数理解析研究所講究録
第 1918 巻 2014 年 161-176

SUMMARY OF STUDIES OF CLOSED/OPEN MIRROR SYMMETRY FOR QUINTIC THREEFOLDS THROUGH LOG MIXED HODGE THEORY

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0. Introduction and Statements

This is a summary of [U14p].

We correct the definitions and descriptions of the integral structures in our previous paper [U14]. We use $\hat{\Gamma}$-integral structure of Iritani in [I11] for A-model. Using the corrected version, we study open mirror symmetry for quintic threefolds through log mixed Hodge theory, especially the recent result on Néron models for admissible normal functions with non-torsion extensions in the joint work [KNU14] with K. Kato and C. Nakayama. We positively use integral structures of local systems with graded polarizations over the boundary points.

In a series of joint works with Kato and Nakayama, we are constructing a fundamental diagram which consists of various kind of partial compactifications of classifying space

2010 *Mathematics Subject Classification.* Primary 14C30; Secondary 14D07, 32G20.
[1]Partially supported by JSPS. KAKENHI (B) No. 23340008.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX of mixed Hodge structures and their relations. We try to understand Hodge theoretic aspects of mirror symmetry in this framework of the fundamental diagram.

Fundamental Diagram

For a classifying space D of Hodge structures of specified type, we have

$$
\begin{tikzcd}
 & & D_{\mathrm{SL}(2),\mathrm{val}} \arrow[r] \arrow[d] & D_{\mathrm{BS},\mathrm{val}} \arrow[d] \\
\Gamma\backslash D_{\Sigma,\mathrm{val}} \arrow[d] & D^{\sharp}_{\Sigma,\mathrm{val}} \arrow[l] \arrow[r] \arrow[d] & D_{\mathrm{SL}(2)} & D_{\mathrm{BS}} \\
\Gamma\backslash D_{\Sigma} & D^{\sharp}_{\Sigma} \arrow[l] & &
\end{tikzcd}
$$

in pure case: [KU99], [KU02], [KU09]. For mixed case, we should extend to an amplified diagram: [KNU08], [KNU09], [KNU11], [KNU13], continuing.

Mirror symmetry for quintic threefolds

Mirror symmetry for the A-model of quintic threefold V and the B-model of its mirror V° was predicted in the famous paper [CDGP91]. We recall two styles of the theorem (1) and (2) below. Every statement in the present paper is near the large radius point q_0 of the complexified Kähler moduli $\mathcal{KM}(V)$ and the maximally unipotent monodromy point p_0 of the complex moduli $\mathcal{M}(V^{\circ})$.

Let $t := y_1/y_0$, $u := t/2\pi i$ be the canonical parameters and $q := e^t = e^{2\pi i u}$ be the canonical coordinate from 2.2 below and the respective ones in 2.3 below.

The following theorem is due to Lian-Liu-Yau [LLuY97].

(1) (*Potential*). The potentials of the two models coincide: $\Phi^{V}_{\mathrm{GW}}(t) = \Phi^{V^{\circ}}_{\mathrm{GM}}(t)$.

The following theorem is formulated by Morrison [M97] and proved by Iritani [I11].
(2) (*Variation of Hodge structure*). The isomorphism $(q_0 \in \overline{\mathcal{KM}}(V)) \xleftarrow{\sim} (p_0 \in \overline{\mathcal{M}}(V^{\circ}))$ of neighborhoods of the compactifications, by the canonical coordinate $q = \exp(2\pi i u)$, lifts to an isomorphism, over the punctured neighborhoods $\mathcal{KM}(V) \xleftarrow{\sim} \mathcal{M}(V^{\circ})$, of polarized $\mathbf{Z}$-variations of Hodge structure with a specified section

$$(\mathcal{H}^{V}, S, \nabla^{\mathrm{even}}, \mathcal{H}^{V}_{\mathbf{Z}}, F; 1) \xleftarrow{\sim} (\mathcal{H}^{V^{\circ}}, Q, \nabla^{\mathrm{GM}}, \mathcal{H}^{V^{\circ}}_{\mathbf{Z}}, F; \tilde{\Omega}).$$

Our (3) below is equivalent to (1) and (2) by a log version [KU09, 2.5.14] of the nilpotent orbit theorem of Schmid [S73] (this part of [U14] is valid).

(3) (*Log Hodge structure, Log period map*). The isomorphism $(q_0 \in \overline{\mathcal{KM}}(V)) \xleftarrow{\sim} (p_0 \in \overline{\mathcal{M}}(V^{\circ}))$ of neighborhoods of the compactifications uniquely lifts to an isomorphism of B-model log variation of polarized Hodge structure with a specified section $\tilde{\Omega}$ for V° and A-model log variation of polarized Hodge structure with a specified section

1 for V, whose restriction over the punctured $\mathcal{KM}(V) \xleftarrow{\sim} \mathcal{M}(V^\circ)$ coincides with the isomorphism of variations of polarized Hodge structure with specified sections in (2).

This rephrases as follows. Let σ be the common monodromy cone, transformed by a level structure into End of a reference fiber of the local system, for the A-model and for the B-model. Then, we have a commutative diagram of horizontal log period maps

$$\begin{array}{ccc} (q_0 \in \overline{\mathcal{KM}}(V)) & \xleftarrow{\sim} & (p_0 \in \overline{\mathcal{M}}(V^\circ)) \\ & \searrow \quad \swarrow & \\ & ([\sigma, \exp(\sigma_{\mathbf{C}})F_0] \in \Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma) & \end{array}$$

with extensions of specified sections in (2), where $(\sigma, \exp(\sigma_{\mathbf{C}})F_0)$ is the nilpotent orbit, regarded as a boundary point, and $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$ is the fine moduli of log Hodge structures of specified type. (For fine moduli $\Gamma(\sigma)^{\mathrm{gp}}\backslash D_\sigma$, or more generally $\Gamma\backslash D_\Sigma$, see [KU09].)

Open mirror symmetry for quintic threefolds

The following theorem is due to Walcher [W07] and Morrison-Walcher [MW09].

(4) (*Inhomogenous solutions*).
Let $\mathcal{L}$ be the Picard-Fuchs differential operator for quintic mirror (cf. 2.2). Let

$$\mathcal{T}_A = \frac{u}{2} \pm \Big(\frac{1}{4} + \frac{1}{2\pi^2} \sum_{d \text{ odd}} n_d q^{d/2}\Big)$$

be the A-model domainwall tension in [MW09], and

$$\mathcal{T}_B = \int_{C_-}^{C_+} \Omega$$

be the B-model domainwall tension, where $C_\pm \subset V^\circ$ are the disjoint smooth curves coming from the two conics in $\{x_1 + x_2 = x_3 + x_4 = 0\} \cap V_\psi \subset \mathbf{P}^4(\mathbf{C})$ [ibid].
Then

$$\mathcal{L}(y_0(z)\mathcal{T}_A(z)) = \mathcal{L}(\mathcal{T}_B(z))\Big(= \frac{15}{16\pi^2}\sqrt{z}\Big) \quad \Big(z = \frac{1}{(5\psi)^5}\Big).$$

Concerning this, we have the following observations.

(5) (*Log mixed Hodge structure, Log normal function*). We describe for B-model. The same holds for A-model by (1)–(3) and the correspondence table in 2.5 below.

Put $\mathcal{H} := \mathcal{H}^{V^\circ}$ and $\mathcal{T} := \mathcal{T}_B$. We use $e^0 \in I^{0,0}$, $e^1 \in I^{1,1}$ which are a part of a basis of $\mathcal{H}_{\mathcal{O}^{\log}}$ respecting the Deligne decomposition at p_0 (see 2.5 (3B)) and a flat sections $s^0 = e^0$, $s^1 = e^1 - ue^0$ (see 2.5 (5B)). To make the local monodromy of $\mathcal{T}$ unipotent, we take a double cover $z^{1/2} \mapsto z$. Let $L_{\mathbf{Q}}$ be the translated local system from the trivial extension $\mathbf{Q} \oplus \mathcal{H}_{\mathbf{Q}}$ by $-(\mathcal{T}/y_0)s^0$ in $\mathcal{E}xt^1(\mathbf{Q}, \mathcal{H}_{\mathbf{Q}})$. Let $J_{L_{\mathbf{Q}}}$ be the Néron model on a neighborhood S of p_0 in the $z^{1/2}$-plane which lies over $L_{\mathbf{Q}}$ in [KNU14]. Then,

$J_{L_{\mathbf{Q}}} = \mathcal{E}xt^1_{\mathrm{LMH}/S}(\mathbf{Z}, \mathcal{H})$ (extension group of log mixed Hodge structures over S) in the present case ([KNU13, III, Corollary 6.1.6], cf. 1.4 below), and we have the following (5.1)–(5.3).

(5.1) The normalized tension $\mathcal{T}/y_0$ is understood as a truncated normal function by $(\mathcal{T}/y_0)s^0$. This extends as a truncated log normal function over the puncture. Then it lifts uniquely to a log normal function $S \to J_{L_{\mathbf{Q}}}$ so that the corresponding exact sequence $0 \to \mathcal{H} \to H \to \mathbf{Z} \to 0$ of log mixed Hodge structures over S is given by the liftings $1_{\mathbf{Z}}$ and 1_F in H of $1 \in \mathbf{Z} \simeq (\mathrm{gr}^W)_{\mathbf{Z}}$ respecting the lattice and the Hodge filtration, respectively, which are defined as follows: $1_{\mathbf{Z}} := 1 - (\mathcal{T}/y_0)s^0$ with $(\mathcal{T}/y_0)s^0 \in \mathcal{H}_{\mathcal{O}^{\log}} = (\mathrm{gr}^W_3)_{\mathcal{O}^{\log}}$, and $1_F - 1_{\mathbf{Z}} := -(\theta(\mathcal{T}/y_0))e^1 + (\mathcal{T}/y_0)e^0$.

(5.2) A splitting of the weight filtration W of the local system $H_{\mathbf{Z}}$, i.e., a splitting compatible with the monodromy of the local system $H_{\mathbf{Z}}$, is given by $1^{\mathrm{spl}}_{\mathbf{Z}} = 1_{\mathbf{Z}} + s^1/2$, and the log normal function over it is given by $1^{\mathrm{spl}}_F - 1^{\mathrm{spl}}_{\mathbf{Z}} = -(\theta(\mathcal{T}/y_0))e^1 + (\mathcal{T}/y_0)e^0$.

(5.3) (4) says that the inverse of the truncated normal function in (5.1) from its image is given by $16\pi^2/15$ times the Picard-Fuchs differential operator $\mathcal{L}$.

Some geometric backgrounds of (5) are explained in Section 3.
We treat Tate twists case by case in this article.

Acknowledgments. The author thanks to Kazuya Kato and Chikara Nakayama for series of joint works on log Hodge theory, from which he learns a lot and enjoys exciting studies. He thanks to Hiroshi Iritani for pointing out insufficient parts in the previous paper [U14]. He also thanks Yukiko Konishi and Satoshi Minabe, together with Iritani, for a stimulating seminar on the present topic from which Section 3 grew up.

1. Log mixed Hodge theory

In this section, we recall some notions and results of log mixed Hodge theory from [KU09], [KNU13] and [KNU14] adapting to the present context.

1.1. Category $\mathcal{B}(\log)$

Let S be a subset of an analytic space Z. The *strong topology of S in Z* is the strongest one among those topologies on S in which, for any analytic space A and any morphism $f : A \to Z$ with $f(A) \subset S$ as sets, $f : A \to S$ is continuous. S is regarded as a local ringed space by the pullback sheaf of $\mathcal{O}_Z$.

Let $\mathcal{B}$ be the category of local ringed spaces S over $\mathbf{C}$ which have an open covering $(U_\lambda)_\lambda$ satisfying the following condition: For each λ, there exist an analytic space Z_λ, and a subset S_λ of Z_λ such that, as local ringed space over $\mathbf{C}$, U_λ is isomorphic to S_λ which is endowed with the strong topology in Z_λ and the inverse image of $\mathcal{O}_{Z_\lambda}$.

A *log structure* on a local ringed space S is a sheaf of monoids M on S together with a homomorphisim $\alpha : M \to \mathcal{O}_S$ such that $\alpha^{-1}\mathcal{O}^\times_S \xrightarrow{\sim} \mathcal{O}^\times_S$. *fs* log structure means, locally on the underlying space, the log structure has a chart which is finitely generated, integral and saturated.

Let $\mathcal{B}(\log)$ be the category of objects of $\mathcal{B}$ endowed with an fs log structure (more precisely, cf. [KU09]).

1.2. Ringed space $(S^{\log}, \mathcal{O}_S^{\log})$

Let $S \in \mathcal{B}(\log)$. As a set define
$S^{\log} := \{(s,h) \mid s \in S,\ h : M_s^{\mathrm{gp}} \to \mathbf{S}^1 \text{ homomorphism s.t. } h(u) = u/|u| \ (u \in \mathcal{O}_{S,s}^{\times})\}$.
Endow $S^{\log}$ with the weakest topology such that the following two maps are continuous.
(1) $\tau : S^{\log} \to S, (s,h) \mapsto s$.
(2) For any open set $U \subset S$ and any $f \in \Gamma(U, M^{\mathrm{gp}})$, $\tau^{-1}(U) \to \mathbf{S}^1$, $(s,h) \mapsto h(f_s)$.

Then, τ is proper and surjective with fiber $\tau^{-1}(s) = (\mathbf{S}^1)^{r(s)}$, where $r(s)$ is the rank of $(M^{\mathrm{gp}}/\mathcal{O}_S^{\times})_s$ which varies with $s \in S$.

For $s \in S$ and $t \in S^{\log}$ lying over s, let $q_j \in M_s^{\mathrm{gp}}$ $(1 \le j \le r(s))$ be elements such that their images in $(M^{\mathrm{gp}}/\mathcal{O}_S^{\times})_s$ form a basis. Let $t_j := \log(q_j)$ and define $\mathcal{O}_{S,t}^{\log}$ to be a polynomial ring $\mathcal{O}_{S,s}[t_j\ (1 \le j \le r(s)]$ over $\mathcal{O}_{S,s}$. Thus $\tau : (S^{\log}, \mathcal{O}_S^{\log}) \to (S, \mathcal{O}_S)$ is a morphism of ringed spaces over $\mathbf{C}$ (more precisely, cf. [KU09]).

1.3. Graded polarized log mixed Hodge structure

Let $S \in \mathcal{B}(\log)$. A *pre-graded polarized log mixed Hodge structure on S* is a tuple $H = (H_{\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w, H_{\mathcal{O}})$ consisting of a local system of $\mathbf{Z}$-free modules $H_{\mathbf{Z}}$ of finite rank on $S^{\log}$, an increasing filtration W of $H_{\mathbf{Q}} := \mathbf{Q} \otimes H_{\mathbf{Z}}$, a nondegenerate $(-1)^w$-symmetric $\mathbf{Q}$-bilinear form $\langle\ ,\ \rangle_w$ on gr_w^W, a locally free $\mathcal{O}_S$-module $H_{\mathcal{O}}$ on S, a specified isomorphism $\mathcal{O}_S^{\log} \otimes_{\mathbf{Z}} H_{\mathbf{Z}} \simeq \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} H_{\mathcal{O}}$ (*log Riemann-Hilbert correspondence*), and a specified decreasing filtration $FH_{\mathcal{O}}$ of $H_{\mathcal{O}}$ such that $F^pH_{\mathcal{O}}$ and $H_{\mathcal{O}}/F^pH_{\mathcal{O}}$ are locally free. Put $F^p := \mathcal{O}_S^{\log} \otimes_{\mathcal{O}_S} F^pH_{\mathcal{O}}$. Then $\tau_*F^p = F^pH_{\mathcal{O}}$. For each integer w, the orthogonality condition $\langle F^p(\mathrm{gr}_w^W), F^q(\mathrm{gr}_w^W)\rangle_w = 0$ $(p+q > w)$ is imposed.

A pre-graded polarized log mixed Hodge structure on S is a *graded polarized log mixed Hodge structure on S* if its pullback to each $s \in S$ is a graded polarized log mixed Hodge structure on s in the following sense.

Let $(H_{\mathbf{Z}}, W, (\langle\ ,\ \rangle_w)_w, H_{\mathcal{O}})$ be a pre-graded polarized log mixed Hodge structure on a log point s. It is a *graded polarized log mixed Hodge structure* if it satisfies the following three conditions.

(1) (Admissibility). For each logarithm N of the local monodromy of the local system $(H_{\mathbf{R}}, W, (\langle\ ,\ \rangle_w)_w)$, there exists a W-relative N-filtration $M(N,W)$.

(2) (Griffiths transversality). For any integer p, $\nabla F^p \subset \omega_s^{1,\log} \otimes F^{p-1}$ is satisfied, where $\omega_s^{1,\log}$ is the sheaf of $\mathcal{O}^{\log}$-module of log differential 1-forms on $(s^{\log}, \mathcal{O}_s^{\log})$, and $\nabla = d \otimes 1_{H_{\mathbf{Z}}} : \mathcal{O}_s^{\log} \otimes H_{\mathbf{Z}} \to \omega_s^{1,\log} \otimes H_{\mathbf{Z}}$ is the log Gauss-Manin connection.

(3) (Positivity). For a point $t \in s^{\log}$ and a $\mathbf{C}$-algebra homomorphism $a : \mathcal{O}_{s,t}^{\log} \to \mathbf{C}$, define a filtration $F(a) := \mathbf{C} \otimes_{\mathcal{O}_{s,t}^{\log}} F_t$ on $H_{\mathbf{C},t}$. Then, $(H_{\mathbf{Z},t}(\mathrm{gr}_w^W), \langle\ ,\ \rangle_w, F(a))$ is a polarized Hodge structure of weight w in the usual sense if a is sufficiently twisted, i.e., for $(q_j)_{1 \le j \le n} \subset M_s$ inducing generators of $M_s/\mathcal{O}_s^{\times}$, $|\exp(a(\log q_j))| \ll 1$ for any j.

1.4. Néron model for admissible normal function

We review some results from [KNU14, Theorem 1.3], [KNU13, III, Section 6.1] and [KNU10, Section 8] adapted to the situation (5) in Introduction.

For a pure case $h^{p,q}=1$ $(p+q=3,\ p,q\geq 0)$ and $h^{p,q}=0$ otherwise, a complete fan is constructed in [KU09, Section 12.3]. For a mixed case $h^{p,q}=1$ (the above (p,q), plus $(p,q)=(2,2)$) and $h^{p,q}=0$ otherwise, over the above fan, a weak fan of Néron model for given admissible normal function is constructed in [KNU14, Theorem 3.1], and we have a Néron model in the following sense.

Let $S\in\mathcal{B}(\log)$, $U:=S_{\rm triv}\subset S$ (consisting of those points with trivial log structure), $H_{(-1)}$ be a polarized variation of Hodge structure of weight -1 (Tate-twisted by 2 from $\mathcal{H}$ in Introduction (5)) on U and $L_{\mathbf{Q}}$ be a local system of $\mathbf{Q}$-vector spaces which is an extension of $\mathbf{Q}$ by $H_{(-1),\mathbf{Q}}$. An admissible normal function over U for $H_{(-1)}$ underlain by the local system $L_{\mathbf{Q}}$ can be regarded as an admissible variation of mixed Hodge structure which is an extension of $\mathbf{Z}$ by $H_{(-1)}$ and lies over local system $L_{\mathbf{Q}}$.

For any given unipotent admissible normal function over U as above, $H_{(-1)}$ and $L_{\mathbf{Q}}$ extend to a polarized log mixed Hodge structure on S and a local system on $S^{\log}$, respectively, denoted by the same symbols, and there is a relative log manifold $J_{L_{\mathbf{Q}}}$ over S (cf. [KU09]) which is strict over S (i.e., endowed with the pullback log structure from S) and which represents the following functor on $\mathcal{B}/S^\circ$ ($S^\circ\in\mathcal{B}$ is the underlying space of S):

$S'\mapsto$ {LMH H on S' satisfying $H(\mathrm{gr}^W_w)=H_{(w)}|_{S'}$ $(w=-1,0)$ and $(*)$ below}/isom.

$(*)$ Locally on S', there is an isomorphism $H_{\mathbf{Q}}\simeq L_{\mathbf{Q}}$ on $(S')^{\log}$ preserving W.

Here $H_{(w)}|_{S'}$ is the pullback of $H_{(w)}$ by the structure morphism $S'\to S^\circ$, and S' is endowed with the pullback log structure from S.

Put $H':=H_{(-1)}$. In the present case, we have $J_{L_{\mathbf{Q}}}=\mathcal{E}\mathrm{xt}^1_{\mathrm{LMH}/S}(\mathbf{Z},H')$ by [KNU13, Corollary 6.1.6]. This is the subgroup of $\tau_*(H'_{\mathcal{O}^{\log}}/(F^0+H'_{\mathbf{Z}}))$ restricted by admissibility condition and log-point-wise Griffiths transversality condition ([KNU10, Section 8], cf. 1.3). Define $\bar{J}_{L_{\mathbf{Q}}}$ as the image of the composite map $J_{L_{\mathbf{Q}}}\to\tau_*(H'_{\mathcal{O}^{\log}}/(F^0+H'_{\mathbf{Z}}))\to\tau_*(H'_{\mathcal{O}^{\log}}/(F^{-1}+\mathcal{H}_{\mathbf{Z}})$. By using the polarization, we have a commutative diagram:

$$\begin{array}{ccccccc}
J_{L_{\mathbf{Q}}} & = & \mathcal{E}\mathrm{xt}^1_{\mathrm{LMH}/S}(\mathbf{Z},H') & \subset & \tau_*(H'_{\mathcal{O}^{\log}}/(F^0+H'_{\mathbf{Z}})) & \xrightarrow[\sim]{\mathrm{pol}} & \tau_*((F^0)^*/H'_{\mathbf{Z}}) \\
\downarrow & & & & \downarrow & & \downarrow \\
\bar{J}_{L_{\mathbf{Q}}} & & & \subset & \tau_*(H'_{\mathcal{O}^{\log}}/(F^{-1}+\mathcal{H}_{\mathbf{Z}}) & \xrightarrow[\sim]{\mathrm{pol}} & \tau_*((F^1)^*/H'_{\mathbf{Z}}).
\end{array}$$

2. Quintic threefolds

In this section, we give a correspondence table of A-model for quintic threefold and B-model for its mirror. This is a correction of our previous [U14, 3] by using $\hat{\Gamma}$-integral structure of Iritani [I11].

2.1. Quintic threefold and its mirror

Let V be a general quintic threefold in $\mathbf{P}^4$.

Let $V_\psi : f := \frac{1}{5}\sum_{j=1}^5 x_j^5 - \psi\prod_{j=1}^5 x_j = 0$ ($\psi \in \mathbf{P}^1$) be a pencil of quintics in $\mathbf{P}^4$. Let μ_5 be the group consisting of the fifth roots of the unity in $\mathbf{C}$. Then the group $G := \{(a_j) \in (\mu_5)^5 \,|\, a_1 \dots a_5 = 1\}$ acts on V_ψ by $x_j \mapsto a_j x_j$. Let V_ψ° be a crepant resolution of quotient singularity of V_ψ/G (cf. [MW09]). Divide further by the action $(x_1, \dots, x_5) \mapsto (a^{-1}x_1, x_2, \dots, x_5)$ ($a \in \mu_5$).

2.2. Picard-Fuchs equation on the mirror V°

Let Ω be a 3-form on V_ψ° with a log pole over $\psi = \infty$ induced from

$$\Big(\frac{5}{2\pi i}\Big)^3 \mathrm{Res}_{V_\psi}\Big(\frac{\psi}{f}\sum_{j=1}^{5}(-1)^{j-1}x_j dx_1 \wedge \cdots \wedge \widehat{dx_j} \wedge \wedge \cdots \wedge dx_5\Big).$$

Let $z := 1/(5\psi)^5$ and $\theta := zd/dz$. Let

$$\mathcal{L} := \theta^4 - 5z(5\theta+1)(5\theta+2)(5\theta+3)(5\theta+4)$$

be the Picard-Fuchs differential operator for Ω, i.e., $\mathcal{L}\Omega = 0$ via the Gauss-Manin connection ∇.

At $z = 0$, the Picard-Fuchs differential equation $\mathcal{L}y = 0$ has the indicial equation $\rho^4 = 0$ (ρ is indeterminate), i.e., maximally unipotent. By the Frobenius method, we have a basis of solutions $y_j(z)$ ($0 \le j \le 3$) as follows. Let

$$\tilde{y}(-z;\rho) := \sum_{n=0}^{\infty}\frac{\prod_{m=1}^{5n}(5\rho+m)}{\prod_{m=1}^{n}(\rho+m)^5}(-z)^{n+\rho}$$

be a solution of $\mathcal{L}(\tilde{y}(-z;\rho)) = \rho^4(-z)^\rho$, and let

$$\tilde{y}(-z;\rho) = y_0(z) + y_1(z)\rho + y_2(z)\rho^2 + y_3(z)\rho^3 + \cdots, \quad y_j(z) := \frac{1}{j!}\frac{\partial^i \tilde{y}(-z;\rho)}{\partial\rho^j}|_{\rho=0}$$

be the Taylor expansion at $\rho = 0$. Then, y_j ($0 \le j \le 3$) form a basis of solutions for the equation $\mathcal{L}y = 0$. We have

$$y_0 = \sum_{n=0}^{\infty}\frac{(5n)!}{(n!)^5}z^n,$$

$$y_1 = y_0 \log z + 5\sum_{n=1}^{\infty}\frac{(5n)!}{(n!)^5}\Big(\sum_{j=n+1}^{5n}\frac{1}{j}\Big)z^n.$$

Define the canonical parameters by $t := y_1/y_0$, $u := t/2\pi i$, and the canonical coordinate by $q := e^t = e^{2\pi i u}$ which is a specific chart of the log structure given by the divisor $(z = 0)$ of $\mathbf{P}^1$ and gives a mirror map.

y_0 is holomorphic in z and invertible at $z = 0$. Write $z = z(q)$ which is holomorphic in q. Then we have

$$\log z = 2\pi i u - \frac{5}{y_0(z(q))} \sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5} \Big(\sum_{j=n+1}^{5n} \frac{1}{j} \Big) z(q)^n.$$

The Gauss-Manin potential of V_z° is

$$\Phi_{\mathrm{GM}}^{V^\circ} = \frac{5}{2} \Big(\frac{y_1}{y_0} \frac{y_2}{y_0} - \frac{y_3}{y_0} \Big).$$

Let $\tilde{\Omega} := \Omega / y_0$. Then, the Yukawa coupling at $z = 0$ is

$$Y := - \int_{V^\circ} \tilde{\Omega} \wedge \nabla_\theta \nabla_\theta \nabla_\theta \tilde{\Omega} = \frac{5}{(1 + 5^5 z) y_0(z)^2}.$$

2.3. A-model of quintic V

Let V be a general quintic hypersurface in $\mathbf{P}^4$. Let $T^2 = H$ be the cohomology class of a hyperplane section of V in $\mathbf{P}^4$, $K(V) = \mathbf{R}_{>0} T^2$ be the Kähler cone of V, and u be the coordinate of $\mathbf{C}T^2$. Put $t := 2\pi i u$. A complexified Kähler moduli is defined as

$$\mathcal{KM}(V) := (H^2(V, \mathbf{R}) + iK(V))/H^2(V, \mathbf{Z}) \xrightarrow{\sim} \Delta^*, \quad uT^2 \mapsto q := e^{2\pi i u}.$$

Let $C \in H_2(V, \mathbf{Z})$ be the homology class of a line on V, and $T^1 \in H^4(V, \mathbf{Z})$ be the cohomology class Poincaré duality isomorphic to C.

For $\beta = dC \in H_2(V, \mathbf{Z})$, define $q^\beta := q^{\int_\beta T^1} = q^d$. The Gromov-Witten potential of V is defined as

$$\Phi_{\mathrm{GW}}^V := \frac{1}{6} \int_V (tT^2)^3 + \sum_{0 \neq \beta \in H_2(V, \mathbf{Z})} N_d q^\beta = \frac{5t^3}{6} + \sum_{d>0} N_d q^d.$$

Here the Gromov-Witten invariant N_d is

$$\overline{M}_{0,0}(\mathbf{P}^4, d) \xleftarrow{\pi_1} \overline{M}_{0,1}(\mathbf{P}^4, d) \xrightarrow{e_1} \mathbf{P}^4,$$
$$N_d := \int_{\overline{M}_{0,0}(\mathbf{P}^4, d)} c_{5d+1}(\pi_{1*} e_1^* \mathcal{O}_{\mathbf{P}^4}(5)).$$

Note that $N_d = 0$ if $d \leq 0$. Let $N_d = \sum_{k|d} n_{d/k} k^{-3}$. Then $n_{d/k}$ is the instanton number.

2.4. Integral structure

Let S^* be $\mathcal{KM}(V)$ for A-model of V and $\mathcal{M}(V^\circ)$ for B-model for V°, and let S be $\overline{\mathcal{KM}}(V)$ for A-model and $\overline{\mathcal{M}}(V^\circ)$ for B-model (see 2.2, 2.3). Endow S with the log structure associated to the divisor $S \setminus S^*$.

The B-model variation of Hodge structure $\mathcal{H}^{V^\circ}$ is the usual variation of Hodge structure arising from the smooth projective family $f : X \to S^*$ of the quintic mirrors over a punctured neighborhood of the maximally unipotent monodromy point p_0. Its integral structure is the usual one $\mathcal{H}^{V^\circ}_{\mathbf{Z}} = R^3 f_* \mathbf{Z}$. This is compatible with the monodromy weight filtration M around p_0. Define $M_{k,\mathbf{Z}} := M_k \cap \mathcal{H}^{V^\circ}_{\mathbf{Z}}$ for all k.

For the A-model $\mathcal{H}^V$ on S^*, the locally free sheaf on S^*, the Hodge filtration, and the monodromy weight filtration M around the large radius point q_0 are given by $\mathcal{H}^V_{\mathcal{O}} := \mathcal{O}_{S^*} \otimes (\bigoplus_{0 \le p \le 3} H^{2p}(V))$, $F^p := \mathcal{O}_{S^*} \otimes H^{\le 2(3-p)}(V)$, and $M_{2p} := H^{\ge 2(3-p)}(V)$, respectively. Iritani defined $\hat{\Gamma}$-integral structure in more general setting in [I11, Definition 3.6]. In the present case, it is characterized as follows. Let H and C be a hyperplane section and a line on V, respectively. Then, in the present case, a basis of the $\hat{\Gamma}$-integral structure is given by $\{s(\mathcal{E}) \mid \mathcal{E} \text{ is } \mathcal{O}_V, \mathcal{O}_H, \mathcal{O}_C, \mathcal{O}_{\mathrm{pt}}\}$ [ibid, Example 6.18], where $s(\mathcal{E})$ is a unique ∇^{even}-flat section satisfying

$$s(\mathcal{E}) \sim (2\pi i)^{-3} e^{-2\pi i u H} \cdot \hat{\Gamma}(T_V) \cdot (2\pi i)^{\deg/2} \operatorname{ch}(\mathcal{E})$$

at the large radius point q_0. Here, for the Chern roots $c(T_V) = \prod_{j=1}^{3}(1+\delta_j)$, the Gamma class $\hat{\Gamma}(T_V)$ is defined by

$$\begin{aligned}\hat{\Gamma}(T_V) := \prod_{j=1}^{3} \Gamma(1+\delta_j) &= \exp(-\gamma c_1(V) + \sum_{k \ge 2} (-1)^k (k-1)! \zeta(k) \operatorname{ch}_k(T_V) \\ &= \exp(\zeta(2) \operatorname{ch}_2(T_V) - 2\zeta(3) \operatorname{ch}_3(T_V))\end{aligned}$$

where γ is the Euler constant, and $\deg|_{H^{2p}(V)} := 2p$. The important point is that this class $\hat{\Gamma}(T_V)$ plays the role of a "square root" of the Todd class in Hirzebruch-Riemann-Roch ([I09, 1], [I11, 1, (13)]). Denote this $\hat{\Gamma}$-integral structure by $\mathcal{H}^V_{\mathbf{Z}}$. This is compatible with the monodromy weight filtration M and we define $M_{k,\mathbf{Z}} := M_k \cap \mathcal{H}^V_{\mathbf{Z}}$ for all k. For a direct definition of $\hat{\Gamma}$-integral structure, see [I11, Definition 3.6].

In both A-model case and B-model case, the integral structures $\mathcal{H}^V_{\mathbf{Z}}$ and $\mathcal{H}^{V^\circ}_{\mathbf{Z}}$ on S^* extend to the local systems of $\mathbf{Z}$-modules over $S^{\log}$ ([O03], [KU09, Proposition 2.3.5]), still denoted $\mathcal{H}^V_{\mathbf{Z}}$ and $\mathcal{H}^{V^\circ}_{\mathbf{Z}}$, respectively.

Consider a diagram:

$$\begin{tikzcd}
\tilde{S}^{\log} := (\mathbf{R} \times i(0,\infty])^r \arrow[d] & \supset & \tilde{S}^* := (\mathbf{R} \times i(0,\infty))^r \arrow[d] \\
S^{\log} \arrow[d, "\tau"'] & \supset & S^* \\
S & &
\end{tikzcd}$$

The coordinate u of $\tilde{S}^*$ extends over $\tilde{S}^{\log}$. Fix base points as $u_0 = 0 + i\infty \in \tilde{S}^{\log} \mapsto b := \bar{0} + i\infty \in S^{\log} \mapsto q = 0 \in S$, where $q = 0$ corresponds to q_0 for A-model and p_0

for B-model. Note that fixing a base point $u = u_0$ on $\tilde{S}^{\log}$ is equivalent to fixing a base point b on $S^{\log}$ and also a branch of $(2\pi i)^{-1}\log q$.

Let $B := \mathcal{H}^V_{\mathbf{Z}}(u_0) = \mathcal{H}^V_{\mathbf{Z}}(b)$ for A-model and $B := \mathcal{H}^{V^\circ}_{\mathbf{Z}}(u_0) = \mathcal{H}^{V^\circ}_{\mathbf{Z}}(b)$ for B-model.

2.5. Correspondence table

In this section, we complete the approximation in the previous paper [U14]. These results will be used in Section 3.

We use (1) and (2) in Introduction. Put $\Phi := \Phi^V_{\mathrm{GW}} = \Phi^{V^\circ}_{\mathrm{GM}}$.

(1A) *Polarization of A-model of V.*

$$S(\alpha,\beta) := (-1)^p \int_V \alpha \cup \beta \quad (\alpha \in H^{p,p}(V), \beta \in H^{3-p,3-p}(V)).$$

(1B) *Polarization of B-model of V°.*

$$Q(\alpha,\beta) := (-1)^{3(3-1)/2} \int_{V^\circ} \alpha \cup \beta = -\int_{V^\circ} \alpha \cup \beta \quad (\alpha, \beta \in H^3(V^\circ)).$$

(2A) $\mathbf{Z}$*-basis compatible with monodromy weight filtration.*

Let $B := \mathcal{H}^V_{\mathbf{Z}}(u_0) = \mathcal{H}^V_{\mathbf{Z}}(b)$. Then we have a basis b^0, b^1, b^2, b^3 of B compatible with the monodromy weight filtration M [I11, Example 6.18].

(2B) $\mathbf{Z}$*-basis compatible with monodromy weight filtration.*

Let $B := \mathcal{H}^{V^\circ}_{\mathbf{Z}}(u_0) = \mathcal{H}^{V^\circ}_{\mathbf{Z}}(b)$. Then we have a basis b^0, b^1, b^2, b^3 of B compatible with the monodromy weight filtration M [ibid].

For both cases (2A) and (2B), we regard B as a constant sheaf endowed with M on $S^{\log}$ and also on S.

(3A) *Specified sections inducing* $\mathbf{Z}$*-basis of* gr^M *for A-model of V.*

$$T^3 := 1 \in H^0(V,\mathbf{Z}), \quad T^2 := H \in H^2(V,\mathbf{Z}),$$
$$T^1 := C \in H^4(V,\mathbf{Z}), \quad T^0 := [\mathrm{pt}] \in H^6(V,\mathbf{Z}),$$

where H is a hyperplane section of V and C is a line on V. Then $S(T^3,T^0) = 1$ and $S(T^2,T^1) = -1$. Hence T^3, T^2, $-T^0$, T^1 form a symplectic base for S in (1A).

(3B) *Specified sections inducing* $\mathbf{Z}$*-basis of* gr^M *for B-model of V°.*

We use Deligne decomposition [D97]. We consider B in (2B) as a constant sheaf on $S^{\log}$. We have locally free $\mathcal{O}_S$-submodules $\mathcal{M}_{2p} := \tau_*(\mathcal{O}^{\log}_S \otimes_{\mathbf{Z}} M_{2p}B)$ and $\mathcal{F}^p$ of $\tau_*(\mathcal{O}^{\log}_S \otimes_{\mathbf{Z}} B) = \mathcal{O}_S \otimes_{\mathbf{Z}} B$. The mixed Hodge structure of Hodge-Tate type $(\mathcal{M},\mathcal{F})$ has decomposition:

$$\mathcal{O}_S \otimes_{\mathbf{Z}} B = \bigoplus_p I^{p,p}, \qquad I^{p,p} := \mathcal{M}_{2p} \cap \mathcal{F}^p \xrightarrow{\sim} \mathrm{gr}^{\mathcal{M}}_{2p}.$$

Transporting the basis b^p $(0 \leq p \leq 3)$ of B in (2B), regarded as sections of the constant sheaf B on $S^{\log}$, via isomorphism

$$I^{p,p} \xrightarrow{\sim} \mathcal{O}_S \otimes_{\mathbf{Z}} \mathrm{gr}^M_{2p} B$$

we define sections $e^p \in I^{p,p}$ $(0 \leq p \leq 3)$. Then e^3, e^2,$-e^0$, e^1 form a symplectic basis for Q in (1B).

Note that $e^3 = \tilde{\Omega}$.

(4A) *A-model connection* $\nabla = \nabla^{\mathrm{even}}$ *of* V.

$$\begin{aligned}
&\nabla_\theta T^0 := 0, \quad \nabla_\theta T^1 := T^0,\\
&\nabla_\theta T^2 := \frac{1}{(2\pi i)^3}\frac{d^3\Phi}{du^3}T^1 = \Big(5 + \frac{1}{(2\pi i)^3}\frac{d^3\Phi_{\mathrm{hol}}}{du^3}\Big)T^1,\\
&\nabla_\theta T^3 := T^2.
\end{aligned}$$

∇ is flat, i.e., $\nabla^2 = 0$.

(4B) *B-model connection* $\nabla = \nabla^{\mathrm{GM}}$ *of* V°.

$$\begin{aligned}
&\nabla_\theta e^0 = 0, \quad \nabla_\theta e^1 = e^0,\\
&\nabla_\theta e^2 = \frac{1}{(2\pi i)^3}\frac{d^3\Phi}{du^3}e^1 = Ye^1 = \frac{5}{(1+5^5)y_0(z)^2}\Big(\frac{q}{z}\frac{dz}{dq}\Big)^3 e^1,\\
&\nabla_\theta e^3 = e^2.
\end{aligned}$$

(5A) ∇*-flat* $\mathbf{Z}$*-basis for* $\mathcal{H}^V_{\mathbf{Z}}$.

$$\begin{aligned}
s^0 &:= T^0,\\
s^1 &:= T^1 - uT^0,\\
s^2 &:= T^2 - \Big(\frac{1}{(2\pi i)^3}\frac{\partial^2\Phi}{\partial u^2} - \frac{11}{2}\Big)T^1 + \Big(\frac{1}{(2\pi i)^3}\frac{\partial\Phi}{\partial u} - \frac{11}{2}u - \frac{25}{12}\Big)T^0,\\
s^3 &:= T^3 - uT^2 + \Big(\frac{1}{(2\pi i)^3}\Big(u\frac{\partial^2\Phi}{\partial u^2} - \frac{\partial\Phi}{\partial u}\Big) - \frac{25}{12}\Big)T^1\\
&\quad - \Big(\frac{1}{(2\pi i)^3}\Big(u\frac{\partial\Phi}{\partial u} - 2\Phi\Big) - \frac{25}{12}u - \frac{25i}{\pi^3}\zeta(3)\Big)T^0.
\end{aligned}$$

Then s^3, s^2,$-s^0$, s^1 form a symplectic basis for S in (1A).

(5B) ∇*-flat* $\mathbf{Z}$*-basis for* $\mathcal{H}^{V^\circ}_{\mathbf{Z}}$.

$$\begin{aligned}
s^0 &:= e^0,\\
s^1 &:= e^1 - ue^0,\\
s^2 &:= e^2 - \Big(\frac{1}{(2\pi i)^3}\frac{\partial^2\Phi}{\partial u^2} - \frac{11}{2}\Big)e^1 + \Big(\frac{1}{(2\pi i)^3}\frac{\partial\Phi}{\partial u} - \frac{11}{2}u - \frac{25}{12}\Big)e^0,\\
s^3 &:= e^3 - ue^2 + \Big(\frac{1}{(2\pi i)^3}\Big(u\frac{\partial^2\Phi}{\partial u^2} - \frac{\partial\Phi}{\partial u}\Big) - \frac{25}{12}\Big)e^1\\
&\quad - \Big(\frac{1}{(2\pi i)^3}\Big(u\frac{\partial\Phi}{\partial u} - 2\Phi\Big) - \frac{25}{12}u - \frac{25i}{\pi^3}\zeta(3)\Big)e^0.
\end{aligned}$$

Then s^3, s^2,$-s^0$, s^1 form a symplectic basis for Q in (1B).

(6A) *Expression of the T^p by the s^p.*
It is computed that T^p are written by the ∇-flat $\mathbf{Z}$-basis s^p of $\mathcal{H}^V_{\mathbf{Z}}$ as follows.

$$
\begin{aligned}
T^0 &= s^0,\\
T^1 &= s^1 + us^0,\\
T^2 &:= s^2 + \Big(\frac{1}{(2\pi i)^3}\frac{\partial^2\Phi}{\partial u^2} - \frac{11}{2}\Big)s^1 + \Big(\frac{1}{(2\pi i)^3}\Big(u\frac{\partial^2\Phi}{\partial u^2} - \frac{\partial\Phi}{\partial u}\Big) + \frac{25}{12}\Big)s^0,\\
T^3 &= s^3 + us^2 + \Big(\frac{1}{(2\pi i)^3}\frac{\partial\Phi}{\partial u} - \frac{11}{2}u + \frac{25}{12}\Big)s^1\\
&\quad + \Big(\frac{1}{(2\pi i)^3}\Big(u\frac{\partial\Phi}{\partial u} - 2\Phi\Big) + \frac{25}{12}u - \frac{25i}{\pi^3}\zeta(3)\Big)s^0.
\end{aligned}
$$

Note that the section $1 = T^3$ varies with respect to the the lattice $\mathcal{H}^V_{\mathbf{Z}}$ as above while the section $[\mathrm{pt}] = T^0 = s^0$ does not.

(6B) *Expression of the e^p by the s^p.*
It is computed that e^p are written by the ∇-flat $\mathbf{Z}$-basis s^p of $\mathcal{H}^{V^\circ}_{\mathbf{Z}}$ as follows.

$$
\begin{aligned}
e^0 &= s^0,\\
e^1 &= s^1 + us^0,\\
e^2 &:= s^2 + \Big(\frac{1}{(2\pi i)^3}\frac{\partial^2\Phi}{\partial u^2} - \frac{11}{2}\Big)s^1 + \Big(\frac{1}{(2\pi i)^3}\Big(u\frac{\partial^2\Phi}{\partial u^2} - \frac{\partial\Phi}{\partial u}\Big) + \frac{25}{12}\Big)s^0,\\
e^3 &= s^3 + us^2 + \Big(\frac{1}{(2\pi i)^3}\frac{\partial\Phi}{\partial u} - \frac{11}{2}u + \frac{25}{12}\Big)s^1\\
&\quad + \Big(\frac{1}{(2\pi i)^3}\Big(u\frac{\partial\Phi}{\partial u} - 2\Phi\Big) + \frac{25}{12}u - \frac{25i}{\pi^3}\zeta(3)\Big)s^0.
\end{aligned}
$$

Note that the normalized holomorphic 3-form $\tilde{\Omega} = \Omega/y_0 = e^3$ varies with respect to the lattice $\mathcal{H}^{V^\circ}_{\mathbf{Z}}$ as above, while the section $e^0 = s^0$ does not.

Idea of proof of (4A) and (4B). We prove (4B). (4A) follows by mirror symmetry theorems (1) and (2) in Introduction.

We improve the proof of [CoK99, Prop. 5.6.1] carefully by a log Hodge theoretic understanding of the relation among a constant sheaf and a local system on $S^{\log}$, of the canonical extension of Deligne on S, and of the Deligne decomposition.

Idea of proofs of (5A), (5B), (6A) and (6B). In [I11, Introduction] (cf. 2.4), the asymptotic condition in the large radius limit is stated for the flat integral section corresponding to $\mathcal{E} = \mathcal{O}_V \in K(V)$ in the situation (5A). Up to Tate twists, this condition coincides with the one in [CDGP91, (5.5)] stated in the situation (6A). By the mirror symmetry in [I11] (cf. (2) in Introduction), this condition is interpreted in the situation (6B). Our previous results in [U14, Sections 3.5–3.6] are insufficient (see Remark below). In order to complete them, we compute here higher approximations in the situation (6B). The result in the situation (5B) is a linear algebraic solution of this.

Remark. The author was pointed out by Hiroshi Iritani that the definitions and the descriptions of integral structures in [U14, 3.5, 3.6] are insufficient. Actually, they were the first approximations of integral structures by means of gr^M, and the second proof in [ibid, 3.9] works well even in this approximation.

3. Discussions on geometries for (5) in Introduction

We discuss here the relation with geometries and local systems considered in [W07] and [MW09]. Forgetting Hodge structures, we consider only local systems corresponding to the monodromy of integral periods and tensions.

Let V_ψ and V_ψ° be a quintic threefold and its mirror from 2.1. Let S be a small neighborhood in the z-plane (z in 2.2) of the maximal unipotent monodromy point p_0 endowed with the log structure associated to the divisor p_0.

We first consider B-model. Let the setting be as in [MW09, 4]. For $z \neq 0$ near 0, i.e., near p_0, let V_z° be the mirror quintic and $C_{+,z} \cup C_{-,z}$ be the disjoint union of smooth rational curves on V_z° coming from the two conics contained in $V_\psi \cap \{x_1 + x_2 = x_3 + x_4 = 0\} \subset \mathbf{P}^4(\mathbf{C})$. From the relative homology sequence for $(V_z^\circ, (C_{+,z} \cup C_{-,z}))$, we have

$$(1) \qquad 0 \to H_3(V_z^\circ; \mathbf{Z}) \to H_3(V_z^\circ, (C_{+,z} \cup C_{-,z}); \mathbf{Z}) \xrightarrow{\partial} \mathbf{Z}([C_{+,z}] - [C_{-,z}]) \to 0,$$

where $\mathbf{Z}([C_{+,z}] - [C_{-,z}])$ is $\mathrm{Ker}(H_2(C_{+,z} \cup C_{-,z}); \mathbf{Z}) \to H_2(V_z^\circ); \mathbf{Z})$. The monodromy T_∞ around p_0 interchanges $C_{+,z}$ and $C_{-,z}$.

Respecting the sequence (1), we take a family of cycles Poincaré duality isomorphic to the flat integral basis s^p $(0 \leq p \leq 3)$ in 2.5 (5B) and a family of chains joining from $C_{-,z}$ to $C_{+,z}$ (a choice up to integral cycles and up to half twists), and over them integrate the family of 3-forms $\Omega(z)$ with log pole over $z = 0$ (z in the punctured disc in the z-plan) in 2.2, then we have a family of vectors $(\eta_0, \eta_1, \eta_2, \eta_3, \mathcal{T})$ consisting of periods and a tension. This corresponds to the data in [W07], [MW09]. Since $T_\infty(\mathcal{T}) = -(\mathcal{T} + \eta_1 + \eta_0)$ by [W07, (3.14)], we find $\mathcal{T} + \frac{1}{2}\eta_1 + \frac{1}{4}\eta_0 = \frac{15}{\pi^2}\tau$ is an eigenvector of the monodromy T_∞ with eigenvalue -1.

The family of sequences (1) $(z \neq 0)$ forms an exact sequence of local systems of $\mathbf{Z}$-modules. To make the monodromy of this system unipotent, we take a double cover $z^{1/2} \mapsto z$. Let S be a neighborhood disc of p_0 in the $z^{1/2}$-plane endowed with log structure associated to the divisor p_0 in S, and let $S^{\log}$ be as in 1.2. Let S^* be the punctured disc $S \smallsetminus \{p_0\}$. Pull back the above local system to S^* and then extend it over $S^{\log}$.

Applying Tate twist (-3) and Poincaré duality isomorphism to the left and the right ends of this exact sequence, we have a local system L' over $S^{\log}$ which is an extension of $\mathbf{Z}(-2)$ by $\mathcal{H}_\mathbf{Z}$:

$$(2) \qquad 0 \to \mathcal{H}_\mathbf{Z} \to L' \to \mathbf{Z}(-2) \to 0.$$

Let $1 \in \mathbf{Z} \simeq \mathrm{gr}_4^W \mathbf{Z}(-2)$, take a lifting $1_{\mathbf{Z}} := 1 - (\mathcal{T}/\eta_0)s^0$ in L' of 1, and extend ∇ on $\mathcal{H}_{\mathbf{Z}}$ over L' by $\nabla(1_{\mathbf{Z}}) = 0$. We look for a T_∞^2-invariant ∇-flat element associated to $1_{\mathbf{Z}}$. This is computed as $1_{\mathbf{Z}}^{\mathrm{spl}} := 1_{\mathbf{Z}} - (s^1/2)$, and we know that L' coincides with $H_{\mathbf{Z}}$ in (5) in Introduction.

For the relative monodromy weight filtration $M = M(N, W)$, we see that $1_{\mathbf{Z}} \in M_4$ and $s^1 \in M_2$ are the smallest filters containing the elements in question. Taking the graded quotients by M of the sequence (2), we have

$$\begin{aligned} &\mathrm{gr}_6^M \mathcal{H}_{\mathbf{Z}} \xrightarrow{\sim} \mathrm{gr}_6^M L', \\ 0 \to\, &\mathrm{gr}_4^M \mathcal{H}_{\mathbf{Z}} \to \mathrm{gr}_4^M L' \to \mathbf{Z}(-2) \to 0, \\ 0 \to\, &\mathrm{gr}_2^M \mathcal{H}_{\mathbf{Z}} \to \mathrm{gr}_2^M L' \to (2\text{-torsion}) \to 0, \\ &\mathrm{gr}_0^M \mathcal{H}_{\mathbf{Z}} \xrightarrow{\sim} \mathrm{gr}_0^M L'. \end{aligned} \tag{3}$$

The 2-torsion in the third sequence of (3) corresponds to a half twist of chains from C_- to C_+. Standing on a half integral point and looking at the integral points nearby, we have two orientations. These correspond to the two orientations of a half twist of the chains, and also correspond to $\mathcal{T}_\pm := \pm(\frac{15}{\pi^2}\tau - \frac{\eta_0}{4}) - \frac{\eta_1}{2}$ in [W07]. $\mathcal{T}_-$ is different from $-\mathcal{T}_+$ by the complementary half twist, i.e., $\mathcal{T}_+ + \mathcal{T}_- = -\eta_1$.

For A-model, we consider the setting in [W07, 2.1]. Let $V = V_\psi$ with $\psi = 0$ from 2.1 be a Fermat quintic threefold in $\mathbf{P}^4(\mathbf{C})$ and $Lg := V \cap \mathbf{P}^4(\mathbf{R})$ be a Lagrangian submanifold of its real locus. From the exact sequence of relative homology for (V, Lg), we have

$$\begin{aligned} &H_6(V;\mathbf{Z}) \xrightarrow{\sim} H_6(V, Lg; \mathbf{Z}), \\ 0 \to\, &H_4(V;\mathbf{Z}) \to H_4(V, Lg;\mathbf{Z}) \to H_3(Lg;\mathbf{Z}) \to 0, \\ 0 \to\, &H_2(V;\mathbf{Z}) \to H_2(V, Lg;\mathbf{Z}) \to H_1(Lg;\mathbf{Z}) \to 0, \\ &H_0(V;\mathbf{Z}) \xrightarrow{\sim} H_0(V, Lg;\mathbf{Z}). \end{aligned} \tag{4}$$

Let $H' = H_\bullet(V)$, $H = H_\bullet(V, Lg)$ and $H'' = H_\bullet(Lg)$, and let

$$H_{\mathrm{even}}(V) := \bigoplus_{0 \le p \le 3} (H')_{2p}, \quad H_{\mathrm{even}}(V, Lg) := \bigoplus_{0 \le p \le 3} H_{2p}, \quad H_{\mathrm{odd}}(Lg) := \bigoplus_{0 \le p \le 1} (H'')_{2p+1}.$$

Then we have an exact sequence

$$0 \to H_{\mathrm{even}}(V) \to H_{\mathrm{even}}(V, Lg) \to H_{\mathrm{odd}}(Lg) \to 0. \tag{5}$$

The weight filtration W is given by $W_3 H_{\mathrm{even}}(V, Lg) := H_{\mathrm{even}}(V)$, $W_4 H_{\mathrm{even}}(V, Lg) := H_{\mathrm{even}}(V, Lg)$, and the relative monodromy weight filtration $M = M(N, W)$ is given by $M_{2p} H_{\mathrm{even}}(V, Lg) = H_{\le 2p}(V, Lg)$ $(0 \le p \le 3)$.

In the above setting, the projection from $\mathbf{P}^4(\mathbf{R})$ to the real hyperplane $\{x_5 = 0\} = \mathbf{P}^3(\mathbf{R})$ with center $(0, 0, 0, 0, 1)$ induces a homeomorphism $Lg \simeq \mathbf{P}^3(\mathbf{R})$. Therefore there are two choices of flat $U(1)$ connections on Lg. Denote Lg endowed with these structures by $Lg_\pm$. Morrison-Walcher [MW09, 3] explain the relation between $Lg_\pm$ for A-model of V and $C_\pm$ for B-model of V°.

After pulling back to the double cover $z^{1/2} \mapsto z$ ($z \neq 0$) and extending over $S^{\log}$, the sequence for A-model (5) and the sequence for B-model (2), and the set of sequences for A-model (4) and the set of sequences for B-model (3), respectively, seem to correspond in mirror symmetry. By Poincaré duality isomorphisms, $H^{\text{even}}(V) = H_{\text{even}}(V)(-3)$ and $H^{\text{even}}(Lg) \simeq H_{\text{odd}}(Lg)$.

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