数理解析研究所講究録
第 1915 巻 2014 年 78-88

On (α, β, γ)-structurable algebras and Dynkin diagrams [1]
——Beyond Lie algebras to triple systems——

NORIAKI KAMIYA
Center for Mathematical Sciences, University of Aizu, 965-8580 Aizuwakamatsu, Japan
E-mail: kamiya@u-aizu.ac.jp

DANIEL MONDOC
Centre for Mathematical Sciences, Lund University, 22 100 Lund, Sweden
E-mail: Daniel.Mondoc@math.lu.se

Abstract. In this paper we introduce the notion of (α, β, γ)-structurable algebras, where $\alpha, \beta, \gamma \in \{-1, 0, 1\}$, and give examples of such structures.

AMS 2000 Mathematics subject classification: Primary 17A40; 17B60

Keywords: (anti-)structurable algebras, triple systems, Lie algebras, Lie superalgebras

1 Introduction

Our start point in a historical setting is the construction of Lie (super)algebras starting from a class of nonassociative algebras. Freudenthal ([18]), Tits ([65]), I.L. Kantor ([43]-[45]) and Koecher ([48]) studied constructions of Lie algebras from nonassociative algebras and triple systems, in particular Jordan algebras. B.N. Allison ([3], [4]) defined structurable algebras which are a class of nonassociative algebras with involution that include Jordan algebras (with trivial involution), associative algebras with involution, and alternative algebras with involution. They are related to generalized Jordan triple systems of 2nd order (or $(-1, 1)$-Freudenthal Kantor triple systems) as introduced and studied in [43], [44] and further studied in [5], [7], [42], [51]-[54], [62]. Their importance lies with constructions of 5-graded Lie algebras $L(\varepsilon, \delta) = \sum_{k=-2}^{2} L_k, [L_i, L_j] \subseteq L_{i+j}$, including the three gradings for the cases $L_{-2} = 0 = L_2$. Recently, we have studied constructions of Lie (super)algebras from triple systems and anti-structurable algebras ([33], [35], [38]-[41]).

As a continuation of [38], [39] we give here further generalization of structurable algebras. Hence within the general framework of (ε, δ)-Freudenthal Kantor triple systems $(\varepsilon, \delta = \pm 1)$ and the standard embedding Lie (super)algebra construction ([14],[15],[22]-[24], [30], [35], [49], [50], [61], [67]) we define (α, β, γ)-structurable algebras as a class of nonassociative algebras with involution which coincides with the class of structurable algebras for $(\alpha, \beta, \gamma) = (1, 1, -1)$. For $(\alpha, \beta, \gamma) = (1, -1, 1)$ the notion coincides with anti-structurable algebras ([38]) that may similarly shed light on of $(-1, -1)$-Freudenthal Kantor triple systems hence, by [14], [15], on the construction of Lie and Jordan superalgebras. While in this paper the

[1] This is an announcement note and the details will be published elsewhere

definition of (α, β, γ)-structurable algebras suppose an underling unital algebra structure we mention the construction of quasi δ-structurable algebras ([40]) when no assumption of existence of unit element is made.

Jordan and Lie (super)algebras ([17], [21]) play an important role in mathematics and physics ([11], [19]-[22], [24], [34], [37], [46], [58], [59], [64], [68], [69]) and the construction and characterization of these algebras can be expressed in terms of triple systems ([28], [32], [35], [36], [47], [60]) by the standard embedding method. Specially, we mention the connection between $N \leq 8$ 3-algebras (or triple systems) with N-supersymmetric 3-dimensional Chern-Simons gauge theories ([2],[8]-[10]) and Lie superalgebra constructions studied in terms of anti-Jordan triple systems and anti-Lie triple systems ([12], [13]).

2 Definitions and preamble, structures and examples

2.1 (ε, δ)-Freudenthal Kantor triple systems, δ-Lie triple systems, and Lie (super)algebras

In this paper triple systems have finite dimension over a field Φ of characteristic $\neq 2$ or 3.

A vector space V over a field Φ endowed with a trilinear operation $V \times V \times V \to V$, $(x, y, z) \longmapsto (xyz)$ is said to be a *GJTS of 2nd order* if the following conditions are fulfilled:

$$(ab(xyz)) = ((abx)yz) - (x(bay)z) + (xy(abz)), \tag{2.1}$$

$$K(K(a,b)x,y) - L(y,x)K(a,b) - K(a,b)L(x,y) = 0, \tag{2.2}$$

where $L(a,b)c := (abc)$ and $K(a,b)c := (acb) - (bca)$.

A *Jordan triple system* (for short JTS) satisfies (2.1) and the condition ([19])

$$(abc) = (cba). \tag{2.3}$$

while an *anti*-JTS satisfies (2.1) and the condition ([35])

$$(abc) = -(cba). \tag{2.4}$$

A *generalized Jordan triple system* (for short GJTS) satisfies only the condition (2.1).

We can generalize the concept of GJTS of 2nd order as follows (see [22], [23], [26]-[30], [67] and the earlier references therein).

For $\varepsilon = \pm 1$ and $\delta = \pm 1$, a triple product that satisfies the identities

$$(ab(xyz)) = ((abx)yz) + \varepsilon(x(bay)z) + (xy(abz)), \tag{2.5}$$

$$K(K(a,b)x,y) - L(y,x)K(a,b) + \varepsilon K(a,b)L(x,y) = 0, \tag{2.6}$$

where

$$L(a,b)c := (abc), \quad K(a,b)c := (acb) - \delta(bca), \tag{2.7}$$

is called an (ε, δ)-*Freudenthal Kantor triple system* (for short (ε, δ)-FKTS).

Remark. We note that $K(b,a) = -\delta K(a,b)$.

Let U be a GJTS of 2-nd order U and let $V_k, k = 1,2,3$, be subspaces of U. We denote by (V_1, V_2, V_3) the subspace of U spanned by elements $(x_1, x_2, x_3), x_k \in V_k, k = 1,2,3$. A subspace V of U is called an ideal of U if the following relations hold $(V, U, U) \subseteq V, (U, V, U) \subseteq V, (U, U, V) \subseteq V$. U is called *simple* if $(\, , \, , \,)$ is not a zero map and U has no nontrivial ideal.
Remark. The concept of GJTS of 2nd order coincides with that of $(-1,1)$-FKTS. Thus we can construct the simple Lie algebras by means of the standard embedding method ([14], [22]-[26], [30], [33], [35], [45], [67]).

An (ε, δ)-FKTS U is called *unitary* if the identity map Id is contained in $\kappa := K(U,U)$ i.e., if there exist $a_i, b_i \in U$, such that $\Sigma_i K(a_i, b_i) = Id$.

For $\delta = \pm 1$, a triple system $(a,b,c) \mapsto [abc], a,b,c \in V$ is called a *δ-Lie triple system* (for short δ-LTS) if the following three identities are fulfilled

$$\begin{aligned} [abc] &= -\delta[bac], \\ [abc] + [bca] + [cab] &= 0, \\ [ab[xyz]] &= [[abx]yz] + [x[aby]z] + [xy[abz]], \end{aligned} \tag{2.8}$$

where $a, b, x, y, z \in V$. An 1-LTS is a *LTS* while a -1-LTS is an *anti-LTS*, by [23].

Proposition 2.1 *([23],[30]) Let $U(\varepsilon,\delta)$ be an (ε,δ)-FKTS. If J is an endomorphism of $U(\varepsilon,\delta)$ such that $J < xyz > = < JxJyJz >$ and $J^2 = -\varepsilon\delta Id$, then $(U(\varepsilon,\delta), [xyz])$ is a LTS (if $\delta = 1$) or an anti-LTS (if $\delta = -1$) with respect to the product*

$$[xyz] := < xJyz > -\delta < yJxz > +\delta < xJzy > - < yJzx > . \tag{2.9}$$

Corollary 2.1 *Let $U(\varepsilon,\delta)$ be an (ε,δ)-FKTS. Then the vector space $T(\varepsilon,\delta) = U(\varepsilon,\delta) \oplus U(\varepsilon,\delta)$ becomes a LTS (if $\delta = 1$) or an anti-LTS (if $\delta = -1$) with respect to the triple product defined by*

$$\left[\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} \right] = \begin{pmatrix} L(a,d) - \delta L(c,b) & \delta K(a,c) \\ -\varepsilon K(b,d) & \varepsilon(L(d,a) - \delta L(b,c)) \end{pmatrix} \begin{pmatrix} e \\ f \end{pmatrix} \tag{2.10}$$

Thus we can obtain the standard embedding Lie algebra (if $\delta = 1$) or Lie superalgebra (if $\delta = -1$), $L(\varepsilon,\delta) = D(T(\varepsilon,\delta), T(\varepsilon,\delta)) \oplus T(\varepsilon,\delta)$, associated to $T(\varepsilon,\delta)$ where $D(T(\varepsilon,\delta), T(\varepsilon,\delta))$ is the set of inner derivations of $T(\varepsilon,\delta)$, i.e.

$$Der\ T := D(T(\varepsilon,\delta), T(\varepsilon,\delta)) := \left\{ \begin{pmatrix} L(a,b) & \delta K(c,d) \\ -\varepsilon K(e,f) & \varepsilon L(b,a) \end{pmatrix} \right\}_{span},$$

$$T(\varepsilon,\delta) := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| x, y \in U(\varepsilon,\delta) \right\}_{span}.$$

Remark. We note that $L(\varepsilon,\delta) := \sum_{k=-2}^{2} L_k$ is the 5-graded Lie (super)algebra, such that $T(\varepsilon,\delta) = L_{-1} \oplus L_1$ and $D(T(\varepsilon,\delta), T(\varepsilon,\delta)) = L_{-2} \oplus L_0 \oplus L_2$ with $[L_i, L_j] \subseteq L_{i+j}$. This Lie (super)algebra construction is one of the reasons to study nonassociative algebras and triple systems.

2.2 (α, β, γ)-structurable algebras with examples

Let $(\mathcal{A}, ^-)$ be a finite dimensional nonassociative unital algebra with involution (involutive anti-automorphism, i.e. $\overline{\overline{x}} = x, \overline{xy} = \overline{y}\,\overline{x}, x, y \in \mathcal{A}$) over Φ. The identity element of $\mathcal{A}$ is denoted by 1. Since char$\Phi \neq 2$, by [3] we have $\mathcal{A} = \mathcal{H} \oplus \mathbf{S}$, where $\mathcal{H} = \{a \in \mathcal{A} | \overline{a} = a\}$ and $\mathbf{S} = \{a \in \mathcal{A} | \overline{a} = -a\}$.

Suppose $x, y \in \mathcal{A}$ and put $[x, y] := xy - yx$. Let the operators L_x and R_x be defined by $L_x(y) := xy, R_x(y) := yx, x, y \in \mathcal{A}$ and for $\alpha, \beta, \gamma \in \{-1, 0, 1\}$ define

$$V_{x,y}^{\alpha,\beta,\gamma} := \alpha L_{L_x(\overline{y})} + \beta R_x R_{\overline{y}} + \gamma R_y R_{\overline{x}}, \tag{2.11}$$

$$B_{\mathcal{A}}^{\alpha,\beta,\gamma}(x, y, z) := V_{x,y}^{\alpha,\beta,\gamma}(z) = \alpha(x\overline{y})z + \beta(z\overline{y})x + \gamma(z\overline{x})y, x, y, z \in \mathcal{A}. \tag{2.12}$$

We call $B_{\mathcal{A}}^{\alpha,\beta,\gamma}(x, y, z)$ the *(α, β, γ)-triple system obtained from the algebra* $(\mathcal{A}, ^-)$ and write for short

$$V_{x,y} := V_{x,y}^{\alpha,\beta,\gamma}, \quad B_{\mathcal{A}} := (B_{\mathcal{A}}^{\alpha,\beta,\gamma}, \mathcal{A}). \tag{2.13}$$

We call an unital non-associative algebra with involution $(\mathcal{A}, ^-)$ an *(α, β, γ)-structurable algebra* if the following identity is fulfilled

$$[V_{u,v}, V_{x,y}] = V_{V_{u,v}(x),y} - V_{x,V_{v,u}(y)}, \quad u, v, x, y \in \mathcal{A}. \tag{2.14}$$

Remark. If $(\alpha, \beta, \gamma) = (1, 1, -1)$ then $(\mathcal{A}, ^-)$ coincides with the notion of *structurable algebra* ([3]). Then, by [44], the triple system $B_{\mathcal{A}}$ is a GJTS and by [16], $B_{\mathcal{A}}$ is a GJTS of 2nd order, i.e. satisfies the identities (2.5) and (2.6). Further, if $(\alpha, \beta, \gamma) = (1, -1, 1)$ then $(\mathcal{A}, ^-)$ coincides with the notion of *anti-structurable algebra* ([38]). If $(\mathcal{A}, ^-)$ is anti-structurable then we call $B_{\mathcal{A}}$ an *anti-GJTS.*

We give now examples of structurable, anti-structurable and $(1, 1, 0)$-structurable algebras over the fields $\mathbb{C}$ and $\mathbb{R}$, respectively and finish withe examples of $(1, -1, 0)$-structurable algebras over $\mathbb{C}$ and $(0, 1, 0)$-structurable algebras over an associative algebra with involution. We use the notations of [7], [42]. With regard to Lie algebras and Dynkin diagrams we refer to [63]. In the case of real Lie algebras we omit the corresponding Satake diagrams aiming for a shorter text. We also recall the following definitions.

A GJTS of 2-nd order is called *exceptional (classical)* if its embedding Lie algebra is exceptional (classical) Lie algebra.

Two GJTSs of 2-nd order (B, U) and (B', U') are called *weakly isomorphic* if there exists bijective linear maps $M, N : U \to U'$ such that $M(B(x, y, z)) = B'(M(x), N(y), M(z))'$, $x, y, z \in U$. A linear map $F : U \to U'$ is called a *homomorphism* if F satisfies the identity $F(B(x, y, z)) = B'(F(x), F(y), F(z))$, for all $x, y, z \in U$. Moreover, if F is bijective, then F is called an *isomorphism.* In this case the GJTSs of 2-nd order (B, U) and (B', U') are said to be *isomorphic.*

Let (B, U) be a GJTS of 2-nd order and let the linear endomorphisms $L_{x,y}, R_{x,y}$ on U be defined by $L_{x,y}(z) := B(x, y, z), R_{x,y}(z) := B(z, x, y), x, y, z \in U$. If (B, U) be a finite dimensional GJTS of 2-nd order then consider the symmetric bilinear form on U defined by $\gamma_B(x, y) := \frac{1}{2}Tr(2R_{x,y} + 2R_{y,x} - L_{x,y} - L_{y,x})$ ([66],[71]), where $Tr(f)$ means the trace of a linear endomorphism f. We shall call the form γ_B the *canonical (trace) form* for the GJTS of 2-nd order (B, U). A finite dimensional GJTS of 2-nd order (B, U) is called *compact* if its canonical form γ_B is positive definite.

We give first examples of structurable over the field $\mathbb{C}$ such that the corresponding standard embedding Lie algebra is simple classical 5-graded. By historical reasons ([44]) the corresponding standard embedding Lie algebra is denoted $\mathfrak{g} = \oplus_{l=-2}^{2}\mathfrak{g}_l$. We also denote

$M_{p,q}(\mathbb{C})$: the vector space of all $p \times q$ complex matrices,

$x^{\top}$: the transposed matrix of a matrix x,

x^*: the transposed conjugate matrix of x,

$\tilde{J}_n = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes J_n = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}$, where $J_n = (a_{i,j})$ is the matrix of order n such that $a_{i,j} = \delta_{i,n+1-j}$ and $\delta_{i,j}$ denotes the Kronecker's delta,

$Alt'_n(\mathbb{C}) = \{x \in M_{n,n}(\mathbb{C}) | x^{\top} J_n + J_n x = 0\}$.

By means of these notation, we have the following.

Proposition 2.2 ([7], [44], [71]) *Let $(U, \{\ \ \})$ be a classical complex simple GJTS of 2-nd order and $\mathfrak{g} = \oplus_{l=-2}^{2}\mathfrak{g}_l$ be the corresponding standard embedding Lie algebra. Then $(U, \{\ \ \})$ are classified (up to weak isomorphism) as follows*

$$\mathbf{1.}\ \mathfrak{g} = \mathfrak{sl}(n, \mathbb{C}), \quad \mathfrak{g}_{-1} = U = \begin{pmatrix} M_{p,q}(\mathbb{C}) \\ M_{q,r}(\mathbb{C}) \end{pmatrix}, \quad 1 \le p \le \left[\frac{n}{2}\right],\ p \le r,\ p+q+r = n,$$

$$\{XYZ\} = \left\{ \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} X_1 Y_1^* Z_1 + Z_1 Y_1^* X_1 - Z_1 X_2 Y_2^* \\ X_2 Y_2^* Z_2 + Z_2 Y_2^* X_2 - Y_1^* X_1 Z_2 \end{pmatrix} \right\},$$

$$\mathbf{2.}\ \mathfrak{g} = \mathfrak{so}(m, \mathbb{C}), \quad \mathfrak{g}_{-1} = U = M_{k,m-2k}(\mathbb{C}), \quad \begin{cases} \mathbf{a.}\ 2 \le k \le n,\ m = 2n+1, \\ \mathbf{b.}\ 2 \le k < n,\ m = 2n\ (n \ge 4), \end{cases}$$

$$\{XYZ\} = XY^*Z + ZY^*X - ZJ_{m-2k}X^{\top}\overline{Y}J_{m-2k},$$

$$\mathbf{3.}\ \mathfrak{g} = \mathfrak{so}(2n, \mathbb{C}), \quad \mathfrak{g}_{-1} = U = \begin{pmatrix} M_{1,n-1}(\mathbb{C}) \\ Alt'_{n-1}(\mathbb{C}) \end{pmatrix}, \quad n \ge 5,$$

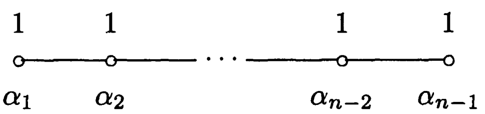
$$\left\{ \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} X_1 Y_1^* Z_1 + Z_1 Y_1^* X_1 - Z_1 X_2 Y_2^* \\ X_2 Y_2^* Z_2 + Z_2 Y_2^* X_2 - Y_1^* X_1 Z_2 - Z_2 J_{n-1} X_1^{\top} \overline{Y}_1 J_{n-1} \end{pmatrix} \right\},$$

$$\mathbf{4.}\ \mathfrak{g} = \mathfrak{sp}(n, \mathbb{C}), \quad \mathfrak{g}_{-1} = U = M_{k,2n-2k}(\mathbb{C}), \quad 1 \le k \le n-1\ (n \ge 3),$$

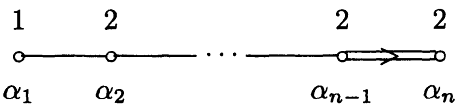
$$\{XYZ\} = XY^*Z + ZY^*X + Z\tilde{J}_{n-k}X^{\top}\overline{Y}\tilde{J}_{n-k}.$$

Moreover the corresponding Dynkin diagrams with the grading roots are given in Table 1.

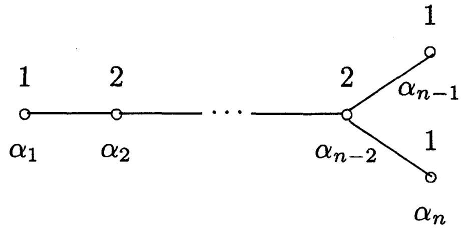
Table 1. Dynkin diagrams with coefficients of highest root and grading roots



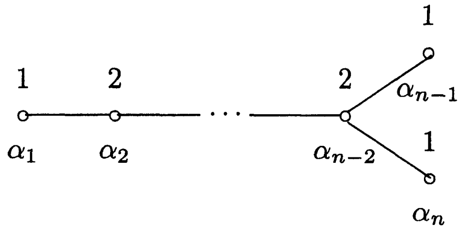
$\{\alpha_p, \alpha_{p+q}\}$ **1.**



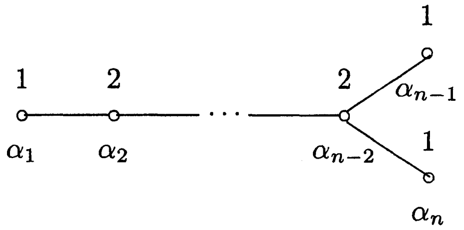
$\{\alpha_k\}$ **2a.**



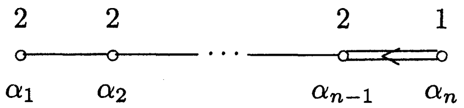
$\{\alpha_k\}$ $(2 \leq k < n-1)$ **2b.**



$\{\alpha_{n-1}, \alpha_n\}$ **2b.**



$\{\alpha_1, \alpha_n\}$ **3.**



$\{\alpha_k\}$ **4.**

Following [70], we can obtain the structurable algebras from the GJTS of 2nd order with a left unital element e such that $eex = x$ for all x.

Thus we have the following theorem (the details will be discussed in future).

Theorem 2.1 *Under the assumpsion as in above, we can obtain the structurable algebras as follows:*

1) case $p = q = r$, $e = (Id_p, Id_p)^T$, $\{\alpha_p,\ \alpha_{2p}\}$, $Der\ T = A_{p-1} \oplus A_{2p-1} \oplus \mathbb{C}$,

2) case $k = m - 2k$, $e = Id_k$, $\{\alpha_k\}$

$Der\ T = D_k \oplus B_{n-k}$ *if* $m = 2n+1$,

$Der\ T = D_k \oplus D_{n-k}$ *if* $m = 2n$,

3) case no example

4) case $k = 2n - 2k$. $e = Id_k$, $\{\alpha_k\}$

$Der\ T = C_k \oplus C_{n-k}$.

Proof.

Indeed, for example (2) (resp. (4)), the structurable algebra obtained from GJTS of 2nd order with e is given by

$$\{xyz\} = (x \cdot \overline{y}) \cdot z + (z \cdot \overline{y}) \cdot x - (z \cdot \overline{x}) \cdot y,$$

where this involution is defined by $\overline{x} = J_k x^T J_k$, (resp. $\overline{x} = -\tilde{J}_{n-k} x^T \tilde{J}_{n-k}$) and we denote the product of structurable by $x \cdot y$.(i.e., $x \cdot y = xy$ *by usual matrix product*).

Hence we have

$$\overline{x \cdot y} = \overline{y} \cdot \overline{x}, \quad \overline{\overline{x}} = x.$$

Also othere case has same proofs.

Remark. For examples of structurable over the field $\mathbb{C}$ such that the corresponding standard embedding Lie algebra is simple exceptional 5-graded we refer to [43], [44], [71] and we can get their examples of structurable by means of choise a left unit element e in the GJTS of 2nd order as same methods to Theorem 2.1.

For the JTS, we can obtain the $(1, 1, 0)$ structurable fom JTS. That is, we note that examples of structurable algebras induced from Jordan triple systems U w.r.t.

$$\{xyz\} = xy^* z + zy^* x$$

can be obtained same as in Theorem 2.1 by means of the element e such that $L(e, e)x = x$ for all $x \in U$.

We give now examples of anti-structurable algebras or $(1, -1, 1)$-structurable algebras over $\mathbb{C}$. Let $U = M_{m,n}(\mathbb{C})$ with the multiplication $\{xyz\} = x\overline{y}^{\top} z - z\overline{y}^{\top} x + z\overline{x}^{\top} y$. Then U is a $(-1, -1)$-FKTS and $U = M_{n,n}(\mathbb{C})$ is an anti-structurable algebra, by [41]. Then by [38], the following construction of Lie superalgebras is obtained by the standard embedding method. If $U = M_{2n,m}(\mathbb{C})$ with the product above, then the corresponding standard embedding Lie superalgebra is $osp(2n|2m) = D(n, m)$ (as defined by [17]), hence the standard embedding Lie superalgebra of the anti-structurable algebra $M_{2n,2n}(\mathbb{C})$ is $osp(2n|4n)$. Similarly, if $U = M_{2n+1,m}(\mathbb{C})$ with the product above, then the corresponding standard embedding Lie superalgebra is $osp(2n+1|2m) = B(n, m)$ (as defined by [17]), hence the standard embedding Lie superalgebra of the anti-structurable algebra $M_{2n+1,2n+1}(\mathbb{C})$ is $osp(2n + 1|4n + 2)$.

Finally, the last two examples refers to $(1, -1, 0)$ and $(0, 1, 0)$-structurable algebras. We start first with examples of $(1, -1, 0)$-structurable algebras over $\mathbb{C}$.

Let $U = M_{n,n}(\mathbb{C})$ with multiplication $\{xyz\} = xy^{\top} z - zy^{\top} x$, $x, y, z \in U$ and set $\{XYZ\} = \begin{pmatrix} 0 & -xy'z + zy'x \\ x'yz' - z'yx' & 0 \end{pmatrix}$, such that x, y, z are symmetric and x', y', z' are skew-symmetric.

Let V be the vector space $V = \left\{ X = \begin{pmatrix} 0 & x \\ x' & 0 \end{pmatrix} \in M_{2n,2n}(\mathbb{C}) \middle| x = x^{\top}, x' = -x'^{\top} \right\}$. Then, by [35], the product $\{XYZ\}$ is an anti-JTS. By [35], we introduce in V a new product $(XYZ) = \{XPYZ\} = \begin{pmatrix} 0 & xy'z - zy'x \\ x'yz' - z'yx' & 0 \end{pmatrix} \in V$, where $P \begin{pmatrix} 0 & -Id \\ Id & 0 \end{pmatrix}$. Then V with the product (XYZ) above is an anti-JTS and the standard embedding Lie superalgebra is $P(n - 1)$ as defined in [17] with a 3-graded structure ([35]).

Let now U be a $(0, 1, 0)$-structurable algebras with product $(xyz) = (z\overline{y})x$ and involution $^{-}$. If $U = A$ in an associative algebra thus $(xyz) = z\overline{y}x$ it is a straightforward calculation to show that (xyz) is a GJTS which is not of second order and not a JTS or anti-JTS.

Remark. For the $(-1, -1)$-FKTS $U := M_{k,k}(\Phi)$ with product $(xyz) = xy^{\top} z - zy^{\top} x + zx^{\top} y$ ([38]), since $K(x, y) = L(y, x) + L(x, y)$ we can easily show that there exists an almost complex structure on the associated anti-LTS $T(-1)$. Moreover, by [38], the standard embedding Lie superalgebra $L(U)$ corresponding to the $(-1, -1)$-FKTS above is $\mathfrak{osp}(k|2k)$

if $k = 2m$ or $k = 2m + 1$, respectively, but by [33], for a structurable algebra with product $(xyz) = xy^\top z + zy^\top x - zx^\top y$ does not exist an associated almost complex structure.

In end of this note,for an application to M-theory of physics, following[72], we note examples of hermitian (-1,-1) Freudenthal-Kantor triple systems as follows.

$a)$ $\{xyz\} = x\bar{y}^T z - z\bar{y}^T x$,

$b)$ $\{xyz\} = x\bar{y}^T z - z\bar{y}^T x + zx^T \bar{y}$,

where for all $x, y, z \in M_{m,n}(\mathbb{C})$.

These details will be descussed in future paper with Dr.M.Sato.

References

[1] Albert A.A., On Jordan algebras of linear transformations, Trans. Amer. Math. Soc. **59** (1946), 524-555.

[2] Aharony O.; O. Bergman O.; Jafferis D. L.; Maldacena J., $N = 6$ superconformal Chern- Simons-matter theories, $M2$-branes and their gravity duals, arXiv:0806.1218.

[3] Allison B.N., A class of nonassociative algebras with involution containing the class of Jordan algebras. Math. Ann. **237** (1978), no. 2, 133-156.

[4] Allison B.N., Models of isotropic simple Lie algebras. Comm. Algebra **7** (1979), no. 17, 1835-1875.

[5] Asano H.; Yamaguti K., A construction of Lie algebras by generalized Jordan triple systems of second order. Nederl. Akad. Wetensch. Indag. Math. **42** (1980), no. 3, 249-253.

[6] Asano H.; Kaneyuki S. On compact generalized Jordan triple systems of the second kind. Tokyo J. Math. **1988**, *11*, 105-118.

[7] Asano H., Classification of non-compact real simple generalized Jordan triple systems of the second kind. Hiroshima Math. J. **21** (1991), 463-489.

[8] Bagger J.; Bruhn G., Three-algebras in $N = 5, 6$ superconformal Chern-Simons theories: representations and relations, arXiv:1006.0040.

[9] Bagger J.; Lambert N., Gauge symmetry and supersymmetry of multiple $M2$-branes, arXiv:0711.0955.

[10] Bagger J.; Lambert N., Three-algebras and $N = 6$ Chern-Simons gauge theories, arXiv:0807.0163.

[11] Bertram W., Complex and quaternionic structures on symmetric spaces - correspondence with Freudenthal-Kantor triple systems. Theory of Lie Groups and Manifolds, ed. R. Miyaoka and H. Tamaru, Sophia Kokyuroku in Math. **45** (2002), 61-80.

[12] Cantarini N.; Kac V. G., Infinite dimensional primitive linearly compact Lie superalgebras, Adv. Math. **207** (2006), no. 1, 328-419.

[13] Cantarini N.; Kac V. G., Classification of linearly compact simple algebraic $N = 6$ 3-algebras, Transform. Groups **16** (2011), no. 3, 649-671.

[14] Elduque A.; Kamiya N.; Okubo S., Simple $(-1, -1)$ balanced Freudenthal Kantor triple systems. Glasg. Math. J. **11** (2003), no. 2, 353-372.

[15] Elduque A.; Kamiya N.; Okubo S., $(-1, -1)$ balanced Freudenthal Kantor triple systems and noncommutative Jordan algebras. J. Algebra **294** (2005), no. 1, 19-40.

[16] Faulkner J.R., Structurable triples, Lie triples, and symmetric spaces. Forum Math. **6** (1994), 637-650.

[17] Frappat L.; Sciarrino A.; Sorba P., *Dictionary on Lie Algebras and Superalgebras*, Academic Press, San Diego, California 92101-4495, USA, 2000, xxii+410 pp.

[18] Freudenthal H., Beziehungen der E_7 und E_8 zur Oktavenebene, I. Indag. Math. (1954), no. 16, 218-230.

[19] Jacobson N., Lie and Jordan triple systems, Amer. J. Math. **71** (1949), 149-170.

[20] Jacobson N., *Structure and representations of Jordan algebras.* Amer. Math. Soc. Colloq. Publ., Vol. XXXIX Amer. Math. Soc., Providence, R.I. 1968, x+453 pp.

[21] Kac V.G., Lie superalgebras. Advances in Math. **26** (1977), no. 1, 8-96.

[22] Kamiya N., A structure theory of Freudenthal-Kantor triple systems. J. Algebra **110** (1987), no. 1, 108-123.

[23] Kamiya N., A construction of anti-Lie triple systems from a class of triple systems, Mem. Fac. Sci. Shimane Univ. **22** (1988), 51-62.

[24] Kamiya N., A structure theory of Freudenthal-Kantor triple systems. II. Comment. Math. Univ. St. Paul. **38** (1989), no. 1, 41-60.

[25] Kamiya N., On (ε, δ)-Freudenthal-Kantor triple systems. *Nonassociative algebras and related topics* (Hiroshima, 1990), 65-75, World Sci. Publ., River Edge, NJ, 1991.

[26] Kamiya N. The construction of all simple Lie algebras over $\mathbb{C}$ from balanced Freudenthal-Kantor triple systems. *Contributions to general algebra*, 7 (Vienna, 1990), 205-213, Holder-Pichler-Tempsky, Vienna, 1991.

[27] Kamiya N. On Freudenthal-Kantor triple systems and generalized structurable algebras. *Non-associative algebra and its applications* (Oviedo, 1993), 198-203, Math. Appl., 303, Kluwer Acad. Publ., Dordrecht, 1994.

[28] Kamiya N., On the Peirce decompositions for Freudenthal-Kantor triple systems. Comm. Algebra **25** (1997), no. 6, 1833-1844.

[29] Kamiya N., On a realization of the exceptional simple graded Lie algebras of the second kind and Freudenthal-Kantor triple systems. Bull. Polish Acad. Sci. Math. 46 (1998), no. 1, 55-65.

[30] Kamiya N.; Okubo S., On δ-Lie supertriple systems associated with $(\varepsilon, \delta)-$ Freudenthal-Kantor supertriple systems. Proc. Edinburgh Math. Soc. **43** (2000), no. 2, 243-260.

[31] Kamiya N.; Okubo S., A construction of Jordan superalgebras from Jordan-Lie triple systems, *Lecture Notes in pure and appl. math.* 211, Nonass. alg. and its appl. (ed. Costa, Peresi, etc.), (2002), 171-176, Marcel Dekker Inc.

[32] Kamiya N.; Okubo S., A construction of simple Jordan superalgebra of F type from a Jordan-Lie triple system. Ann. Mat. Pura Appl. (4) **181** (2002), no. 3, 339-348.

[33] Kamiya N.; Okubo S., Construction of Lie superalgebras $D(2,1;\alpha), G(3)$ and $F(4)$ from some triple systems. Proc. Edinburgh Math. Soc. **46** (2003), no. 1, 87-98.

[34] Kamiya N.; Okubo S., On generalized Freudenthal-Kantor triple systems and Yang-Baxter equations, *Proc. XXIV Inter. Colloq. Group Theoretical Methods in Physics*, IPCS, vol. 173 (2003), 815-818.

[35] Kamiya N.; Okubo S., A construction of simple Lie superalgebras of certain types from triple systems. Bull. Austral. Math. Soc. **69** (2004), no. 1, 113-123.

[36] Kamiya N., Examples of Peirce decomposition of generalized Jordan triple system of second order-Balanced cases-. *Noncommutative geometry and representation theory in mathematical physics*, 157-165, Contemp. Math., 391, AMS, Providence, RI, 2005.

[37] Kamiya N.; Okubo S., Composition, quadratic, and some triple systems. *Non-associative algebra and its applications*, 205-231, Lect. Notes Pure Appl. Math., 246, Chapman & Hall/CRC, Boca Raton, FL, 2006.

[38] Kamiya N.; Mondoc D., A new class of nonassociative algebras with involution. Proc. Japan Acad. Ser. A **84** (2008), no. 5, 68-72.

[39] Kamiya N.; Mondoc D.; Okubo S., A structure theory of $(-1,-1)$-Freudenthal Kantor triple systems, Bull. Australian Math. Soc. **81** (2010), 132-155.

[40] Kamiya N.; Mondoc D.; Okubo S., A review on δ-structurable algebras, Proc. Alg. Geom. Math. Phys.-5, 2009 (Eds. V. Abramov, J. Fuchs, E. Paal, A. Stolin, A. Tralle, P. Urbanski), Banach Center Publ. **93** (2011), 59-67.

[41] Kamiya N.; Mondoc D., On anti-structurable algebras and extended Dynkin diagrams, J. Gen. Lie Theory Appl. **3** (2009), no. 3, 185-192.

[42] Kaneyuki S.; Asano H., Graded Lie algebras and generalized Jordan triple systems. Nagoya Math. J. **112** (1988), 81-115.

[43] Kantor I.L., Graded Lie algebras. Trudy Sem. Vect. Tens. Anal. **15** (1970), 227-266.

[44] Kantor I.L., Some generalizations of Jordan algebras. Trudy Sem. Vect. Tens. Anal. **16** (1972), 407-499.

[45] Kantor I.L. Models of exceptional Lie algebras. Soviet Math. Dokl. Vol. **14** (1973), no. 1, 254-258.

[46] Kantor I.L. A generalization of the Jordan approach to symmetric Riemannian spaces. *The Monster and Lie algebras* (Columbus, OH, 1996), 221-234, Ohio State Univ. Math. Res. Inst. Publ., 7, de Gruyter, Berlin, 1998.

[47] Kantor I.L.; Kamiya N., A Peirce decomposition for generalized Jordan triple systems of second order. Comm. Algebra **31** (2003), no. 12, 5875-5913.

[48] Koecher M., Embedding of Jordan algebras into Lie algebras I. II., Amer. J. Math **89** (1967), 787-816 and **90** (1968), 476-510.

[49] Lister W.G., A structure theory of Lie triple systems, Trans. Amer. Math. Soc. **72** (1952), 217-242.

[50] Meyberg K., Lectures on algebras and triple systems, *Lecture Notes*, The University of Virginia, Charlottesville, Va., 1972, v+226 pp.

[51] Mondoc D., Models of compact simple Kantor triple systems defined on a class of structurable algebras of skew-dimension one. Comm. Algebra **34** (2006), no. 10, 3801-3815.

[52] Mondoc D., On compact realifications of exceptional simple Kantor triple systems. J. Gen. Lie Theory Appl. **1** (2007), no. 1, 29-40.

[53] Mondoc D., Compact realifications of exceptional simple Kantor triple systems defined on tensor products of composition algebras. J. Algebra **307** (2007), no. 2, 917-929.

[54] Mondoc D., Compact exceptional simple Kantor triple systems defined on tensor products of composition algebras. Comm. Algebra **35** (2007), no. 11, 3699-3712.

[55] Neher E., Klassifikation der einfachen reellen speziellen Jordan-tripel-systeme. Manuscripta Math. **31** (1980), 197-215.

[56] Neher E., Klassifikation der einfachen reellen Ausnahme-Jordan-tripel-systeme. J. Reine Angew. Math. **322** (1981), 145-169.

[57] Neher E., On the classification of Lie and Jordan triple systems. Comm. Algebra **13** (1985), 2615-2667.

[58] Neher E., Jordan triple systems by the grid approach. *Lecture Notes in Mathematics*, 1280, Springer-Verlag, Berlin, 1987. xii+193 pp.

[59] Okubo S., Introduction to octonion and other non-associative algebras in physics. *Montroll Memorial Lecture Series in Mathematical Physics, 2.* Cambridge University Press, Cambridge, 1995, xii+136 pp.

[60] Okubo S.; Kamiya N., Jordan-Lie superalgebra and Jordan-Lie triple system. J. Algebra **198** (1997), no. 2, 388-411.

[61] Okubo S.; Kamiya N., Quasi-classical Lie superalgebras and Lie supertriple systems. Comm. Algebra **30** (2002), no. 8, 3825-3850.

[62] Okubo S., Symmetric triality relations and structurable algebras. Linear Algebra Appl. **396** (2005), 189-222.

[63] Osnishchik A.L.; Vinberg E.B., *Lie groups and algebraic groups*, Reference chapter: §2 Tables. Berlin: Springer-Verlag, 1990.

[64] Scheunert M., The theory of Lie superalgebras. An introduction. *Lecture Notes in Mathematics*, 716. Springer, Berlin, 1979, x+271 pp.

[65] Tits J., Une classe d'algèbres de Lie en relation avec les algèbres de Jordan. Nederl. Acad. Wetensch. Proc. Ser. A **65** (1962), 530-535.

[66] Yamaguti K. On the metasymplectic geometry and triple sytems. Kokyuroku, RIMS, Kyoto Univ. **1977**, *308*, 55-92 (in Japanese).

[67] Yamaguti K.; Ono A. On representations of Freudenthal-Kantor triple systems $U(\varepsilon, \delta)$, Bull. Fac. School Ed. Hiroshima Univ., **7** (1984), no. II, 43-51.

[68] Zhevlakov K.A.; Slinko A.M.; Shestakov I.P.; Shirshov A.I., Rings that are Nearly Associative, *Academic Press, Inc., New York-London*, 1982, xi+371 pp.

[69] Zelmanov E., Primary Jordan triple systems. Sibirsk. Mat. Zh. (1983), no. 4, 23-37.

[70] Elduque A., Kamiya N., Okubo S., left unital Kantor triple systems and structurable algebras, Linear and multilinear algebras,to appear.

[71] Kamiya N., A report of triple systems and nonassociative algebras, Aizu Univ. 2010,2012.

[72] Kamiya N, Sato M., hermitian triple systems and an application to physics, preprint.