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KKR TYPE BIJECTION AND FUTURE PERSPECTIVES

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ABSTRACT. The Kerov-Kirillov-Reshetikhin bijection can be viewed as a combinatorial proof of Bethe's fermionic formula. We start with the historial background, see how it is generalized to arbitrary root systems, and recognize the role of Kashiwara's crystal basis theory. We then introduce the $X=M$ conjecture and summarize the settled cases. Finally, we discuss the future perspectives including open problems.

1. HISTORICAL BACKGROUND

1.1. **Bethe's fermionic formula.** In his paper [3] in 1931, Bethe showed the following formula.

$$[V_1^{\otimes N}:V_l]=\sum_{\{m_i\}}\prod_{i\geq 1}\binom{p_i+m_i}{m_i} \tag{1}$$

The l.h.s. stands for the multiplicity of the representation V_l of spin $l/2$ of the Lie algebra sl_2 in the N-fold tensor product of the representation V_1 of spin $1/2$. In the r.h.s. the sum is taken over all nonnegative integer sequences $\{m_i\}_{i\geq 1}$ satisfying

$$\sum_{i\geq 1} im_i=\frac{N-l}{2}. \tag{2}$$

Note that by the constraint (2) there are only finite number of sequences $\{m_i\}$ so the sum in (1) is finite, and for each $\{m_i\}$ there are only finite number of i such that $m_i>0$ so the product is practically finite.

He considered to diagonalize the Hamiltonian of a one-dimensional quantum spin chain, called the XXX model. He made an Ansatz on the eigenvectors, obtained algebraic equations (Bethe equations) for parameters as the condition that the provisional eigenvectors are true ones. He then put a hypothesis (string hypothesis) for the roots of the Bethe equations. Counting the number of such roots in the complex plane, and finally he obtained the formula (1). The r.h.s. is now called a fermionic formula, since it is derived from the counting problem obeying fermionic exclusion rules. See [22, §1.1].

1.2. **Kerov-Kirillov-Reshetikhin.** It is known that the l.h.s. of (1) is equal to the number of standard tableaux of shape $(\frac{N+l}{2},\frac{N-l}{2})$. Hence, if one finds an appropriate combinatorial object whose total number equals the r.h.s., one may establish the formula by finding a bijection between these two combinatorial objects. Actually, Kerov, Kirillov and Reshetikhin did it in [18]. Namely, they introduced an object, called rigged configuration. The total number of rigged configurations is by definition equal to the r.h.s. of (1). They then constructed an explicit bijection

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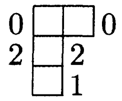
between standard tableaux and rigged configurations by induction on the number of boxes of tableau. We call it the KKR bijection.

Example 1. An example of the KKR bijection. The left object is a rigged configuration corresponding to the right standard tableau.



1	2	4	6
3	5	7	8

To be precise what KKR did was the sl_n $(n \geq 2)$ case. The l.h.s. is replaced by $[V(\Lambda_1)^{\otimes N} : V(\lambda)]$ where Λ_1 is the first fundamental weight of sl_n and $V(\lambda)$ is the irreducible highest weight representation of highest weight λ. In this case rigged configurations have $n-1$ Young diagrams.

1.3. **Generalization.** Let us review representation theory of the simple Lie algebra sl_n. Any irreducible finite-dimensional representation of sl_n is in one-to-one correspondence to a partition λ (or Young diagram) of length $\ell(\lambda)$ less than n. The highest weight of the corresponding representation is given by $\sum_{i=1}^{n-1}(\lambda_i - \lambda_{i+1})\Lambda_i$ where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ $(\lambda_n = 0)$ and Λ_i is the i-th fundamental weight. Let λ, μ be partitions. We assume $\ell(\lambda) < n$. Set

$$K_{\lambda\mu} = [\bigotimes_{i=1}^{\ell(\mu)} V(\mu_i \Lambda_1) : V(\lambda)]. \tag{3}$$

$K_{\lambda\mu}$ is called the Kostka number.

As a generalization of [18] Kirillov and Reshetikhin [19] obtained the fermionic formula for (3). Namely, they considered the case where the tensor components are representations corresponding to single rows. The Kostka number $K_{\lambda\mu}$ has a q-analogue called Kostka (Kostka-Foulkes) polynomial $K_{\lambda\mu}(q)$. It appears in many places of mathematics. For instance, $K_{\lambda\mu}(q)$ can be defined as the transition coefficent between the Schur function $s_\lambda(x)$ and Hall-Littlewood polynomial $H_\mu(x;q)$. See [28, Chapter III]. In [19] Kirillov and Reshetikhin also obtained the fermionic formula for $K_{\lambda\mu}(q)$, which looks as follows.

$$K_{\lambda\mu}(q) = \sum_{\{m_i^{(a)}\}} q^{c(\{m_i^{(a)}\})} \prod_{1 \leq a \leq n-1, i \geq 1} \begin{bmatrix} p_i^{(a)} + m_i^{(a)} \\ m_i^{(a)} \end{bmatrix}$$

$c(\{m_i^{(a)}\})$ is some quadratic form and $\begin{bmatrix} p+m \\ m \end{bmatrix}$ is the q-binomial coefficient. It is known in [26] that $K_{\lambda\mu}(q)$ can be written as the generating function of semistandard tableaux by the weight called charge. Kirillov and Reshetikhin defined a charge also on rigged configurations and showed that their bijection preserves the charge.

Subsequently, a generalization of [19] was made by Kirillov, Schilling and Shimozono [21]. It corresponds to the situation where the l.h.s. (at $q = 1$) is given by

$$[\bigotimes_{i=1}^{N} V(s_i \Lambda_{r_i}) : V(\lambda)],$$

or all tensor components are of rectangular shape. One might ask what happens if tensor components are of arbitrary shape. We do not think there is a neat formula as above. A part of the reason is mentioned in next section.

2. Kirillov-Reshetikhin conjecture and $X = M$ conjecture

We present a generalization to what we wrote in the previous section.

2.1. **Kirillov-Reshetikhin conjecture.** Let $\mathfrak{g}$ be an affine algebra and I the index set of its Dynkin nodes. Let $\mathfrak{g}_0$ be the finite-dimensional simple Lie algebra obtained by removing the node 0 from I, where 0 is specified as in [15], and set $I_0 = I \setminus \{0\}$. Although we can present a statement for all affine algebra cases, we deal for simplicity with the case when $\mathfrak{g}$ is simply-laced, i.e., $\mathfrak{g} = A_n^{(1)}, D_n^{(1)}, E_{6,7,8}^{(1)}$. Let $U_q(\mathfrak{g})$ be the quantized enveloping algebra [6, 14] associated to an affine algebra $\mathfrak{g}$ and $U'_q(\mathfrak{g})$ its subalgebra without the degree operator q^d. They contain the quantum enveloping algebra $U_q(\mathfrak{g}_0)$ associated to $\mathfrak{g}_0$ as a subalgebra.

In their paper [20] Kirillov and Reshetikhin proposed a remarkable conjecture. Let $V(\lambda)$ be the irreducible highest weight $U(\mathfrak{g}_0)$-module of highest weight λ.

Theorem 1 (Kirillov-Reshetikhin conjecture). *There exists a family of finite-dimensional $U'_q(\mathfrak{g})$-modules $\{W^{r,s}\}_{r\in I_0, s\in\mathbb{Z}_{>0}}$ such that*

$$[\bigotimes_{j=1}^{N} W^{r_j,s_j} : V(\lambda)] = \sum_{\{m_i^{(a)}\}} \prod_{a\in I_0, i\in\mathbb{Z}_{>0}} \binom{p_i^{(a)} + m_i^{(a)}}{m_i^{(a)}}$$

These family of finite-dimensional modules are called the Kirillov-Reshetikhin (KR) modules.

Remark 2. To be precise, they conjectured the existence of a family of finite-dimensional Yangian ($Y(\mathfrak{g}_0)$) modules $\{W^{r,s}\}_{r\in I_0, s\in\mathbb{Z}_{>0}}$. At that time, it was a folkfore that representation theory of $Y(\mathfrak{g}_0)$ and $U'_q(\mathfrak{g})$ are more or less equivalent. Such correspondence was clarified recently by Gautam and Toledano Laredo in [8].

Let us explain the meaning of the r.h.s. in more detail when $\mathfrak{g}$ is simply-laced. Let α_a be a simple root of $\mathfrak{g}$ and $\overline{\Lambda}_a$ a fundamental weight of $\mathfrak{g}_0$. For $a \in I_0, i \in \mathbb{Z}_{>0}$ set

$$\begin{aligned} L_i^{(a)} &= \sharp\{j \mid (r_j, s_j) = (a, i), 1 \le j \le N\}, \\ p_i^{(a)} &= \sum_{j\in\mathbb{Z}_{>0}} L_j^{(a)} \min(i,j) - \sum_{b\in I_0, j\in\mathbb{Z}_{>0}} (\alpha_a|\alpha_b) \min(i,j) m_j^{(b)}. \end{aligned} \tag{4}$$

There are two interpretations of the r.h.s. The first one is to take the summation $\sum_{\{m_i^{(a)}\}}$ over all nonnegative integers $m_i^{(a)}$ ($a \in I_0, i \in \mathbb{Z}_{>0}$) satisfying

$$\sum_{a\in I_0, i\in\mathbb{Z}_{>0}} i\, m_i^{(a)} \alpha_a = \sum_{a\in I_0, i\in\mathbb{Z}_{>0}} i\, L_i^{(a)} \overline{\Lambda}_a - \lambda \tag{5}$$

and $p_i^{(a)} \ge 0$ for all a, i. We refer to this case as (C(ombinatorial)). The second one is to take the summation over all nonnegative integers $m_i^{(a)}$ satisfying (5) and allow $p_i^{(a)}$ to become negative. Note that the binomial coefficient may become 0 or negative in this case. We refer to this case as (N(on)C(ombinatorial)). Originally, Kirillov and Reshetikhin derived the formula for (C) by counting physical states via Bethe Ansatz, but the both formulas are valid.

There are three stages of the proof. To explain it, we need to introduce the Q-system, algebraic relations satisfied by the characters of KR modules. For simply-laced types it is expressed as

$$Q_j^{(a)^2} = Q_{j+1}^{(a)} Q_{j-1}^{(a)} + \prod_{b \sim a} Q_j^{(b)}$$

where $b \sim a$ means that the nodes $a, b \in I_0$ are connected by a line in the Dynkin diagram of $\mathfrak{g}$. The proof of Theorem 1 goes as follows.

I. If $Q_s^{(r)} = \operatorname{ch} W^{r,s}(a)$ satisfies the Q-system, then the formula (NC) holds [19, 11, 24].
II. $\operatorname{ch} W^{r,s}(a)$ satisfies the Q-system [30, 12].
III. The formula (NC) is equal to (C) [5]. (Di Francesco and Kedem only deal with the untwisted cases. Hence, twisted cases are still open.)

2.2. **Kirillov-Reshetikhin crystal and $X = M$ conjecture.** In the previous subsection we saw that Bethe's fermionic formula was generalized to an arbitrary affine root systems. Here we propose a q-analogue of the Kirillov-Reshetikhin conjecture. For this purpose we need the notion of crystal bases by Kashiwara [16]. To be precise KR modules have another parameter a taking values in $\mathbb{Q}(q)$. It is denoted by $W^{r,s}(a)$. It is known that for nonexceptional types, with suitable $a^\dagger$ the KR module $W^{r,s}(a^\dagger)$ has a crystal base $B^{r,s}$ [34, 39]. The crystal structure of the KR crystal $B^{r,s}$ is also known [7]. Let B be a tensor product of KR crystals $B = B^{r_1,s_1} \otimes B^{r_2,s_2} \otimes \cdots \otimes B^{r_N,s_N}$, and for a subset J of I set $\mathrm{hw}_J(B) = \{b \in B \mid \tilde{e}_i b = 0 \text{ for any } i \in J\}$ where $\tilde{e}_i$ is the so-called Kashiwara operator [16] acting on the crystal B.

As a natural q-analogue of the Kirillov-Reshetikhin conjecture we proposed the $X = M$ conjecture in [11, 10].

Conjecture 2 ($X = M$ conjecture).

$$\sum_{b \in \mathrm{hw}_{I_0}(B), \mathrm{wt}\, b = \lambda} q^{D(b)} = \sum_{\{m_i^{(a)}\}} q^{c(\{m_i^{(a)}\})} \prod_{a \in I_0, i \in \mathbb{Z}_{>0}} \begin{bmatrix} p_i^{(a)} + m_i^{(a)} \\ m_i^{(a)} \end{bmatrix}_{q^{t_a^\vee}}$$

The l.h.s. is denoted by $X_{\lambda,B}(q)$ and the r.h.s. by $M_{\lambda,\boldsymbol{L}}(q)$. B and $\boldsymbol{L} = (L_i^{(a)})$ are related by (4). Noting that the l.h.s. at $q = 1$ is equal to the l.h.s. of the Kirillov-Reshetikhin conjecture and $\begin{bmatrix} m \\ n \end{bmatrix}_q$ is a q-analogue of $\binom{m}{n}$, it is easy to see that this conjecture is a q-analogue of the Kirillov-Reshetikhin conjecture. For details we refer to [11, 10, 33], but we only mention the representation-theoretical meaning of D. Let ℓ be the minimum of the levels of B^{r_j,s_j}. (For the definition of the level of a KR crystal, see [11].) Let $B(\mu)$ be the irreducible highest weiht crystal of $U_q(\mathfrak{g})$ with highest weight μ. Then we know

$$B(\ell\Lambda_0) \otimes B \simeq \bigoplus_j B(\mu_j).$$

Let u be the highest weight vector of $B(\ell\Lambda_0)$ and $b \in B$. We have

$$D(b) = -\langle d, \text{affine weight of } u \otimes b \rangle$$

where d is the degree operator in $\mathfrak{g}$.

3. Settled cases

We review settled cases of the $X = M$ conjecture.

3.1. **Similar method to KKR.** As we see in §1.3 the $X = M$ conjecture was settled for $\mathfrak{g} = A_n^{(1)}$ in full generality [21]. By a similar method, namely, constructing an explicit bijection from $\mathrm{hw}_{I_0}(B)$ to rigged configurations, the following cases were also settled.

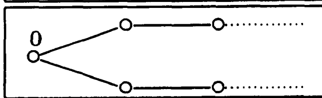
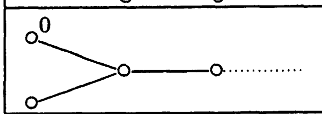
(1) $B = \bigotimes_j B^{1,s_j}$ of all nonexceptional types [40, 41],

(2) $B = \bigotimes_j B^{r_j,1}$ for type $D_n^{(1)}$ [42],

(3) $B = (B^{1,1})^{\otimes N}$ for type $E_6^{(1)}$ [38].

The bijection in (3) is constructed by looking at the crystal graph of $B^{1,1}$. We expect that the same algorithm works also for the crystal graph where it is connected as I_0-crystal and any i-string has length 1 for any $i \in I_0$.

3.2. **Large rank case when $\mathfrak{g}$ is nonexceptional.** Let $\mathfrak{g}$ be of nonexceptional type. Suppose the rank of $\mathfrak{g}$ is sufficiently large. Then it is known that both $X_{\lambda,B}(q)$ and $M_{\lambda,L}(q)$ depends only on the attachment of the node 0 to the rest of the Dynkin diagram. Hence, we only have four "stable" $X^{\diamondsuit}_{\lambda,B}(q)$ and $M^{\diamondsuit}_{\lambda,L}(q)$ as shown in the table below.

Dynkin	$\mathfrak{g}$	$\diamondsuit$
	$A_n^{(1)}$	$\varnothing$
	$B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$
	$C_n^{(1)}$	$\square\square$
	$A_{2n}^{(2)}, D_{n+1}^{(2)}$	$\square$

Remark 3. The symbol $\diamondsuit$ has a representation-theoretical meaning. If the node r is not related to a spin representation, we have

$$W^{r,s}(a) \simeq \bigoplus_{\lambda} V(\lambda) \quad \text{as } U_q(\mathfrak{g}_0)\text{-modules.}$$

Here λ runs over all partitions that can be obtained from (s^r) by removing $\diamondsuit$. See e.g. [4, 39].

Shimozono and Zabrocki [43, 44] conjectured that for $\diamondsuit \neq \varnothing$, $X^{\diamondsuit}_{\lambda,B}(q)$ is expressed as sums of $X^{\varnothing}_{\nu,B}(q)$.

$$X^{\diamondsuit}_{\lambda,B}(q) = q^{\frac{|\lambda|-|B|}{|\diamondsuit|}} \sum_{\mu\in\mathcal{P}^{\diamondsuit}_{|B|-|\lambda|},\,\nu\in\mathcal{P}^{\square}_{|B|}} c^{\nu}_{\lambda\mu} X^{\varnothing}_{\nu,B}(q^{\frac{2}{|\diamondsuit|}}).$$

Here $|B| = \sum_{j=1}^{N} r_j s_j, \mathcal{P}^{\diamondsuit}_N$ = set of partitions of N tiled from $\diamondsuit$, and $c^{\nu}_{\lambda\mu}$ stands for the Littlewood-Richardson coefficient. This conjecture was settled in [27]. Hence, to show the $X = M$ conjecture for large rank case when $\mathfrak{g}$ is of nonexceptional type, it suffices to show the same equality holds with X replaced by M. This is settled in [36].

3.3. **Naoi's approach.** Naoi introduced a new approach to tackle the $X = M$ conjecture. Let $\mathfrak{g}$ be an untwisted affine algebra. Then $\mathfrak{g}_0$ covers all finite-dimensional simple Lie algebras. He investigated certain graded modules of the current algebra $\mathfrak{g}_0 \otimes \mathbb{C}[t]$ and showed that their graded character is, on one hand, is equal to X, and on the other hand, is equal to M. He settled the case when $\mathfrak{g}$ is an arbitrary untwisted affine algebra and $s_j = 1$ for any j in [31], and when $\mathfrak{g} = A_n^{(1)}, D_n^{(1)}$ in [32].

4. Future perspectives

In this section we discuss what we think we should do on the KKR type bijection and related topics in near future.

4.1. **KR crystals.** The existence of a Kirillov-Reshetikhin crystal $B^{r,s}$ is settled only when $s = 1$ for an arbitrary affine algebra $\mathfrak{g}$ [17] and when $\mathfrak{g}$ is of nonexceptional type [34, 39]. Hence, the existence problem of the KR crystal for exceptional types is widely open. The inverse problem, namely, the determination of the finite-dimensional modules having crystal bases, is still open even when $\mathfrak{g} = A_n^{(1)}$.

4.2. **KKR type bijection for other root systems.** Although Naoi's approach using the current algebra gives a proof of the $X = M$ conjecture, we have enough reason to stick to the proof by an explicit bijection of KKR type. There is an ultra-discrete integrable system, or a soliton cellular automaton, or called a box-ball system in simplest cases, constructed from KR crystals. (See a nice review [13] for this topic.) There the bijection acquires an important physical meaning, separation of variables into action-angle variables. See [23].

Next case we should try to solve is type D. If B is a single KR crystal $B^{r,s}$, it is solved in [37]. We should go forward. Another case which we think promising is the case when B is a tensor product of the so called adjoint crystal [2]. For any affine algebra the adjoint crystal exists. For type $E_6^{(1)}$ a conjectural KKR type bijection is given in [29].

In [35] it was shown that KR crystals have a similarity property. Under the similarity map we expect that the bijection behaves in a simple way. This would also be an interesting problem to solve.

4.3. **Beyond the $X = M$ conjecture.** As we see in §1 the M side has an origin in Bethe Ansatz in physics. The X side also has an origin in physics, more precisely, Baxter's corner transfer matrix method in two-dimensional solvable lattice models [1]. Intriguingly, we can apply this method not only to KR modules but also to any finite-dimensional $U_q'(\mathfrak{g})$-modules, and experiments predict that for type A X for not necessarily KR modules coincides with Lascoux-Leclerc-Thibon (LLT) polynomials [25]. It is also known that for KR module cases LLT polynomials agree with the l.h.s. X of the $X = M$ conjecture [9]. It would be a challenging problem to show that X coincides with the LLT polynomial in full generality.

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