda}(x_1 z_{\alpha_1}, x_1 z_{\alpha_2}; q, t) Q_{\lambda}(y_{\beta_1}, y_{\beta_2}; q, t) \) in (3.6) and (3.8), the following identity must hold:
\[
0 \leq \rho_f \leq \rho_1 \leq \rho_0 \sum_{\rho_1, \ldots, \rho_f} \Phi_{0}^{f} (\rho, \theta; q, t)
\]
\[
\times \frac{b_{(\theta_0 - \rho_0; q, t)}}{b_{(\theta_0, \rho_0; q, t)}} f(l_i; 0) f(l_i + r_i; 0)
\]
where \( (\theta_1, \ldots, \theta_f) \) is determined from \( \theta_0 \) and \( (\rho_1, \ldots, \rho_f) \) by using the equations \( \theta_i = \rho_{i-1} + \theta_{i-1} + r_{i-1} - \rho_i \) for \( i = 1, \ldots, f \). Since (2.1) implies
\[
b_{(\theta_0, \rho_0)} = f(\theta_0 - \rho_0; 1) f(\theta_0 - \rho_0; 1) f(\rho_0; 0),
\]
we obtain
\[
\frac{b_{(\theta_0 - \rho_0; q, t)}}{b_{(\theta_0, \rho_0; q, t)}} = \frac{f(\theta_0 - \rho_0; 1) f(\theta_0 - \rho_0; 1) f(\rho_0; 0) f(\theta_0; 1)}{f(\rho_0; 0) f(\theta_0; 1)}.
\]
Hence it is enough to prove
\[
\sum_{\rho_1, \ldots, \rho_f} \Phi_{0}^{f} (\rho, \theta; q, t) = \sum_{\rho_1, \ldots, \rho_f} \sum_{l_i \geq 0} \prod_{i=1}^f f(l_i; 0) f(l_i + r_i; 0).
\]
In fact a more general formula holds. If we prove the following theorem, then the proof of (4.1) are done.

**Theorem 4.1.** Let \( m \) and \( n \) be nonegative integers. Let \( k_0, \rho_0, \theta_0 \) be integers such that \( 0 \leq k_0 \leq
\[ \rho_0 \leq \theta_0, \text{ and let } \gamma_1, \ldots, \gamma_n \text{ be nonnegative integers. Then we have} \]
\[ \sum_{k_0 \leq \rho_1 \leq \cdots \leq \rho_n \leq \rho_0} f(\rho_n - k_0; 0) f(\theta_n - k_0; m + n) \times \prod_{i=1}^{n} \frac{f(\rho_{i-1} - \rho_i; i + m - 1)}{f(\theta_{i-1} - \rho_i; i + m - 1)} \times \frac{f(\theta_{i} - \rho_{i}; 0)}{f(\theta_{i} - \rho_{i}; i + m - 1)} \]
\[ = \sum_{\gamma_1, \ldots, \gamma_n \geq 0} f(\rho_0 - \sum_{i=0}^{n} k_i; 0) f(\theta_0 - \sum_{i=0}^{n} k_i; m) \prod_{i=1}^{n} f(k_i; 0) f(k_i + \gamma_i; 0), \quad (4.2) \]

where the sum on the left-hand side runs over all \( n \)-tuples \((\rho_1, \ldots, \rho_n)\) of nonnegative integers such that \( k_0 \leq \rho_n \leq \cdots \leq \rho_1 \leq \rho_0 \), and the sum on the right-hand side runs over all \( n \)-tuples \((k_1, \ldots, k_n)\) of nonnegative integers which satisfy \( k_1 + \cdots + k_n \leq \rho_0 - \rho_{m+1} \), and \( \theta_i \) is determined from \( \rho_i, \rho_{i-1} \) and \( \theta_{i-1} \) by \( \theta_i = \gamma_i + \theta_{i-1} + \rho_{i-1} - \rho_i \) for \( i = 1, \ldots, n \).

Before we prove this theorem, we need the following lemma which is a special case (i.e., \( n = 1 \)) of this theorem.

**Lemma 4.2.** Let \( m \) be a nonnegative integer. Let \( k_0, \rho_0 \) and \( \theta_0 \) be integers such that \( 0 \leq k_0 \leq \rho_0 \leq \theta_0 \), and let \( \gamma \) be a nonnegative integer. Then we have
\[ \sum_{\rho = k_0}^{\rho_0} f(\rho - k_0; 0) f(\theta - k_0; m + 1) \frac{f(\rho_0 - \rho; 0) f(\theta_0 - \rho; m) f(\theta_0 - \rho_0; m) f(\theta - \theta_0; 0)}{f(\theta - \rho; m) f(\theta_0 - \rho_0; m + 1)} = \sum_{k=0}^{\rho_0 - k_0} f(\rho_0 - k_0 - k; 0) f(\theta_0 - k_0 - k; m) f(k; 0) f(k + \gamma; 0), \quad (4.3) \]

where \( \theta = \gamma + \rho_0 + \theta_0 - \rho_0 \).

**Proof.** Set \( S_1 \) to be the left-hand side of (4.3). If one puts \( k = \rho_0 - \rho \), then \( \rho = \rho_0 - k \) and \( \theta = k + \gamma + \theta_0 - \rho_0 \). Hence one obtains
\[ S_1 = \sum_{k=0}^{\rho_0 - k_0} f(\rho_0 - k_0 - k; 0) f(k + \gamma + \theta_0 - k_0; m + 1) \times \frac{f(k; 0) f(k + \gamma + \theta_0 - \rho_0; m) f(k + \theta_0 - \rho_0; m)}{f(2k + \gamma + \theta_0 - \rho_0; m) f(2k + \gamma + \theta_0 - \rho_0; m + 1)}. \]

If we use
\[ (\alpha; q)_{2k} = (\alpha^q; q)_k (-\alpha^q; q)_k (\alpha^q; q)_k (-\alpha^q; q)_k, \]
then the factors in the denominator are written as
\[ f(2k + \gamma + \theta_0 - \rho_0; m) = f(\gamma + \theta_0 - \rho_0; m) \frac{(\alpha^q; q)_{2k} (\alpha^q; q)_k (\alpha^q; q)_k (\alpha^q; q)_k)}{(-\alpha^q; q)_k (-\alpha^q; q)_k (\alpha^q; q)_k (\alpha^q; q)_k), \]
and
\[ f(2k + \gamma + \theta_0 - \rho_0; m + 1) = f(\gamma + \theta_0 - \rho_0; m + 1) \frac{(\alpha^q; q)_{2k} (\alpha^q; q)_k (\alpha^q; q)_k (\alpha^q; q)_k)}{(-\alpha^q; q)_k (-\alpha^q; q)_k (\alpha^q; q)_k (\alpha^q; q)_k). \]

Meanwhile, the factors in the numerator are
\[ f(\rho_0 - k_0 - k; 0) = f(\rho_0 - k_0; 0) \frac{(\alpha^q; q)_{\rho_0 - k_0} (\alpha^q; q)_k}{(\alpha^q; q)_{\rho_0 - k_0 + 1} (\alpha^q; q)_k}, \]
\[ f(k + \gamma + \theta_0 - \rho_0; m) = f(\gamma + \theta_0 - \rho_0; m) \frac{(\alpha^q; q)_{k + \gamma + \theta_0 - \rho_0} (\alpha^q; q)_k}{(\alpha^q; q)_{k + \gamma + \theta_0 - \rho_0 + 1} (\alpha^q; q)_k}, \]
\[ f(k + \theta_0 - \rho_0; m) = f(\theta_0 - \rho_0; m) \frac{(\alpha^q; q)_{k + \theta_0 - \rho_0} (\alpha^q; q)_k}{(\alpha^q; q)_{k + \theta_0 - \rho_0 + 1} (\alpha^q; q)_k}. \]

Hence, substituting these factors, we obtain
\[ S_1 = C \cdot 12 W_{11} \left( \frac{bc}{d}; (bcq/ad)^q + q(bcq/ad)^q + q(bcq/d)^q + (bcq/d)^q, ab/d, ac/d, a, b, c; q/q_a \right). \]
where \( a = t, b = tq^\gamma, c = q^{-\rho_0 + k_0}, d = t^{-m}q^{-\theta_0 + k_0} \) and
\[
C = \frac{f(\rho_0 - k_0; 0)f(\gamma + \theta_0 - k_0; m + 1)f(\theta_0 - \rho_0; m)f(\gamma; 0)}{f(\gamma + \theta_0 - \rho_0; m + 1)}.
\]
On the other hand, Set \( S_2 \) to be the right-hand side of (4.3). If we use \( f(\rho_0 - k_0 - k; 0) = f(\rho_0 - k_0; 0) \frac{(q^{-\rho_0 + k_0}; q)_{k}}{(tq; q)_{k} (t^(-1)q^{-\rho_0 + k_0}; q)_{k} (t^(-1)q^{-\rho_0 + k_0}; q)_{k} (t^(-1)q^{-\rho_0 + k_0}; q)_{k}} \) and \( f(k + \gamma; 0) = f(k + \gamma; 0) \frac{(tq^\gamma; q)_{k}}{(q^\gamma; q)_{k}} \), then we obtain
\[
S_2 = f(\rho_0 - k_0; 0)f(\theta_0 - k_0; m)f(\gamma; 0)\phi_3 \left[ \frac{q^{-\rho_0 + k_0}, t^{-m}q^{-\theta_0 + k_0}, t, tq^\gamma}{t^{-1}q^{-\rho_0 + k_0 + 1}, t^{-m-1}q^{-\theta_0 + k_0 + 1}, q^{-\rho_0 + k_0 + 1}, \frac{q^2}{t^2}} \right].
\]
Hence Gasper’s formula (1.2) proves that \( S_1 = S_2 \). The details are left to the reader. This completes our proof.

**Proof of Theorem 4.1.** We proceed by induction on \( n \). If \( n = 1 \), then (4.2) is nothing but (4.3). Let \( n \geq 2 \) and assume (4.2) is true for \( n - 1 \). If we set \( S \) to be the left-hand side of (4.2), then we have
\[
S = \sum_{\rho_1 = k_0}^{\rho_0} \frac{f(\rho_0 - \rho_1; 0)f(\theta_0 - \rho_1; m)f(\theta_1 - \rho_0; m)f(\theta_0 - \rho_1; m + 1)}{f(\theta_1 - \rho_1 - \sum_{i=2}^{n} k_i - m + 1)}
\times \sum_{k_0 \leq k_1 \leq \cdots \leq k_n \leq \rho_0 - \rho_1 - \sum_{i=2}^{n} k_i} f(\rho_0 - \rho_1; 0)f(\theta_0 - \rho_1; m)f(\theta_1 - \rho_0; m)f(\theta_1 - \theta_0; 0). \]
We can use our induction hypothesis to obtain
\[
S = \sum_{k_2 \ldots k_n \geq 0} \prod_{i=2}^{n} f(k_i; 0)f(k_i + \gamma_i; 0)
\times \sum_{\rho_1 = k_0 + \sum_{i=2}^{n} k_i}^{\rho_0} f(\rho_1 - k_0 - \sum_{i=2}^{n} k_i; 0)f(\theta_1 - k_0 - \sum_{i=2}^{n} k_i - m + 1)
\times \frac{f(\rho_0 - \rho_1; 0)f(\theta_0 - \rho_1; m)f(\theta_1 - \rho_0; m)f(\theta_1 - \theta_0; 0)}{f(\theta_1 - \rho_1; m)f(\theta_1 - \rho_1; m + 1)}. \]
If we use (4.3) again, then we obtain
\[
S = \sum_{k_2 \ldots k_n \geq 0} \prod_{i=2}^{n} f(k_i; 0)f(k_i + \gamma_i; 0)
\times \sum_{0 \leq k_1 \leq k_0 - \sum_{i=2}^{n} k_i} f(\rho_0 - \sum_{i=0}^{n} k_i; 0)f(\theta_0 - \sum_{i=0}^{n} k_i, m)f(k_1, 0)f(k_1 + \gamma_1, 0)
\]
which equals the right-hand side of (4.2). This completes our proof.

**References**


