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Notes on Some Orthogonal Polynomials Having the Brenke Type Generating Functions

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Abstract

The main purpose of this note is to summarize part of results on a classical problem, originally, posed by Brenke in 1945 [12] and on its related topics. A full description in detail is given in [6].

1 Preliminaries

Let $\{P_n(x)\}$ be a system of monic polynomials, $P_n(x)$ of degree n, and functions $h(x), \rho(t)$ and $B(t)$ be analytic around the origin, $h(x)=\sum_{n=0}^{\infty}h_nx^n$, $\rho(t)=\sum_{n=1}^{\infty}r_nt^n$ and $B(t)=\sum_{n=0}^{\infty}b_nt^n$ with $h_n\neq 0$ for $n\geq 0$ and $h(0)=B(0)=\rho'(0)=1$ just for normalizations. Suppose that a generating function $\psi(t,x)$ of $\{P_n(x)\}$ has the following form,

$$\psi(t,x):=h(\rho(t)x)B(t)=\sum_{n=0}^{\infty}h_nP_n(x)t^n. \tag{1.1}$$

$\psi(t,x)$ is called a generating function of the *Boas-Buck* type [8].

On the other hand, it is known [15] that $\{P_n(x)\}$ is the orthogonal polynomials with respect to a probability measure μ on $\mathbb{R}$ with finite moments of all orders if and only if there exists a pair of sequences $\alpha_0, \alpha_1\ldots\in\mathbb{R}$ and $\omega_1,\omega_2,\ldots>0$ satisfying the recurrence relation

$$\begin{cases}P_0(x)=1,\ P_1(x)=x-\alpha_0,\\ P_{n+1}(x)=(x-\alpha_n)P_n(x)-\omega_nP_{n-1}(x),\ n\geq 1\end{cases} \tag{1.2}$$

where $P_{-1}(x)=0$ by convention. A pair of sequences $\{\alpha_n,\omega_n\}$ is called the *Jacobi-Szegö parameters* in this note.

It is quite natural to ask if one can

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determine all orthogonal polynomials having the Boas-Buck type generating functions in Eq.(1.1).

It has been remained as a longstanding open problem, although particular cases have been considered as follows.

Example 1.1 (Classical Meixner type). If $h(x) = \exp(x)$, Eq.(1.1) is called the (classical) *Meixner* type, which is also called the orthogonal *Sheffer* type. This type provides the classical Meixner class of orthogonal polynomials, which consists of Hermite, Charlier, Laguerre, Meixner, and Meixner-Pollaczek polynomials. If we restrict our consideration to $h(x) = \exp(x)$ with $\rho(t) = t$, Eq.(1.1) is called the *Appell* type. It is known that orthogonal polynomials obtained from this type contain the Hermite polynomials only.

Example 1.2 (Free Meixner type). If $h(x) = (1-x)^{-1}$, let us call Eq.(1.1) the *free Meixner* type, because this choice provides the free analogue of the classical Meixner class. More generally, the case $h(x) = (1-x)^{-\alpha}$ for $\alpha > 0$ has been considered. We do not mention this case in this note.

See [6] for relevant papers on classical, free Meixner, and other classes.

Example 1.3 (Brenke type). Eq.(1.1) with $\rho(t) = t$ is called the *Brenke* type. The Brenke type provides Hermite, Laguerre, and (Szegö's) generalized Hermite polynomials. Moreover, this type generates some q-orthogonal polynomials such as Al-Salam-Carlitz (I and II), little q-Laguerre (Wall), q-Laguerre (generalized Stieltjes-Wigert), and discrete q-Hermite (I and II) polynomials. These polynomials will be appeared in Section 2 and 3.

Let us prepare minimum notations from q-calculus for later use in Section 3 (see [18][19], for example). In this paper, we always assume that $0 < q < 1$ for simplicity. The *q-shifted factorials* (q-analogue of the *Pochhammer symbol* $(\cdot)_n$ defined by (2.2)) are defined by

$$(a;q)_n = \begin{cases} 1, & n = 0, \\ \prod_{k=1}^{n}(1-aq^{k-1}), & n = 1, 2, \ldots, \infty, \end{cases}$$

and the *multiple q-shifted factorials* are by

$$(a_1, a_2, \ldots, a_k; q)_n := \prod_{j=1}^{k}(a_j; q)_n.$$

The *q-hypergeometric series* ${}_0\phi_0, {}_0\Phi_0, {}_0\Phi_1$ are defined respectively by[1]

$$
\begin{cases}
{}_0\phi_0\left(-;-;q,z\right) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n q^{\binom{n}{2}} z^n}{(q;q)_n}, \\
{}_0\Phi_0\left(-;-;q,z\right) = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{(q;q)_n}, \\
{}_0\Phi_1\left(-;b_1;q,z\right) = \displaystyle\sum_{n=0}^{\infty} \frac{z^n}{(q,b_1;q)_n}.
\end{cases}
\tag{1.3}
$$

It can be shown that

$$
\begin{cases}
\displaystyle\lim_{q\to 1^-} {}_0\Phi_0\left(-;-;q,(1-q)z\right) = e^z, \\
\displaystyle\lim_{q\to 1^-} {}_0\phi_0\left(-;-;q,(1-q)z\right) = e^{-z}.
\end{cases}
\tag{1.4}
$$

It is convenient to introduce notations $e_q(z)$ and $E_q(z)$ by

$$
\begin{cases}
e_q(z) := {}_0\Phi_0\left(-;-;q,z\right), \\
E_q(z) := {}_0\phi_0\left(-;-;q,z\right).
\end{cases}
\tag{1.5}
$$

From the consequence of (1.4), $e_q(z)$ and $E_q(z)$ are usually considered as q *exponential* functions known as *Euler's formulas.*

Remark 1.4. One should be careful about a definition of q-exponential when referring other literatures. In [16][18][19], $E_q(z)$ is defined as

$$
E_q(-z) := {}_0\phi_0\left(-;-;q,z\right).
$$

On the other hand, in [2],

$$
\begin{cases}
e_q(z) := {}_0\Phi_0\left(-;-;q,(1-q)z\right), \\
E_q(-z) := {}_0\phi_0\left(-;-;q,(1-q)z\right).
\end{cases}
$$

are adopted. Our definitions in (1.5) are slightly different from theirs. However, these differences do not make any essential effects on our discussion in Section 3.

2 The Brenke-Chihara Problem

As mentioned in Example 1.3, we shall consider another subclass of the Boas-Buck type generating functions with $\rho(t) = t$, that is,

$$
\psi(t,x) = h(tx)B(t). \tag{2.1}
$$

[1]The second and third series in (1.3) are special cases of the *old basic hypergeometric series* ${}_r\Phi_s$ defined by Bailey [7]. The first series is a particular case of so-called the *basic hypergeometric series* ${}_r\phi_s$.

Such a generating function $\psi(t,x)$ is named the *Brenke* type after the pioneer work by Brenke [12]. He tried to determine all orthogonal polynomials $\{P_n(x)\}$ generated from the Brenke type generating functions (2.1), explicitly. Geronimus [17] independently considered a slightly more general problem (see Remark in Section 4). However, they could not solve the problem. Chihara [13][14] examined it and claimed that the Brenke type generating functions are classified into the four classes, Class I, II, III, and IV in terms of the Jacobi-Szegö parameters. However, it is quite difficult for us to follow his results in details because papers were written in a very sketchy way, that is, no complete proofs were presented. Moreover, no general forms of the Jacobi-Szegö parameters were given, for instance. One cannot fill up them by simple and routine calculations. Therefore, it was one of our motivation [6] to fill up these gaps by reformulating the problem as follows.

The Brenke-Chihara Problem

Determine all orthogonal polynomials $\{P_n(x)\}$ generated from the Brenke type generating functions (2.1) satisfying the recurrence relation in (1.2) and compute $\{\alpha_n, \omega_n\}$ and $(h(x), B(t))$, explicitly.

Let us first give classical examples in the Brenke class. Non-trivial examples will be given in Section 3.

2.1 Classical Examples

In this section, we shall give three examples generated from the generating functions of the Brenke type by Eq.(2.1).

Example 2.1 (Hermite polynomials). It is well known that

$$\psi(t,x) = \exp\left(tx - \frac{1}{2}t^2\right)$$

is a generating function of the standard Hermite polynomials $\{H_n(x)\}$ for $N(0,1)$. It is clear to see that $h(x) = \exp(x)$ and $B(t) = \exp(-t^2/2)$. The Jacobi-Szegö parameters are $\alpha_n = 0$, $\omega_n = n$. We remark that this example is in the intersection of the Appell, Meixner and Brenke types.

Example 2.2 (Laguerre polynomials). Let $(\kappa)_n, \kappa > 0$, be the *Pochhammer symbol* defined by

$$(\kappa)_n = \begin{cases} 1, & n = 0, \\ \kappa(\kappa+1)\cdots(\kappa+n-1), & n \geq 1, \end{cases} \tag{2.2}$$

and a hypergeometric function ${}_0F_1(-;\kappa;x)$ by

$${}_0F_1(-;\kappa;x) := \sum_{n=0}^{\infty} \frac{1}{n!(\kappa)_n} x^n.$$

It is known that

$$\psi(t,x) = {}_0F_1(-;\kappa;tx)\exp(-t) \tag{2.3}$$

is a generating function of the Laguerre polynomials $\{L_n^{(\kappa-1)}(x)\}$ (see [19] for example). It is also clear to see that $h(x) = {}_0F_1(-;\kappa;x)$ and $B(t) = \exp(-t)$. The Jacobi-Szegö parameters are $\alpha_n = 2n+\kappa$, $\omega_n = n(n+\kappa-1)$. The corresponding orthogonality measure is the Gamma distribution of a parameter $\kappa > 0$.

On the other hand, as mentioned in Example 1.1, the Laguerre polynomials can be obtained from the Meixner type, too. In fact, if $h(x) = \exp(x)$ and $\rho(t) = t(1+t)^{-1}$ and $B(t) = (1+t)^{-\kappa}$, then the Laguerre polynomials can be also generated from

$$h(\rho(t)x)B(t) = \exp\left(\frac{tx}{1+t}\right)(1+t)^{-\kappa},$$

which has a different form from the Brenke type of Eq.(2.3).

Example 2.3 (Generalized Hermite polynomials). For $k > 0$, consider

$$h(x) = {}_0F_1\left(-;k;\frac{1}{4}x^2\right) + {}_0F_1\left(-;k+1;\frac{1}{4}x^2\right)\frac{x}{2k} \tag{2.4}$$

and $B(t) = \exp\left(-\frac{1}{2}t^2\right)$. The above $(h(x), B(t))$ provides (Szegö's) generalized Hermite polynomials $\{H_n^{(k)}(x)\}$, with respect to the generalized symmetric Gamma distribution of parameter 2 with the density,

$$\frac{1}{2^k\Gamma(k)}|x|^{2k-1}\exp\left(-\frac{1}{2}x^2\right), \quad x \in \mathbb{R}.$$

The above density with $k = 1$ is called the two sided Rayleigh distribution. The Jacobi-Szegö parameters are $\omega_{2n} = 2n$, $\omega_{2n+1} = 2n+2k$. Obviously, one can get the standard Hermite polynomials $\{H_n(x)\}$ for $N(0,1)$ in Example 2.1 if $k = 1/2$ is taken. In addition, substitute $k = 1/2$ into Eq.(2.4), one can see

$$\begin{aligned} h(x) &= {}_0F_1\left(-;\frac{1}{2};\frac{1}{4}x^2\right) + {}_0F_1\left(-;\frac{3}{2};\frac{1}{4}x^2\right)x \\ &= \cosh x + \sinh x \\ &= \exp(x) \end{aligned}$$

as in Example 2.1.

2.2 General forms of the Jacobi-Szegö parameters in the Brenke class ([6])

In this section, we shall present general forms of the Jacobi-Szegö parameters associated with the Brenke type generating functions in Asai-Kubo-Kuo[6].

First of all, due to the expression in Eq.(2.1), it is easy to obtain the expression,

$$P_n(x) = \sum_{k=0}^{n} \frac{h_k}{h_n} b_{n-k} x^k. \tag{2.5}$$

By substituting Eq.(2.5) into Eq.(1.2), we derive the relation

$$\frac{h_m}{h_{n+1}} b_{n-m+1} - \frac{h_{m-1}}{h_n} b_{n-m+1} + \alpha_n \frac{h_m}{h_n} b_{n-m} + \omega_n \frac{h_m}{h_{n-1}} b_{n-m-1} = 0 \tag{2.6}$$

for $0 \leq m \leq n$ with the convention $h_{-1} = b_{-1} = b_{-2} = 0$. For $m = n$, we have

$$\frac{h_n}{h_{n+1}} b_1 - \frac{h_{n-1}}{h_n} b_1 + \alpha_n = 0.$$

Hence, we get

$$\alpha_n = -b_1 \left(\frac{h_n}{h_{n+1}} - \frac{h_{n-1}}{h_n} \right) \quad \text{for} \quad n \geq 0 \tag{2.7}$$

with the convention $h_{-1} = 0$. From Eq.(2.7) and put $A_n = \sum_{j=0}^{n} \alpha_j$, we have

$$b_1 \frac{h_n}{h_{n+1}} = -A_n. \tag{2.8}$$

Proposition 2.4. *If $b_1 \neq 0$, then the Jacobi-Szegö parameter ω_n is given by*

$$\omega_n = -\frac{A_{n-1}}{b_1^2} \left(b_2(\alpha_n + \alpha_{n-1} - b_1^2 \alpha_n) \right). \tag{2.9}$$

Moreover, the following three terms recurrence relation holds:

$$(b_3 - 2b_1 b_2 + b_1^3)\alpha_n - (b_3 - b_1 b_2)\alpha_{n-1} + b_3 \alpha_{n-2} = 0. \tag{2.10}$$

Proof. Due to Eq.(2.8), one has

$$h_n = (-b_1)^n \prod_{i=0}^{n-1} \frac{1}{A_i}, \quad A_n = \sum_{j=0}^{n} \alpha_j \neq 0.$$

In particular, $h_1 = -b_1/\alpha_0$. Putting $m = n - 1$ in Eq.(2.6), we have

$$\frac{h_{n-1}}{h_{n+1}} b_2 - \frac{h_{n-2}}{h_n} b_2 + \alpha_n \frac{h_{n-1}}{h_n} b_1 + \omega_n = 0. \tag{2.11}$$

By Eqs.(2.7), (2.8), and (2.11), one can get the first assertion in Eq.(2.9),

Next, we shall derive the second assertion. From Eqs.(2.6), (2.7) and (2.9), we obtain

$$\left(\frac{h_n}{h_{n+1}} - \frac{h_{m-1}}{h_m} \right) b_{n-m+1} + \alpha_n b_{n-m} + \left(\frac{b_2}{b_1}(\alpha_n + \alpha_{n-1}) - b_1 \alpha_n \right) b_{n-m-1} = 0,$$

which implies

$$b_{n-m+1}\left(\sum_{j=m}^{n}\alpha_j\right)-b_1b_{n-m}\alpha_n-b_{n-m-1}\big(b_2(\alpha_n+\alpha_{n-1})-b_1^2\alpha_n\big)=0. \quad (2.12)$$

By setting $m=n-2$, we have

$$b_3(\alpha_n+\alpha_{n-1}+\alpha_{n-2})-b_1b_2\alpha_n-b_1b_2(\alpha_n+\alpha_{n-1})+b_1^3\alpha_n=0.$$

Hence, we obtain the recurrence relation in Eq.(2.10). □

Proposition 2.5. *Let* $\Omega_n := \sum_{j=1}^{n}\omega_j$. *If* $b_1=0$, *then* $\alpha_n=0$ *for any* $n\geq 0$. *Moreover, one can obtain*

$$\begin{cases}\omega_{2n}=\dfrac{\omega_{2n-1}}{\Omega_{2n-1}}\left(\Omega_{2n-1}-\omega_1\displaystyle\prod_{j=1}^{n-1}\frac{\Omega_{2j+1}}{\Omega_{2j}}\right),\\ \omega_{2n+1}=\dfrac{\omega_{2n-1}}{\Omega_{2n-1}}\left(\Omega_{2n}-\omega_1\displaystyle\prod_{j=1}^{n}\frac{\Omega_{2j}}{\Omega_{2j-1}}\right)\end{cases}$$

for $n\geq 2$ *and given* $\omega_1,\omega_2,\omega_3>0$.

Proof. The first assertion is due to Eq.(2.7). Lemma 3.1, 3.2, and 3.5 in [20] and Eq.(2.6) can provide our second assertion. See our paper [6] in details. □

3 Results on the Problem

In this note we will not describe the full derivation of our results for readers to avoid being disgusted with technical computations with a q-deformation parameter and many other parameters. Those who would like to go into details can refer to our paper [6], which is more complete and general than that in [13][14].

In conclusion, one can say

for $q\neq 1$

> the Brenke type generating functions generate four classes of q-orthogonal polynomials

and

for $q=1$

> the Brenke type generating functions generate Laguerre, shifted Hermite and generalized Hermite polynomials, essentially.

Remark 3.1. The reason why a q-deformation parameter is appeared originates in a solution of the recurrence relation on $\{\alpha_n\}$ given by Eq.(2.10). In this sense, q in the Brenke class is not an artificially added object, but an intrinsic parameter.

We simply give typical forms of the Jacobi-Szegö parameters and $(h(x), B(t))$ for each class.

3.1 Class I

(1) For $0 < q < 1$ a particular choice of parameters as in Remark 4.2 for Theorem 4.1 of [6] gives us

$$\begin{cases} h(x) = {}_0\Phi_1(-; aq; q; x), \\ B(t) = E_q(t), \\ \alpha_n = q^n \left(1 + a(1 - q^n - q^{n+1})\right), \\ \omega_n = aq^{2n-1}(1 - q^n)(1 - aq^n). \end{cases}$$

Due to the equality (see Appendix in [6]),

$$\frac{1}{(x;q)_\infty}\, {}_0\phi_1(-; a; q, ax) = {}_0\Phi_1(-; a; q, x), \tag{3.1}$$

one can get

$$\psi(t, x) = {}_0\Phi_1(-; aq; q, tx)E_q(t). \tag{3.2}$$

It is a generating function of the *little q-Laguerre polynomials* (see [19], for example). Moreover, if $0 < q < 1, 0 < a < q^{-1}$, then the corresponding orthogonality measure is uniquely given by

$$\mu = \sum_{k=0}^{\infty} \frac{(aq;q)_\infty (aq)^k}{(q;q)_k} \delta_{q^k}.$$

Remark 3.2. As soon as our paper [6] was published, M. Ismail kindly informed the author that the formula (3.1) is closely related with the relationship between *q-Bessel functions* (Jackson, 1905) of the first kind $J_\nu^{(1)}(z;q)$ and second kind $J_\nu^{(2)}(z;q)$ of a parameter ν,

$$J_\nu^{(1)}(z;q) = \frac{J_\nu^{(2)}(z;q)}{(-z^2/4;q)_\infty} \tag{3.3}$$

in Theorem 14.1.3 of [18] (see also page 23 in [19]). That is, the formula (3.1) with $a = q, x = -z^2/2$ is nothing but the formula (3.3) with $\nu = 0$.

(2) The case of $q = 1$ ends up with *Laguerre polynomials* by a special choice of parameters as in Remark 4.4 for Theorem 4.3 of [6]. See Example 2.2.

3.2 Class II

(1) For $0 < q < 1$, a certain choice of parameters as in Remark 5.2 for Theorem 5.1 [6] provides us

$$\begin{cases} h(x) = {}_0\Phi_1\left(-;a;q^2,x^2\right) - \frac{x}{1-a}\,{}_0\Phi_1\left(-;aq^2;q^2,x^2\right), \\ B(t) = (1+t)\ E_{q^2}(q^2t^2), \\ \alpha_{2n} = (1-a)q^{2n},\ \alpha_{2n+1} = (a-q^2)q^{2n}, \\ \omega_{2n} = q^{2n}(1-q^{2n}),\ \omega_{2n+1} = aq^{2n}\left(1-aq^{2n}\right). \end{cases}$$

Chihara [14] gave the unique corresponding orthogonality measure μ as

$$\mu = \sum_{k=0}^{\infty} \frac{a^k\left(a;q^2\right)_\infty}{2\left(q^2;q^2\right)_k}\left((1-q^k)\delta_{-q^k} + (1+q^k)\delta_{q^k}\right).$$

We do not know whether or not a particular name of orthogonal polynomials has been given to this example.
(2) The case $q = 1$ is reduced to Class IV.

3.3 Class III

(1) If $0 < q < 1$, a particular choice of parameters as in Remark 6.3 for Theorem 6.1 of [6] gives us

$$\begin{cases} h(x) = e_q(x), \\ B(t) = E_q(t)E_q(at), \\ \alpha_n = (1+a)q^n, \\ \omega_n = -aq^{n-1}(1-q^n). \end{cases}$$

If $0 < q < 1$ and $a < 0$, then the generating function

$$\psi(t,x) = e_q(tx)E_q(t)E_q(at)$$

generates *Al-Salam-Carlitz I polynomials* and its corresponding orthogonality measure is uniquely given by

$$\mu = \sum_{n=0}^{\infty}\left(\frac{q^n}{\left(q,q/a;q\right)_n(a;q)_\infty}\delta_{q^n} + \frac{q^n}{\left(q,aq;q\right)_n(1/a;q)_\infty}\delta_{aq^n}\right)$$

on the interval $[a,1]$. See [1][19].
(2) The case of $q = 1$ ends up with *shifted Hermite polynomials* for the Gaussian measure $N(a,1)$, $a \neq 0$, by a special choice of parameters as in Theorem 6.4 of [6]. Characteristic quantities are given by

$$\begin{cases} h(x) = \exp(x), \\ B(t) = \exp\left(\frac{1}{2}t^2 - at\right),\ a \neq 0, \\ \alpha_n = a,\ \omega_n = n. \end{cases}$$

Because of $a \neq 0$, Example 2.1 cannot be recovered from this class.

3.4 Class IV

(1) For $0 < q < 1$, a special choice of parameters as in Remark 7.3 for Theorem 7.2 of [6] gives us

$$\begin{cases} h(x) = {}_0\Phi_1(-; aq; q^2, x^2) + \frac{x}{1-aq}\, {}_0\Phi_1(-; aq^3; q^2, x^2), \\ B(t) = E_{q^2}(t^2), \\ \alpha_n = 0, \\ \omega_{2n} = aq^{2n-1}(1-q^{2n}),\ \omega_{2n+1} = q^{2n}(1-aq^{2n+1}). \end{cases}$$

If $a = 1$ is taken under the condition $0 < q < 1$, then

$$\alpha_n = 0, \quad \omega_n = q^{n-1}(1-q^n), \quad h(x) = e_q(x), \quad B(t) = E_{q^2}(t^2).$$

Note that the equality,

$${}_0\Phi_1(-; q; q^2, x^2) + \frac{x}{1-q}\, {}_0\Phi_1(-; q^3; q^2, x^2) = e_q(x), \tag{3.4}$$

has been used to derive the expression of $h(x)$. The derivation of Eq.(3.4) can be found in Appendix of [6]. Thus we obtain the generating function

$$\psi(t, x) = e_q(tx) E_{q^2}(t^2)$$

of *discrete q-Hermite I polynomials.* The corresponding orthogonality measure is uniquely given by

$$\mu = \sum_{k=0}^{\infty} \frac{\left(q^{k+1}, -q^{k+1}; q\right)_\infty q^k}{\left(q, -1, -q; q\right)_\infty} \left(\delta_{q^k} + \delta_{-q^k}\right).$$

See [19]. This is a special case of Al-Salam-Carlitz I polynomials with $a = -1$ in Class III.

Remark 3.3. In Bożejko-Kümmerer-Speicher [10], "q-Hermite" polynomials play a key role to realize a q-Brownian motion on a certain q-Fock space, which interpolates Fermion ($q = -1$), Free ($q = 0$), and Boson ($q = 1$) Fock spaces. Their "q-Hermite" polynomials mean that the Jacobi-Szegö parameters are given by

$$\begin{cases} \omega_n = [n]_q := 1 + q + \cdots + q^{n-1} = \dfrac{1-q^n}{1-q}, \quad q \in [-1, 1], \\ \alpha_n = 0. \end{cases}$$

and the corresponding orthogonality measure $\nu_q(dx)$ (Szegö, 1926) is given by

$$\nu_q(dx) = \frac{1}{\pi}\sqrt{1-q}\sin\theta \prod_{n=1}^{\infty}(1-q^n)|1-q^n e^{2i\theta}|^2 dx,$$

on the interval $[-2/\sqrt{1-q}, 2/\sqrt{1-q}]$ where $x = \frac{2}{\sqrt{1-q}}\cos\theta$ for $\theta \in [0, \pi]$. Therefore, discrete q-Hermite polynomials are different from "q-Hermite" polynomials.

(2) The case of $q = 1$ ends up with (Szegö's) generalized Hermite polynomials (see Example 2.3) by a special choice of parameters as in Remark 7.6 for Theorem 7.5 of [6].

4 Additional Remark

(1) As mentioned in Section 1, it is still open to characterize orthogonal polynomials associated with the Boas-Buck type generating functions.

(2) It is open to determine all orthogonal polynomials, explicitly, of the form

$$h_n P_n(x) = b_n + \sum_{k=1}^{n} h_k b_{n-k} \prod_{i=1}^{k} (x - x_i).$$

This is called the *Geronimus problem* ([17]). The Brenke-Chihara problem solves it if $x_i = 0$ for $i \geq 1$. See Eq.(2.5).

(3) Throughout this note, we have considered $0 < q \leq 1$ just for simplicity. One can start the Brenke-Chihara problem under a more general assumption on q and include other examples of q-orthogonal polynomials such as q-Laguerre (generalized Stieltjes-Wigert), Al-Salam-Carlitz II, and discrete q-Hermite II polynomials if $q > 1$ (see [6]). In general, a range of a parameter q contains delicate analytical roles when one may discuss the existence of a probability measure and the uniqueness of a moment problem associated with the Jacobi-Szegö parameters, and so on.

(4) One can ask how about the case of $q = 0$. It is a difficult question. The derivation and classification of orthogonal polynomials in this case seem to be open. The Brenke class of orthogonal polynomials for the case $q = 0$ is different from the free Meixner class (see [3][9][22] for the free Meixner class). Our q-parameter plays different roles from that of q-deformed quantum stochastic calculus in the sense of Bożejko-Kümmerer-Speicher [10][11].

(5) It would be interesting to construct q-deformed Bargmann measures associated with the Brenke class along the line with [4][5].

(6) A probabilistic role of the Brenke class has not been well-understood. It would be interesting to pursue it from the non-commutative (algebraic) probabilistic viewpoint in a sense.

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