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On the Cauchy problem for differential equations with double characteristics and the strong Gevrey hyperbolicity

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1 Introduction

In [6], Ivrii and Petkov introduced the notion of fundamental matrix,[1] which now called the Hamilton map, and proved that if the Cauchy problem is C^{∞} well-posed for any lower order term then the characteristics are at most double and at every double characteristic point the Hamilton map has non-zero real eigenvalues, now called *effectively hyperbolic.* If the Hamilton map has no non-zero real eigenvalue, that is *noneffectively hyperbolic* case, they also proved, under some restrictions, in order that the Cauchy problem is C^{∞} well-posed the subprincipal symbol must lie in some interval on the real line, which depends on the reference double characteristic point. This was a breakthrough in researches on hyperbolic operators with multiple characteristics. They conjectured that effectively hyperbolic operator is strongly hyperbolic, that is if the Hamilton map has non-zero real eigenvalues at every double characteristic then the Cauchy problem is C^{∞} well-posed for any lower order term. This conjecture has been proved affirmatively in [9], [10], [11], [12]. On the other hand, the necessary condition for the C^{∞} well-posedness for noneffectively hyperbolic operators, mentioned above was completed in [5] by removing the restrictions and now called the Ivrii-Petkov-Hörmander condition (we abbreviate to IPH condition in the following).

Let P be a differential operator of order m with the principal symbol p. Then the Hamilton map F_p is the linearization of the Hamilton field H_p along the double characteristic set Σ, assumed to be a C^{∞} manifold. The positive trace Tr^+F_p is defined by $\mathrm{Tr}^+F_p=\sum\mu_j$ where $i\mu_j$ are the eigenvalues of F_p on the positive imaginary axis repeated according to their multiplicities. Now $\mathsf{Im}P_{sub}=0$, $|\mathsf{Re}P_{sub}|\leq\mathrm{Tr}^+F_p$ is the IPH condition.

[1] one of the authors of [6] told me the history of the word "fundamental matrix" as follows: At this time I was a grad student and among mathematical students we had the following definitions: *Derivative* of the drunken party is the party financed through deposit bottles. [i.e. if I remember correctly the cheap booze was 1.52 per bottle, while returning the bottle intact to the store one could recover 0.12, so multiplier was 12/152 and in order to be able to get one bottle in the second round one should consume 13 in the first.]

Fundamental drunken party is a party with non-zero second derivative.

If $\mathrm{Ker}F_p^2 \cap \mathrm{Im}F_p^2 = \{0\}$ on the doubly characteristic manifold Σ, the Cauchy problem for noneffectively hyperbolic operator P is C^∞ well-posed under the strict IPH condition which is a classical result proved in [8], [5]. When $\mathrm{Tr}^+ F_p = 0$ on Σ then the IPH condition is reduced to the Levi condition and is necessary and sufficient for the C^∞ well-posedness ([5]). Thus to understand the well-posedness of the Cauchy problem for differential operators with double characteristics the main remaining question is that when $\mathrm{Ker}F_p^2 \cap \mathrm{Im}F_p^2 \neq \{0\}$ on Σ whether we need new necessary conditions for the C^∞ well-posedness or not.

2 Noneffectively hyperbolic operators

It has been recognized that what is crucial to the C^∞ well-posedness is not only the Hamilton map but also the behavior of null bicharacteristics of p near the double characteristic manifold and the Hamilton map itself is not enough to determine completely the behavior of the null bicharacteristics. In the case $\mathrm{Ker}F_p^2 \cap \mathrm{Im}F_p^2 \neq \{0\}$ on Σ, strikingly enough, if there is a null bicharacteristic which lands tangentially on the double characteristic manifold then the Cauchy problem is not C^∞ well-posed even though we assume the Levi conditions, only well-posed in the Gevrey class $1 \leq s < 5$ as proved in [1]. On the other hand if there is no such null bicharacteristic then the above mentioned result still holds; the Cauchy problem is C^∞ well-posed under the strict IPH condition. If $\mathrm{Tr}^+ F_p = 0$ on Σ then the Levi condition is also necessary and sufficient for the C^∞ well-posedness of the Cauchy problem ([13]).

Here considering the following model operator we explain this rather striking phonomenon. Let us consider

$$P(x,D) = -D_0^2 + 2x_1 D_0 D_2 + D_1^2 + x_1^3 D_2^2. \tag{2.1}$$

It is worthwhile to note that if we make the change of coordinates

$$y_j = x_j,\ j = 0,1,\ \ y_2 = x_2 + x_0 x_1$$

which preserves the initial planes $x_0 = const.$, the operator P is written in these coordinates as

$$P = -D_0^2 + (D_1 + x_0 D_2)^2 + (x_1\sqrt{1+x_1}D_2)^2 = -D_0^2 + A^2 + B^2$$

which is of so called "divergence free" from. Here we have $A^* = A$ and $B^* = B$ while $[D_0, A] \neq 0$ and $[A, B] \neq 0$.

Let us denote by $p(x,\xi)$ the symbol of $P(x,D)$ then it is clear that the double characteristic manifold near the double characteristic point $\bar{\rho} = (0,(0,0,1)) \in \mathbb{R}^3 \times \mathbb{R}^3$ is given by

$$\Sigma = \{(x,\xi) \in \mathbb{R}^{2(n+1)} \mid \xi_0 = 0, x_1 = 0, \xi_1 = 0\}$$

and it is not difficult to see

$$\mathrm{Ker}\, F_p^2(\rho) \cap \mathrm{Im}\, F_p^2(\rho) \neq \{0\},\quad \rho \in \Sigma.$$

The main feature of p is that the Hamilton flow H_p lands tangentially on Σ. Indeed the integral curve of H_p

$$x_1 = -\frac{x_0^2}{4},\ x_2 = \frac{x_0^5}{8},\ \xi_0 = 0,\ \xi_1 = \frac{x_0^3}{8},\ \xi_2 = c \neq 0,\ |x_0| > 0 \tag{2.2}$$

parametrized by x_0 lands on Σ tangentially as $\pm x_0 \downarrow 0$.

We are now concerned with the Cauchy problem for P.

Definition 2.1 *We say that the Cauchy problem for P is locally solvable in the Gevrey class s at the origin if for any $\Phi = (u_0, u_1)$ taken in the Gevrey class s, there exists a neighborhood U_Φ of the origin such that the Cauchy problem*

$$\begin{cases} Pu = 0 \quad \text{in } U_\Phi, \\ D_0^j u(0,x') = u_j(x'),\ j = 0, 1,\ \ x' \in U_\Phi \cap \{x_0 = 0\} \end{cases}$$

has a solution $u(x) \in C^\infty(U_\Phi)$.

We can prove the next result following [1], modifying the argument there about the existence of zeros with "negative imaginary part" of some Stokes multiplier (see also [13]).

Theorem 2.1 *If $s > 5$ then the Cauchy problem for P is not locally solvable in the Gevrey class s. In particular the Cauchy problem for P is not C^∞ well-posed.*

Denoting $W = \operatorname{Im} F_p^2 \cap \operatorname{Ker} F_p^2$ the results about C^∞ well-posedness of the Cauchy problem for differential operators with double characteristics can be summarized in the following table:

eigenvalues of F_p	W	geometry of the null bicharacteristics near Σ	C^∞ well-posedness
there are non-zero real eigenvalues	$W = \{0\}$	at every point on Σ two null bicharacteristics intersect Σ transversally	C^∞ well-posed for any lower order term
there are no non-zero real eigenvalue	$W \neq \{0\}$	no null bicharacteristic with limit points in Σ	C^∞ well-posed under Levi (if $\mathrm{Tr}^+ F_p = 0$) or strict IPH condition (if $\mathrm{Tr}^+ F_p > 0$)
		exists a null bicharacteristic with limit points in Σ	Gevrey 5 well-posed with Levi condition (if $\mathrm{Tr}^+ F_p = 0$)(optimal). What happens when $\mathrm{Tr}^+ F_p > 0$?

The missing part in the table is

Assume that $W \neq \{0\}$ and there is a null bicharacteristic landing on Σ tangentially and $\mathrm{Tr}^+ F_p > 0$. Then what happens ?

We exhibit the main difficulty to answer this question by considering the following model operator

$$P(x,D) = -D_0^2 + 2x_1 D_0 D_2 + D_1^2 + x_1^3 D_2^2 + a(x_3^2 D_2^2 + D_3^2) \tag{2.3}$$

where $a > 0$ is a positive constant. It is easy to check that $\mathrm{Tr}^+ F_p = a$ and the double characteristic manifold is given by $\Sigma = \{\xi_0 = \xi_1 = \xi_3 = 0, x_1 = x_3 = 0\}$. Since $P_{sub} = 0$ the IPH condition is satisfied obviouly. If we define a curve $x_0 \mapsto (x'(x_0), \xi(x_0))$ where $(x_1(x_0), x_2(x_0), \xi_0(x_0), \xi_1(x_0), \xi_2(x_0))$ is given by (2.2) and $x_3(x_0) = \xi_3(x_0) = 0$ then this curve is a null bicharacteristic of p even for $a \neq 0$. From the view point of "classical mechanics" it is supposed that the non well-posedness of the Cauchy problem is caused by this singular orbit (2.2) of the Hamilton flow. On the other hand from the view point of "quantum mechanics" it is prohibited from taking $x_3 = 0$, $\xi_3 = 0$ by the Heisenberg uncertainty principle.

3 Strong Gevrey hyperbolicity

Let

$$P = P_m + P_{m-1} + \cdots + P_0$$

be a differential operator of order m where P_j denotes the homogeneous part of degree j. We denote $p(x,\xi) = P_m(x,\xi)$. Motivated by the Gevrey 5 well-posedness results in Section 2 we introduce the following definitions:

Definition 3.1 *Let $s \geq 1$. We say that P (or p) is strongly Gevrey s hyperbolic if for any differential operator Q of order less than m the Cauchy problem for $P + Q$ is locally solvable in the Gevrey class s.*

Definition 3.2 *We define the strong Gevrey hyperbolicity index $G(p)$ of p (or P) by*

$$G(p) = \sup\{s \mid P \text{ is strongly Gevrey } s \text{ hyperbolic}\}.$$

We now consider differential operators with double characteristics. We assume that the doubly characteristic set Σ is a C^∞ manifold of codimension 3. We also assume that

$$\begin{aligned} &\mathrm{rank}\,(d\xi \wedge dx) = \text{constant on } \Sigma, \\ &\text{either } W = \{0\} \text{ or } W \neq \{0\} \text{ throughout } \Sigma. \end{aligned} \tag{3.1}$$

Then the following table sums up a picture of the strong Gevrey hyperbolicity for differential operators with double characteristics ([2], [3]):

eigenvalues of F_p	W	geometry of the null bicharacteristics near Σ	$G(p)$
there are non-zero real eigenvalues	$W = \{0\}$	at every point on Σ two null bicharacteristics intersect Σ transversally	$G(p) = \infty$
there are no non-zero real eigenvalue	$W \neq \{0\}$	no null bicharacteristic with limit points in Σ	$G(p) = 4$
		exists a null bicharacteristic with limit points in Σ	$G(p) = 3$
	$W = \{0\}$	no null bicharacteristic with limit points in Σ	$G(p) = 2$

This implies that, assuming the condition (3.1), the strong Gevrey hyperbolicity index completely characterizes the spectral properties of the Hamilton map and the geometry of null bicharacteristics and vice versa for differential operators with double characteristics.

We turn to consider differential operators with characteristics of higher order. Let $\rho = (0, \bar{\xi})$ be a characteristic of order m. Then the localization of p at ρ is defined by

$$p(\rho + \mu X) = \mu^m \big(p_\rho(X) + o(1)\big), \; X = (x, \xi), \quad \mu \to 0$$

which is nothing but the first non-vanishing part in the Taylor expansion of p around ρ. Denote by Σ the set of characteristics of order m which is assumed to be a C^∞ manifold. Note that p_ρ is a function on $\mathbb{R}^{2(n+1)}/T_\rho\Sigma$ because $p_\rho(X+Y) = p_\rho(X)$ for any $Y \in T_\rho\Sigma$. If $m = 2$ then $p_\rho(X)$ is always strictly hyperbolic on $\mathbb{R}^{2(n+1)}/T_\rho\Sigma$. Taking this fact into account we assume that

$$\begin{aligned} &p_\rho \text{ is strictly hyperbolic in } \mathbb{R}^{2(n+1)}/T\Sigma, \\ &\operatorname{rank}(d\xi \wedge dx) = \text{constant on } \Sigma. \end{aligned} \tag{3.2}$$

A natural question is

For differential operators P with characteristics of order m (≥ 3) verifying (3.2) *the strong Gevrey hyperbolicity index $G(p)$ plays the same role as in the case $m = 2$?*

To investigate this question we first recall a classical result due to Bronshtein [4].

Theorem 3.1 ([4]) *Let P be a differential operator of order m with real characteristics. Then for any differential operator Q of order less than m, the Cauchy problem for $P + Q$ is well-posed in the Gevrey class $m/(m-1)$.*

This implies that for a differential operator P with characteristics of order m we have

$$G(p) \geq m/(m-1).$$

We also recall a result in [7] which bound $G(p)$ from above.

Theorem 3.2 ([7]) *Let P be a differential operator of order m with real analytic coefficients and let $\bar{\xi} = (0, ..., 0, 1) \in \mathbb{R}^{n+1}$. Assume that p verifies*

$$\partial_\xi^\alpha \partial_x^\beta p(0, \bar{\xi}) = 0 \ \ for \ \ |\alpha + \beta| < m, \quad \partial_{\xi_0}^m p(0, \bar{\xi}) \neq 0.$$

Then if the Cauchy problem for P is well-posed near the origin in the Gevrey class κ we have

$$\partial_\xi^\alpha \partial_x^\beta P_s(0, \bar{\xi}) = 0$$

for $|\alpha + \beta| < m - 2(m - s)\kappa/(\kappa - 1)$.

Assume that $(0, \bar{\xi})$ is a characteristic of order m. If P is strongly Gevrey κ hyperbolic then we have $\kappa \leq m/(m-2)$. Indeed if $\kappa > m/(m-2)$ and hence $m - 2\kappa/(\kappa - 1) > 0$ then from Theorem 3.2 it follows that for the Cauchy problem to be well-posed in the Gevrey class κ we have $P_{m-1}(0, \bar{\xi}) = 0$. That is one can not take P_{m-1} arbitrary so that P is not strongly Gevrey κ hyperbolic. This proves

$$G(p) \leq m/(m-2)$$

and hence

$$\frac{m}{m-1} \leq G(p) \leq \frac{m}{m-2}.$$

In a special case that $p(x, \xi) = q(x, \xi)^m$ where $q(x, D)$ is a first order differential operator it is known that

$$G(p) = \frac{m}{m-1}.$$

From the results for differential operators with double characteristics above it is natural to ask

Question 1 *Assume that* (3.2) *is verified. If* rank$(d\xi \wedge dx) = 0$ *on* Σ *then* $G(p) = m/(m-1)$*?.*

Question 2 *Assume that* (3.2) *is verified. If every null bicharacteristic is transversal to* Σ *then* $G(p) = m/(m-2)$*?.*

The next one seems to be much more difficult to answer.

Question 3 *Assume that* (3.2) *is verified. Then* $G(p)$ *takes only discrete values?.*

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