数理解析研究所講究録
第 1913 巻 2014 年 81-92



# Alpha determinants and symmetric polynomials*

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**Abstract**

In the paper, we introduce a family of symmetric polynomials, which are analogous to the ordinary Schur polynomials, via wreath determinants and show that they have a Giambelli-type determinantal formula.

## 1 Introduction

*Schur polynomial* $s_\lambda(x_1,\dots,x_n)$ associated to a partition $\lambda=(\lambda_1,\dots,\lambda_n)$, which is of at most length $n$, in $n$ variables $x_1,\dots,x_n$ is a symmetric polynomial of degree $|\lambda|=\lambda_1+\cdots+\lambda_n$ defined as a quotient of Vandermonde-type determinants by

$$s_\lambda(x_1,\dots,x_n):=\frac{\det\Big(x_j^{\lambda_i+n-i}\Big)_{1\le i,j\le n}}{\det\Big(x_j^{n-i}\Big)_{1\le i,j\le n}}. \tag{1.1}$$

In the paper we define an analogous symmetric polynomial associated to each partition $\lambda$ of certainly limited length, which will be called *wreath Schur polynomials*, via *wreath determinants* (which will be introduced later) in place of ordinary determinants. We prove that such symmetric polynomials have a Giambelli-type formula, that is, any wreath Schur polynomial is expressed as a 'determinant' of the ones associated to *hooks*.

We first introduce the *$\alpha$-determinants*, which are parametric deformation of the ordinary determinants (but different from the so called $q$-determinants), and give several basic properties. Next we define the *wreath determinant* as a $\alpha$-determinant of special type. We show that the wreath determinants have nice properties such as right relative invariance. In the last section, we give the definition of the wreath Schur polynomials and prove a Giambelli-type formula for them.

## 2 Alpha determinant

### 2.1 Definition

For a given permutation $\sigma\in\mathfrak{S}_n$, we define

$$\nu(\sigma):=\sum_{j\ge1}(j-1)m_j(\sigma)=n-\sum_{j=1}^{n}m_j(\sigma),$$

*RIMS Workshop "Algebraic Combinatorics Related to Young Diagrams and Statistical Physics," August 6–10, 2012.

†Partially supported by Grand-in-Aid for Scientific Research (C) No. 22540028.

where $m_j(\sigma)$ denotes the number of $j$-cycles in the cycle decomposition of $\sigma$. The function $\nu$ is, hence, a *class function.*

The *$\alpha$-determinant* $\det_\alpha A$ of an $n$ by $n$ matrix $A = (a_{ij})_{1\le i,j\le n}$ is defined by

$$\det_\alpha A = \sum_{\sigma\in\mathfrak{S}_n} \alpha^{\nu(\sigma)} a_{\sigma(1)1} a_{\sigma(2)2} \dots a_{\sigma(n)n}.$$

Namely, we simply replace the signature $\operatorname{sgn}\sigma$ by $\alpha^{\nu(\sigma)}$ in the definition of the ordinary determinant $\det A$.

It is immediate to see that

$$\det_{-1} A = \det A, \qquad \det_1 A = \operatorname{per} A, \qquad \det_0 A = a_{11}a_{22}\dots a_{nn}.$$

Thus the $\alpha$-determinant is a parametric family of polynomial functions on square matrices including the determinant and permanent. We note that $\alpha^{\nu(\cdot)}$ gives the sign character when $\alpha = -1$, the trivial character when $\alpha = 1$, and a constant multiple of the character for the regular representation when $\alpha = 0$.

**Example 2.1.** For instance, 2 by 2 and 3 by 3 $\alpha$-determinants are given as follows.

$$\det_\alpha \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} + \alpha a_{21}a_{12},$$

$$\det_\alpha \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = a_{11}a_{22}a_{33} + \alpha^2 a_{21}a_{32}a_{13} + \alpha^2 a_{31}a_{12}a_{23}$$
$$+ \alpha\, a_{21}a_{12}a_{33} + \alpha\, a_{11}a_{32}a_{23} + \alpha\, a_{31}a_{22}a_{13}.$$

*Remark.* It is well known that any permutation $\sigma$ can be expressed as a product of *transpositions.* The minimum number of transpositions in such an expression of $\sigma$ is equal to $\nu(\sigma)$. For instance, if $\sigma = (1\,2\,3)(4\,5\,6)(7\,8)(9) \in \mathfrak{S}_9$, then we have $\nu(\sigma) = 9 - 4 = 5$. On the other hand, $\sigma = (1\,2)(2\,3)(4\,5)(5\,6)(7\,8)$ is one of the shortest expression of $\sigma$ as a product of transpositions, whose length is 5.

*Remark.* The $\alpha$-determinant is *different* from the so-called $q$-determinant

$$\det_q A = \sum_{\sigma\in\mathfrak{S}_n} (-q)^{l(\sigma)} \prod_{i=1}^{n} a_{\sigma(i)i},$$

where $l(\sigma)$ denotes the *inversion number* of $\sigma$, that is, the length of shortest expression of $\sigma$ as a product of *simple transpositions.*

## 2.2 Basic properties

By definition, it is clear that the $\alpha$-determinant $\det_\alpha A$ is multilinear with respect to rows and columns of $A$, and is invariant under transposition $A \to {}^tA$. It is also notable that the $\alpha$-determinant of block triangular matix equals the product of those of diagonal blocks:

$$\det_\alpha \begin{pmatrix} A_1 & * & * \\ & \ddots & * \\ & & A_m \end{pmatrix} = \det_\alpha A_1 \cdots \det_\alpha A_m.$$

The $\alpha$-determinant is not central (i.e. $\det_\alpha AB = \det_\alpha BA$ is not true) in general, but we have

$$\det_\alpha AP(\sigma) = \det_\alpha P(\sigma)A \qquad (\sigma \in \mathfrak{S}_n)$$

as a special case, where $P(\sigma) = (\delta_{i\sigma(j)})$ is the permutation matrix for $\sigma$.

**Proposition 2.1** (Laplace expansion [1]). *Let $A = (a_{ij})_{1\le i,j\le n}$ be a given $n$ by $n$ matrix. For any integers $p, q$ such that $1 \le p, q \le n$, define an $n-1$ by $n-1$ matrix $A_{pq}$ by exchanging the $p$-th column and $q$-th column in $A$ and then eliminating the $p$-th row and $p$-th column of it. Then it follows that*

$$\det_\alpha A = \sum_{k=1}^{n} a_{ik}\widetilde{a}_{ik}, \qquad \widetilde{a}_{ik} = \alpha^{1-\delta_{ik}} \det_\alpha A_{ik}. \tag{2.1}$$

*Similar expansion also holds for each rows.* □

**Example 2.2.** Let $\mathbf{1}_n$ be the $n$ by $n$ all-one matrix, that is, $n$ by $n$ matrix whose entries are all one. Since $(\mathbf{1}_n)_{pq} = \mathbf{1}_{n-1}$ for any $p$ and $q$, we have

$$\det_\alpha \mathbf{1}_n = \sum_{k=1}^{n} 1 \times \alpha^{1-\delta_{ik}} \det_\alpha \mathbf{1}_{n-1} = (1 + (n-1)\alpha) \det_\alpha \mathbf{1}_{n-1},$$

from which we obtain

$$\det_\alpha \mathbf{1}_n = (1+\alpha)(1+2\alpha)\dots(1+(n-1)\alpha). \tag{2.2}$$

## 2.3 Origin of the alpha determinant

Vere-Jones [7] proved the idenfity

$$\det\bigl(I_n - \alpha AT\bigr)^{-\frac{1}{\alpha}} = \sum_{d=0}^{\infty} \sum_{1\le i_1,\dots,i_d\le n} \frac{t_{i_1}\dots t_{i_d}}{d!} \det_\alpha \begin{pmatrix} a_{i_1i_1} & \dots & a_{i_1i_d} \\ \vdots & \ddots & \vdots \\ a_{i_di_1} & \dots & a_{i_di_d} \end{pmatrix} \tag{2.3}$$

for $n$ by $n$ matrices $A = (a_{ij})$ and $T = \operatorname{diag}(t_1, \dots, t_n)$. This formula can be regarded as a generalization (or parametric deformation) of *Macmahon's Master Theorem* [3]. Indeed, Macmahon's Master Theorem is equivalent to (2.3) for $\alpha = 1$. We also note that (2.3) gives an expansion of the characteristic polynomial of $A$ when $\alpha = -1$ and $T = zI_n$. The Vere-Jones formula (2.3) is used to study certain multivariate probability distributions [8], as well as to construct certain point processes [6]. We also note that the notion of $\alpha$-Pfaffian is also introduced by Matsumoto [5].

## 2.4 A view from the representation theory

How *far* is it from the ordinary determinant/permanent to the $\alpha$-determinant when the value of the parameter $\alpha$ is apart from $\pm 1$? We formulate this naive question as a problem of representation theory as follows. Let $\mathcal{A}$ be the algebra of polynomials in $n^2$ variables

$x_{ij}$ $(1 \le i,j \le n)$. This becomes a $U(\mathfrak{gl}_n)$-module by defining the action of the standard basis $E_{ij}$ as

$$E_{ij} \cdot f(X) = \sum_{k=1}^{n} x_{ik} \frac{\partial f}{\partial x_{jk}}(X),$$

where $X = (x_{ij})$ and we write an element in $\mathcal{A}$ like $f(X)$ as above.

Then we consider the cyclic module $V_n(\alpha) := U(\mathfrak{gl}_n) \cdot \det_\alpha(X)$. Notice that

$$V_n(-1) = \mathbb{C} \cdot \det X \cong \wedge^n(\mathbb{C}^n), \qquad V_n(1) \cong S^n(\mathbb{C}^n)$$

are both *irreducible*. We think that the *complexity* of the module $V_n(\alpha)$ reflects the *distance* from the ordinary determinant/permanent to the $\alpha$-determinant.

The irreducible decomposition of $V_n(\alpha)$ is determined by Matsumoto and Wakayama [4] as follows.

**Theorem 2.2.** *Let us identify the highest weights for $U(\mathfrak{gl}_n)$ and partitions whose length is at most $n$. For each partition $\lambda$ with $\ell(\lambda) \le n$, $\boldsymbol{E}_n^\lambda$ denotes the irreducible $U(\mathfrak{gl}_n)$-module with highest weight $\lambda$. The cyclic module $V_n(\alpha)$ is decomposed as*

$$V_n(\alpha) \cong \bigoplus_{\substack{\lambda \vdash n \\ f_\lambda(\alpha) \neq 0}} (\boldsymbol{E}_n^\lambda)^{\oplus f^\lambda} = \begin{cases} \displaystyle\bigoplus_{\substack{\lambda \vdash n \\ \ell(\lambda) \le k}} (\boldsymbol{E}_n^\lambda)^{\oplus f^\lambda} & \alpha = \dfrac{1}{k} \quad (1 \le k < n), \\ \displaystyle\bigoplus_{\substack{\lambda \vdash n \\ \ell(\lambda') \le k}} (\boldsymbol{E}_n^\lambda)^{\oplus f^\lambda} & \alpha = -\dfrac{1}{k} \quad (1 \le k < n), \\ \displaystyle\bigoplus_{\lambda \vdash n} (\boldsymbol{E}_n^\lambda)^{\oplus f^\lambda} & \textit{otherwise}, \end{cases}$$

*where $f^\lambda$ denotes the number of standard tableaux with shape $\lambda$, and $f_\lambda(x) = \prod_{(i,j)\in\lambda}(1 - (j-i)x)$ is the modified content polynomial for $\lambda$.* □

**Example 2.3.** The irreducible decomposition of $V_3(\alpha)$ is given as

$$V_3(\alpha) \cong \begin{cases} \boldsymbol{E}_3^{(3)} & \alpha = 1 \\ \boldsymbol{E}_3^{(3)} \oplus (\boldsymbol{E}_3^{(21)})^{\oplus 2} & \alpha = \frac{1}{2}, \\ (\boldsymbol{E}_3^{(21)})^{\oplus 2} \oplus \boldsymbol{E}_3^{(111)} & \alpha = -\frac{1}{2}, \\ \boldsymbol{E}_3^{(111)} & \alpha = -1 \\ \boldsymbol{E}_3^{(3)} \oplus (\boldsymbol{E}_3^{(21)})^{\oplus 2} \oplus \boldsymbol{E}_3^{(111)} & \text{otherwise}, \end{cases}$$

since the modified content polynomials are given by

$$f_{(3)}(x) = (1+x)(1+2x), \qquad f_{(21)}(x) = (1+x)(1-x), \qquad f_{(111)}(x) = (1-x)(1-2x).$$

The result suggests that the $\alpha$-determinant may acquire some special properties when the value of $\alpha$ is a root of a certain modified content polynomial, that is, $\alpha = \pm 1/k$ for some $k$ $(1 \le k < n)$. In fact, in the next chapter, we utilize the $\alpha$-determinant for $\alpha = -1/k$ to define a determinant-like function on rectangular matrices.

# 3 Wreath determinant

## 3.1 Definition

Let $k, n$ be positive integers, $A$ be a $kn$ by $n$ matrix, $\mathbf{1}_{p,q}$ is the $p$ by $q$ all-one matrix. Define

$$\begin{aligned}\operatorname{wrdet}_k A &= \det_{-1/k}(A \otimes \mathbf{1}_{1,k}) \\ &= \det_{-1/k}(\underbrace{\boldsymbol{a}_1, \dots, \boldsymbol{a}_1}_{k\text{-times}}, \dots, \underbrace{\boldsymbol{a}_n, \dots, \boldsymbol{a}_n}_{k\text{-times}}) \quad (A = (\boldsymbol{a}_1, \dots, \boldsymbol{a}_n)),\end{aligned}$$

which we call the *$k$-wreath determinant* of $A$. We also use the notation

$$|A|_k = \operatorname{wrdet}_k A$$

for convenience.

**Example 3.1.**

$$\begin{vmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{vmatrix}_k = \det_{-1/k}\begin{pmatrix} a_1 & \dots & a_1 \\ \vdots & \ddots & \vdots \\ a_k & \dots & a_k \end{pmatrix} = \frac{k!}{k^k} a_1 \cdots a_k.$$

**Example 3.2.**

$$\begin{aligned}\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}_2 &= \frac{1}{4}a_{11}a_{21}a_{32}a_{42} - \frac{1}{8}a_{11}a_{31}a_{22}a_{42} - \frac{1}{8}a_{11}a_{41}a_{22}a_{32} \\ &\quad - \frac{1}{8}a_{21}a_{31}a_{12}a_{42} - \frac{1}{8}a_{21}a_{41}a_{12}a_{32} + \frac{1}{4}a_{31}a_{41}a_{12}a_{22} \\ &= \frac{1}{8}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}\begin{vmatrix} a_{21} & a_{22} \\ a_{41} & a_{42} \end{vmatrix} + \frac{1}{8}\begin{vmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{vmatrix}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.\end{aligned}$$

## 3.2 Basic properties

We summarize the basic properties of the wreath determinants. See [1] for their proofs.

**Proposition 3.1.** (1) $\operatorname{wrdet}_k(A)$ *is multilinear with respect to the row vectors in* $A$.

(2) $\operatorname{wrdet}_k(AP) = \operatorname{wrdet}_k(A) \times (\det P)^k$ *for any* $n$ *by* $n$ *matrix* $P$.

(3) $\operatorname{wrdet}_k(P(g)A) = (\operatorname{sgn}\tau)^k \operatorname{wrdet}_k(A)$ *holds for any* $g = (\sigma_1, \dots, \sigma_n; \tau) \in \mathfrak{S}_k \wr \mathfrak{S}_n = \mathfrak{S}_k^n \rtimes \mathfrak{S}_n \subset \mathfrak{S}_{kn}$.

□

We notice that Proposition 3.1 (2) says that we can calculate the wreath determinants by utilizing the elementary *column* operations.

**Proposition 3.2.** *The three conditions in Proposition 3.1 characterize the k-wreath determinant up to constant multiple.* □

**Proposition 3.3.** *Put*

$$\mathcal{D}(A) = \prod_{r=1}^{k} \det\bigl(a_{k(i-1)+r,j}\bigr)_{1\le i,j\le n}$$

*for kn by n matrix A. Then*

$$\mathrm{wrdet}_k(A) = \frac{1}{k^{kn}} \sum_{g\in\mathfrak{S}_k^n} \mathcal{D}(P(g)A).$$

□

**Example 3.3.** The case where $k = n = 2$:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{vmatrix}_2 = \frac{1}{16}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}\begin{vmatrix} a_{21} & a_{22} \\ a_{41} & a_{42} \end{vmatrix} + \frac{1}{16}\begin{vmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{vmatrix}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$
$$+ \frac{1}{16}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}\begin{vmatrix} a_{11} & a_{12} \\ a_{41} & a_{42} \end{vmatrix} + \frac{1}{16}\begin{vmatrix} a_{21} & a_{22} \\ a_{41} & a_{42} \end{vmatrix}\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix}.$$

# 4 Wreath Schur polynomials

We refer to [2] for basic facts on symmetric functions.

## 4.1 Vandermonde-type wreath determinant

For $\alpha = (\alpha_1, \dots, \alpha_{kn}) \in \mathbb{N}^{kn}$ ($\mathbb{N} = \{0, 1, 2, \dots\}$), put

$$a_\alpha(x) = \mathrm{wrdet}_k\bigl(x_j^{\alpha_i}\bigr)_{\substack{1\le i\le kn \\ 1\le j\le n}} = \begin{vmatrix} x_1^{\alpha_1} & x_2^{\alpha_1} & \dots & x_n^{\alpha_1} \\ x_1^{\alpha_2} & x_2^{\alpha_2} & \dots & x_n^{\alpha_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\alpha_{kn}} & x_2^{\alpha_{kn}} & \dots & x_n^{\alpha_{kn}} \end{vmatrix}_k,$$

where $x = (x_1, \dots, x_n)$. In particular, we have

$$a_\delta(x) = \left(\frac{k!}{k^k}\right)^n \Delta_n(x)^k, \qquad \Delta_n(x) = \prod_{1\le i<j\le n} (x_i - x_j)$$

for

$$\delta := (\overbrace{n-1, \dots, n-1}^{k}, \dots, \overbrace{1, \dots, 1}^{k}, \overbrace{0, \dots, 0}^{k}).$$

In general, $a_\alpha(x)$ is divisible by $a_\delta(x)$ due to Proposition 3.1 (2) (see also §4.4), and the quotient $a_\alpha(x)/a_\delta(x)$ is a symmetric polynomial in $n$ variables.

## 4.2 Definition

For a partition $\lambda = (\lambda_1, \ldots, \lambda_n)$, define

$$\mathcal{S}_\lambda(x) = \frac{a_{\widetilde{\lambda}+\delta}(x)}{a_\delta(x)},$$

where

$$\widetilde{\lambda} = (\lambda_1, \underbrace{0, \ldots, 0}_{k-1}, \lambda_2, \underbrace{0, \ldots, 0}_{k-1}, \ldots, \lambda_n, \underbrace{0, \ldots, 0}_{k-1}) \in \mathbb{N}^{kn}.$$

We call $\mathcal{S}_\lambda(x)$ the *k-wreath Schur polynomial.* Notice that $\mathcal{S}_\lambda(x) = s_\lambda(x)$ when $k = 1$.

**Example 4.1.** When $k = n = 2$, we have

$$\mathcal{S}_{(\lambda_1,\lambda_2)}(x_1, x_2) = \begin{vmatrix} x_1^{\lambda_1+1} & x_2^{\lambda_1+1} \\ x_1 & x_2 \\ x_1^{\lambda_2} & x_2^{\lambda_2} \\ 1 & 1 \end{vmatrix}_2 \Bigg/ \begin{vmatrix} x_1 & x_2 \\ x_1 & x_2 \\ 1 & 1 \\ 1 & 1 \end{vmatrix}_2$$

$$= \frac{1}{2}\Big\{ s_{(\lambda_1,\lambda_2)}(x_1, x_2) - s_{(\lambda_1,0)}(x_1, x_2) s_{(\lambda_2-1,1)}(x_1, x_2) \Big\}.$$

## 4.3 Giambelli-type formulas

The wreath Schur polynomials have the Giambelli-type formula as follows.

**Theorem 4.1.** *It holds that*

$$\mathcal{S}_\lambda(x) = \det_{-1/k}\Big( (-1)^{j-1} s_{(\lambda_i - i \,|\, j-1)}(x) \Big)_{1 \le i,j \le n}. \tag{4.1}$$

*Here* $(\lambda_i - i \,|\, j-1)$ *represents the partition* $(\lambda_i - i + 1, 1^{j-1})$ *when* $\lambda_i - i \geq 0$ *(Frobenius notation). We also set*

$$s_{(a \,|\, b)} = (-1)^b \delta_{a+b,-1} \tag{4.2}$$

*for* $a < 0$. *In particular, we have* $\mathcal{S}_{(a \,|\, b)}(x) = k^{-b} s_{(a \,|\, b)}$.

**Theorem 4.2.** *For* $(\alpha \,|\, \beta) = (\alpha_1, \ldots, \alpha_r \,|\, \beta_1, \ldots, \beta_r)$, *there exists* $m \in \mathbb{N}$ *and* $g \in \mathfrak{S}_r$ *such that*

$$\mathcal{S}_{(\alpha \,|\, \beta)}(x) = (-k)^m \det_{-1/k}\Big( \mathcal{S}_{(\alpha_i \,|\, \beta_{g(j)})}(x) \Big)_{1 \le i,j \le r}. \tag{4.3}$$

*Remark.* When $k = 1$, Theorem 4.2 gives the familiar formula

$$s_{(\alpha \,|\, \beta)} = \det\big( s_{(\alpha_i \,|\, \beta_j)} \big)_{1 \le i,j \le r} \tag{4.4}$$

for Schur polynomials.

*Remark.* The theorems above also shows that the sequence $\{\mathcal{S}_\lambda(x_1, \ldots, x_n)\}_{n \ge 1}$ defines a symmetric function for each partition $\lambda$, since the ordinary Schur polynomials do.

## 4.4 Proofs

Regarding each monomial $x_j^m$ as a complete symmetric polynomial $h_m(x_j)$ in one variable $x_j$, we have

$$a_\alpha(x) = \mathrm{wrdet}_k\big(h_{\alpha_i}(x_j)\big)_{\substack{1\le i\le kn\\ 1\le j\le n}}.$$

By using the relation

$$h_m(y,u) - h_m(z,u) = (y-z)h_{m-1}(y,z,u),$$

where $u = (u_1,\dots,u_r)$, we have

$$a_\alpha(x) = \Delta_n(x)^k\,\mathrm{wrdet}_k\Big(h_{\alpha_i-n+j}(x_j,\dots,x_n)\Big)_{\substack{1\le i\le kn\\ 1\le j\le n}}. \tag{4.5}$$

Especially we have

$$\begin{aligned} a_\delta(x) &= \Delta_n(x)^k\,\mathrm{wrdet}\begin{pmatrix} \mathbf{1}_{k,1} & * & * & * \\ & \mathbf{1}_{k,1} & * & * \\ & & \ddots & * \\ & & & \mathbf{1}_{k,1} \end{pmatrix} \\ &= \Delta_n(x)^k\,\mathrm{det}_{-1/k}\begin{pmatrix} \mathbf{1}_{k} & * & * & * \\ & \mathbf{1}_{k} & * & * \\ & & \ddots & * \\ & & & \mathbf{1}_{k} \end{pmatrix} = \left(\frac{k!}{k^k}\right)^n \Delta_n(x)^k, \end{aligned}$$

and it then follows that

$$\mathcal{S}_\lambda(x) = \left(\frac{k^k}{k!}\right)^n \mathrm{wrdet}_k\Big(h_{\widetilde{\lambda}_i+\delta_i-n+j}(x_j,\dots,x_n)\Big)_{\substack{1\le i\le kn\\ 1\le j\le n}}. \tag{4.6}$$

If we define $f_{ij}(x)$ by the conditions

$$\begin{aligned} F &= \Big(f_{ij}(x)\Big)_{1\le i,j\le n} := HP^{-1}, \\ H &:= \Big(h_{\lambda_i-i+j}(x_j,\dots,x_n)\Big)_{1\le i,j\le n}, \\ P &:= \Big(h_{-i+j}(x_j,\dots,x_n)\Big)_{1\le i,j\le n}, \end{aligned}$$

then, by the relative invariance of the wreath determinant (Proposition 3.1 (2)), we have

$$\begin{aligned} &\mathrm{wrdet}_k\Big(h_{\widetilde{\lambda}_i+\delta_i-n+j}(x_j,\dots,x_n)\Big)_{\substack{1\le i\le kn\\ 1\le j\le n}} \\ &= \begin{vmatrix} H_{11} & H_{12} & \dots & H_{1n} \\ P_{11}\mathbf{1} & P_{21}\mathbf{1} & \dots & P_{2n}\mathbf{1} \\ H_{21} & H_{22} & \dots & H_{2n} \\ P_{21}\mathbf{1} & P_{22}\mathbf{1} & \dots & P_{2n}\mathbf{1} \\ \vdots & \vdots & \ddots & \vdots \\ H_{n1} & H_{n2} & \dots & H_{nn} \\ P_{n1}\mathbf{1} & P_{n2}\mathbf{1} & \dots & P_{nn}\mathbf{1} \end{vmatrix}_k = \begin{vmatrix} f_{11}(x) & f_{12}(x) & \dots & f_{1n}(x) \\ \mathbf{1} & \mathbf{0} & \dots & \mathbf{0} \\ f_{21}(x) & f_{22}(x) & \dots & f_{2n}(x) \\ \mathbf{0} & \mathbf{1} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}(x) & f_{n2}(x) & \dots & f_{nn}(x) \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{1} \end{vmatrix}_k \end{aligned}$$

since $\det P = 1$. Here $\mathbf{1} = \mathbf{1}_{k-1,1}$ and $\mathbf{0}$ is the $k-1$ by 1 zero matrix, $H_{ij}$ and $P_{ij}$ are $(i,j)$-entries of $H$ and $P$ respectively. Now the proof of Theorem 4.1 is reduced to the following two lemmas.

**Lemma 4.3.** $f_{ij}(x) = (-1)^{j-1} s_{(\lambda_i - i \,|\, j-1)}(x)$.

*Proof.* Put $u = (x_1, \ldots, x_{j-1})$, $v = (x_j, \ldots, x_n)$. It is enough to show

$$h_{m+j}(v) = \sum_{p=1}^{j} (-1)^{p-1} s_{(m \,|\, p-1)}(u, v) h_{j-p}(v) \tag{4.7}$$

for any $m \geq 0$. The righthand side of (4.7) is

$$\sum_{p=1}^{j} (-1)^{p-1} \sum_{\mu,\nu} c_{\mu\nu}^{(m \,|\, p-1)} s_\mu(u) s_\nu(v) h_{j-p}(v)$$
$$= \sum_{\mu} \left( \sum_{p=1}^{j} (-1)^{p-1} s_{(m \,|\, p-1)/\mu}(v) h_{j-p}(v) \right) s_\mu(u).$$

When $\mu = \emptyset$, the coefficient of $s_\mu(u)$ in the sum above is

$$\sum_{p=1}^{j} (-1)^{p-1} s_{(m \,|\, p-1)}(v) h_{j-p}(v) = h_{m+j}(v)$$

by the Jacobi-Trudi formula $s_\xi(v) = \det(h_{\xi_i - i + j}(v))$ for $\xi = (m \,|\, j-1)$. When $\mu \neq \emptyset$, $s_{(m \,|\, p-1)/\mu}(v)$ is zero if $\mu$ is not contained in $(m \,|\, p-1)$, and $s_\mu(v) = 0$ if $\ell(\mu) \geq j$ since $v$ consists of $j-1$ variables. Suppose that $\mu = (r \,|\, q-1)$ for some $r$ and $q$ such that $r \leq m$ and $q < j$. Then the coefficient of $s_\mu(u)$ in the sum above is

$$\sum_{p=1}^{j} (-1)^{p-1} s_{(m \,|\, p-1)/(r \,|\, q-1)}(v) h_{j-p}(v) = \sum_{p=q}^{j} (-1)^{p-1} s_{(m-r)}(v) s_{(1^{p-q})}(v) h_{j-p}(v)$$
$$= h_{m-r}(v) \sum_{p=q}^{j} (-1)^{p-1} e_{p-q}(v) h_{j-p}(v) = (-1)^{q-1} \delta_{jq} h_{m-r}(v) = 0.$$

Therefore it follows that the righthand side of (4.7) equals $h_{m+j}(v)$, which is the desired conclusion. □

**Lemma 4.4.** *For any $n$ by $n$ matrix $A = (a_{ij})$, put*

$$A^\sharp = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \mathbf{1} & \mathbf{0} & \ldots & \mathbf{0} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \mathbf{0} & \mathbf{1} & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{1} \end{pmatrix}.$$

*Then*

$$\det_\alpha \left(A^\sharp \otimes \mathbf{1}_{1,k}\right) = (\det_\alpha \mathbf{1}_k)^n \det_\alpha A$$

*holds.*

*Proof.* Apply the Laplace expansion repeatedly to the rows in $A^\sharp \otimes \mathbf{1}_{1,k}$ contains only 1 and 0. □

To derive Theorem 4.2 from Theorem 4.1, we need the fact that

$$\{1,2,\dots,n\} \setminus \{i-\lambda_i \mid r < i \leq n\} = \{\beta_j + 1 \mid 1 \leq j \leq r\} \tag{4.8}$$

for $\lambda = (\lambda_1,\dots,\lambda_n) = (\alpha_1,\dots,\alpha_r \mid \beta_1,\dots,\beta_r)$.

If $i > r$, then $\lambda_i - i < 0$ so that we have

$$(-1)^{j-1} s_{(\lambda_i - i \mid j-1)}(x) = \delta_{i-\lambda_i, j}$$

by (4.2). In other words, in the matrix $((-1)^{j-1} s_{(\lambda_i-i \mid j-1)}(x))_{1\leq i,j\leq n}$, $(i, i-\lambda_i)$-entry is 1 and other entries in the $i$-th row is 0. Hence, by using the Laplace expansion for each row below the $r$-th row and use the fact $\mathcal{S}_{(a \mid b)}(x) = k^{-b} s_{(a \mid b)}$, we get the theorem.

**Example 4.2.** Let us look at the case where $\lambda = (4,3,3,3,1) = (3,1,0 \mid 4,2,1)$. We see that

$$\{i - \lambda_i \mid 3 < i \leq 5\} = \{1,4\}, \qquad \{\beta_j + 1 \mid 1 \leq j \leq 3\} = \{5,3,2\}.$$

We have $\mathcal{S}_\lambda = \det_{-1/k} A$, where

$$A = \begin{pmatrix} s_{(3\mid 0)} & -s_{(3\mid 1)} & s_{(3\mid 2)} & -s_{(3\mid 3)} & s_{(3\mid 4)} \\ s_{(1\mid 0)} & -s_{(1\mid 1)} & s_{(1\mid 2)} & -s_{(1\mid 3)} & s_{(1\mid 4)} \\ s_{(0\mid 0)} & -s_{(0\mid 1)} & s_{(0\mid 2)} & -s_{(0\mid 3)} & s_{(0\mid 4)} \\ \boldsymbol{s_{(-1\mid 0)}} & \underline{-s_{(-1\mid 1)}} & \underline{s_{(-1\mid 2)}} & \underline{-s_{(-1\mid 3)}} & \underline{s_{(-1\mid 4)}} \\ \underline{s_{(-4\mid 0)}} & \underline{-s_{(-4\mid 1)}} & \underline{s_{(-4\mid 2)}} & \boldsymbol{-s_{(-4\mid 3)}} & \underline{s_{(-4\mid 4)}} \end{pmatrix},$$

$\boldsymbol{s_{(-1\mid 0)}} = \boldsymbol{-s_{(-4\mid 3)}} = 1$, and underlined ones are all zero. Thus, by using the Laplace expansion formula twice, we have

$$\begin{aligned} \det_{-1/k} A &= \left(-\frac{1}{k}\right)^2 \det_{-1/k} \begin{pmatrix} s_{(3\mid 4)} & -s_{(3\mid 1)} & s_{(3\mid 2)} \\ s_{(1\mid 4)} & -s_{(1\mid 1)} & s_{(1\mid 2)} \\ s_{(0\mid 4)} & -s_{(0\mid 1)} & s_{(0\mid 2)} \end{pmatrix} \\ &= (-k)^5 \det_{-1/k} \begin{pmatrix} \mathcal{S}_{(3\mid 4)} & \mathcal{S}_{(3\mid 1)} & \mathcal{S}_{(3\mid 2)} \\ \mathcal{S}_{(1\mid 4)} & \mathcal{S}_{(1\mid 1)} & \mathcal{S}_{(1\mid 2)} \\ \mathcal{S}_{(0\mid 4)} & \mathcal{S}_{(0\mid 1)} & \mathcal{S}_{(0\mid 2)} \end{pmatrix}. \end{aligned}$$

In this case, the permutation $g \in \mathfrak{S}_3$ in the theorem is equal to the transposition $(2\,3)$.

## 4.5 Schur-positivity

A symmetric function $f$ is called *Schur-positive* if all the expansion coefficient with respect to Schur functions are non-negative. When is $\mathcal{S}_\lambda$ Schur-positive? We give several examples below.

**Example 4.3.** For a hook $\lambda = (a \,|\, b)$, we have $\mathcal{S}_{(a\,|\,b)} = k^{-b} s_{(a\,|\,b)}$, which is Schur-positive for any $k \geq 1$.

**Example 4.4.** For $\lambda = (m, 2, 2)$ $(m \geq 2)$, we have

$$\mathcal{S}_{(m,2,2)} = \frac{1}{k} s_{(m,2,2)} + \frac{k-1}{k^2} \left\{ s_{(m+1,2,1)} + s_{(m+1,1^3)} + s_{(m,2,1^2)} + s_{(m,1^4)} \right\},$$

which is Schur-positive for any $k \geq 1$.

**Example 4.5.** For $\lambda = (m, 2)$ $(m \geq 2)$, we have

$$\mathcal{S}_{(m,2)} = \frac{1-k}{k} \left\{ s_{(m+1,1)} + s_{(m,1,1)} \right\} + \frac{1}{k} s_{(m,2)},$$

which is Schur-positive only when $k = 1$.

**Example 4.6.**

$$\begin{aligned}
\mathcal{S}_{(3,3,3)} = \frac{1}{k} s_{(3,3,3)} &+ \frac{2(k-1)}{k^2} \left( s_{(4^2,1)} + s_{(3,2^3)} \right) - \frac{(k-1)(k-3)}{k^2} \left( s_{(5,3,1)} + s_{(3,2^2,1^2)} \right) \\
&- \frac{(k-1)(k-2)}{k^2} \left( s_{(6,2,1)} + s_{(4,1^5)} + s_{(3,2,1^4)} + s_{(6,1^3)} + 2 s_{(5,1^4)} \right) \\
&- \frac{(k-1)(k-4)}{k^2} \left( s_{(5,2^2)} + s_{(4,3,2)} + s_{(3^2,2,1)} + s_{(3^2,1^3)} \right) \\
&- \frac{(k-1)(2k-7)}{k^2} \left( s_{(4,3,1^2)} + s_{(4,2^2,1)} \right) - \frac{(k-1)(3k-7)}{k^2} \left( s_{(5,2,1^2)} + s_{(4,2,1^3)} \right).
\end{aligned}$$

This is Schur-positive only when $k = 1, 2$. If $k = 1$, then $\mathcal{S}_{(3,3,3)} = s_{(3,3,3)}$. If $k = 2$, then

$$\begin{aligned}
\mathcal{S}_{(3,3,3)} = \frac{1}{2} s_{(3,3,3)} &+ \frac{1}{2} \left( s_{(4^2,1)} + s_{(3,2^3)} \right) + \frac{1}{4} \left( s_{(5,3,1)} + s_{(3,2^2,1^2)} \right) \\
&+ \frac{1}{2} \left( s_{(5,2^2)} + s_{(4,3,2)} + s_{(3^2,2,1)} + s_{(3^2,1^3)} \right) \\
&+ \frac{3}{4} \left( s_{(4,3,1^2)} + s_{(4,2^2,1)} \right) + \frac{1}{4} \left( s_{(5,2,1^2)} + s_{(4,2,1^3)} \right).
\end{aligned}$$

If $k \geq 3$, then the coefficient of $s_{(6,2,1)}$, for instance, is negative.

## Acknowledgement

The author would like to thank Masao Ishikawa and Soichi Okada, the organizers of the RIMS workshop "Algebraic Combinatorics Related to Young Diagrams and Statistical Physics," for giving him an opportunity to present a talk at the workshop.

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