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Semi-classical Asymptotics for the Partition Function of an Abstract Bose Field Model

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Semi-classical asymptotics for the partition function of an abstract Bose field model is considered.

Keywords: semi-classical asymptotics, Bose field, partition function, second quantization, Fock space.

I. Introduction

In quantum mechanics, in which a physical constant $\hbar := h/2\pi$ (h : the Planck constant) plays an important role, the limit $\hbar \to 0$ for various quantities (if it exists) is called the classical limit. Trace formulas in the abstract boson Fock space and the classical limit for the trace $Z(\beta\hbar)$ (the partition function) of the heat semigroup of a perturbed second quantization operator were derived by Arai [2], where $\beta > 0$ denotes the inverse temperature. Generally speaking, the classical limit is regarded as the zero-th order approximation in $\hbar$. From this point of view, it is interesting to derive higher order asymptotics of various quantities in $\hbar$. Such asymptotics are called semi-classical asymptotics. In this paper the asymptotic formula for $Z(\beta\hbar)$ is stated, which is derived in [1].

II. A Classical Limit in The Abstract Boson Fock Space

In this section we review a classical limit for the trace of a perturbed second quantization operator and some fundamental facts related to it.

Let $\mathscr{H}$ be a real separable Hilbert space, and A be a strictly positive self-adjoint operator acting in $\mathscr{H}$. We denote by $\{\mathscr{H}_s(A)\}_{s\in\mathbb{R}}$ the Hilbert scale associated with A [3]. For all $s \in \mathbb{R}$, the dual space of $\mathscr{H}_s(A)$ can be naturally identified with $\mathscr{H}_{-s}(A)$.

We denote by $\mathscr{I}_1(\mathscr{H})$ the ideal of the trace class operators on $\mathscr{H}$. Let $\gamma > 0$ be fixed. Throughout this paper, we assume the following.

Assumption I. $\quad A^{9-\gamma} \in \mathscr{I}_1(\mathscr{H})$.

Under Assumption I, the embedding mapping of $\mathscr{H}$ into

$$E := \mathscr{H}_{-\gamma}(A)$$

is Hilbert-Schmidt. Hence, by Minlos' theorem, there exists a unique probability measure μ on $(E, \mathscr{B})$ such that the Borel field $\mathscr{B}$ is generated by $\{\phi(f) | f \in \mathscr{H}_\gamma(A)\}$ and

$$\int_E e^{i\phi(f)} d\mu(\phi) = e^{-\|f\|_{\mathscr{H}}^2/2}, \quad f \in \mathscr{H}_\gamma(A),$$

where $\|\cdot\|_{\mathscr{H}}$ denotes the norm of $\mathscr{H}$.

The complex Hilbert space $L^2(E, d\mu)$ is canonically isomorphic to the boson Fock space over $\mathscr{H}$, which is called the Q-space representation of it [3]. We denote by $d\Gamma(A)$ the second quantization of A and set

$$H_0 = d\Gamma(A).$$

Then for all $\beta > 0$, $e^{-\beta H_0} \in \mathscr{I}_1(L^2(E, d\mu))$.

Definition 2.1. *A mapping V of a Banach space X into a Banach space Y is said to be polynomially continuous if there exists a polynomial P of two real variables with positive coefficients such that*

$$\|V(\phi) - V(\psi)\| \leq P(\|\phi\|, \|\psi\|)\|\phi - \psi\|, \quad \phi, \psi \in X.$$

Let V be a real valued function on E. Throughout this paper, we assume the following.

Assumption II. *The function V is bounded from below, 3-times Fréchet differentiable, and V, V', V'', V''' are polynomially continuous.*

For $\hbar > 0$, we define $V_\hbar$ by

$$V_\hbar(\phi) := V(\sqrt{\hbar}\,\phi), \quad \phi \in E.$$

and set

$$H_\hbar := H_0 \dotplus \frac{1}{\hbar} V_\hbar,$$

where $\dotplus$ denotes the quadratic form sum.

Under Assumption I, II, for all $\beta > 0$, $e^{-\beta H_\hbar} \in \mathscr{I}_1(L^2(E, d\mu))$ [2].

Theorem 2.2. [2]. *Let $\beta > 0$. Then*

$$\lim_{\hbar \to 0} \frac{\mathrm{Tr}\, e^{-\beta\hbar H_\hbar}}{\mathrm{Tr}\, e^{-\beta\hbar H_0}} = \int_E \exp\left(-\beta V\left(\sqrt{\frac{2}{\beta}} A^{-1/2}\phi\right)\right) d\mu(\phi).$$

III. A Class of Locally Convex Spaces

In this section we introduce a class of locally convex spaces, which gives a general framework for the asymptotic analysis discussed in this paper.

We denote by $\mathbb{R}_+$ the set of the nonnegative real numbers.

DEFINITION 3.1. *A mapping f from $\mathbb{R}_+$ to a locally convex space X is said to be locally bounded if for all $\delta > 0$ and every continuous seminorm p on X,*

$$p_\delta(f) := \sup_{0 \le \varepsilon \le \delta} p(f(\varepsilon)) < \infty.$$

We denote by $(X^{\mathbb{R}_+})_{\mathrm{l.b.}}$ the linear space of the locally bounded mappings from $\mathbb{R}_+$ to X. The topology defined by the seminorms $\{p_\delta\}_{p,\delta}$ turns $(X^{\mathbb{R}_+})_{\mathrm{l.b.}}$ into a locally convex space. If X is a Fréchet space, $(X^{\mathbb{R}_+})_{\mathrm{l.b.}}$ is a Fréchet space.

Let $\{E_n\}_{n\in\mathbb{N}}$ be a family of Banach spaces with the property that

$$E_{n+1} \subset E_n, \ \|\phi\|_n \le \|\phi\|_{n+1}, \quad \phi \in E_{n+1},$$

for all $n \in \mathbb{N}$, where $\|\cdot\|_n$ denotes the norm of E_n. Then, the topology defined by the norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$ turns $\bigcap_{n\in\mathbb{N}} E_n$ into a Fréchet space.

Let (X, P) be a probability space and Y be a Banach space. We denote by $L^p(X, dP; Y)$ the Banach space of the Y-valued L^p-functions on (X, P). Then $\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y)$ can be provided with the structure of Fréchet space.

DEFINITION 3.2. *Let f be a mapping from $\mathbb{R}_+$ to $\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y)$. We say that f is in $(\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y))^{\mathbb{R}_+}_{\mathrm{u.i.}}$ if and only if for each $\delta > 0$, there exists a nonnegative function $g \in \bigcap_{p\in\mathbb{N}} L^p(X, dP)$ such that*

$$\sup_{0 \le \varepsilon \le \delta} \|f(\varepsilon)(x)\|_Y \le g(x),$$

P-a.e.x.

The set $(\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y))^{\mathbb{R}_+}_{\mathrm{u.i.}}$ is a linear subspace of $(\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y))^{\mathbb{R}_+}_{\mathrm{l.b.}}$. In what follows, we omit x in $f(\varepsilon)(x)$.

Let $X_1, \cdots, X_n$ and Z be non-empty sets and G be a real-valued function on $X_1 \times \cdots \times X_n$ and F_j be a mapping from Z to X_j, $j = 1, \cdots, n$. We define $G(F_1, \cdots, F_n)$, the real-valued function on Z, by

$$G(F_1, \cdots, F_n)(z) = G(F_1(z), \cdots, F_n(z)), \quad z \in Z.$$

Then we can prove the following propositions.

PROPOSITION 3.3. *Let Q be a polynomial of n real valuables. Then the mapping* $(F_1, \cdots, F_n) \longmapsto Q(\|F_1\|, \cdots, \|F_n\|)$ *from* $\left((\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y))_{\text{u.i.}}^{\mathbb{R}_+}\right)^n$ *to* $\left(\bigcap_{p\in\mathbb{N}} L^p(X, dP)\right)_{\text{u.i.}}^{\mathbb{R}_+}$ *is continuous.*

PROPOSITION 3.4. *Let Z_j be a Banach space ($j = 1, \cdots, n$), L be a continuous multilinear form on $Z_1 \times \cdots \times Z_n$, and V_j be a polynomially continuous mapping from Y to $Z_j (j = 1, \cdots, n)$. Then the mapping* $(F_1, \cdots, F_n) \longmapsto L(V_1 \circ F_1, \cdots, V_n \circ F_n)$ *from* $\left((\bigcap_{p\in\mathbb{N}} L^p(X, dP; Y))_{\text{u.i.}}^{\mathbb{R}_+}\right)^n$ *to* $\left(\bigcap_{p\in\mathbb{N}} L^p(X, dP)\right)_{\text{u.i.}}^{\mathbb{R}_+}$ *is continuous.*

IV. AN ASYMPTOTIC FORMULA

Let $\{\lambda_n\}_{n=1}^{\infty}$ be the eigenvalues of A, and $\{e_n\}_{n=1}^{\infty}$ be the complete orthonormal system (CONS) of $\mathscr{H}$ with $Ae_n = \lambda_n e_n$, and

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^{\gamma-9}} < \infty \tag{4.1}$$

Let φ be a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. For all $n, m \in \mathbb{N}$, we set $f_{n,m} = e_{\varphi(n,m)}$. Then $\{f_{n,m}\}_{n,m=1}^{\infty}$ is a CONS of $\mathscr{H}$. For all $\phi \in E$, we define

$$\phi_n := \phi(e_n), \quad \phi_{n,m} := \phi(f_{n,m}).$$

Then $\{\phi_n\}_n$ and $\{\phi_{n,m}\}_{n,m}$ are families of independent Gaussian random variables such that for all $n, m, n', m' \in \mathbb{N}$,

$$\int_E \phi_n d\mu(\phi) = 0, \quad \int_E \phi_n \phi_m d\mu(\phi) = \delta_{nm} \tag{4.2}$$

$$\int_E \phi_{n,m} \phi_{n',m'} d\mu(\phi) = \delta_{nn'} \delta_{mm'}. \tag{4.3}$$

For all $m_1, \cdots, m_p \in \mathbb{N}$, we have

$$\sup_{n_1,\cdots,n_p\in\mathbb{N}} \int_E |\phi_{n_1}|^{m_1} \cdots |\phi_{n_p}|^{m_p} d\mu(\phi) < \infty. \tag{4.4}$$

For all $N, M \in \mathbb{N}$, we set

$$\begin{aligned} F_{N,M}(\varepsilon, \omega, s) &= \sqrt{\frac{2}{\beta}} \sum_{n=1}^{N} \frac{\phi_n}{\sqrt{\lambda_n}} e_n + \sum_{n=1}^{N} \sum_{m=1}^{M} \sqrt{\frac{4\varepsilon^2 \lambda_n}{\beta(\varepsilon^2 \lambda_n^2 + (2\pi m)^2)}} (\psi_{n,m} \cos(2\pi ms) \\ &+ \theta_{n,m} \sin(2\pi ms)) e_n, \quad \varepsilon \geq 0, \ \omega = (\phi, \psi, \theta) \in \Omega, \ 0 \leq s \leq 1. \end{aligned} \tag{4.5}$$

Then we have

$$\frac{\mathrm{Tr} e^{-\beta\hbar H_\hbar}}{\mathrm{Tr} e^{-\beta\hbar H_0}} = \lim_{N,M\to\infty} \int_\Omega \exp\left(-\beta \int_0^1 V\left(F_{N,M}(\varepsilon, \omega, s)\right) ds\right) d\nu(\omega), \tag{4.6}$$

where $\varepsilon = \beta\hbar$ (See [2], Lemma 5.2, Lemma 5.3.).

We set

$$Z(\varepsilon) = \lim_{N,M\to\infty} \int_\Omega \exp\left(-\beta \int_0^1 F_{N,M}(\varepsilon, \omega, s) ds \right) d\nu(\omega), \quad \varepsilon \geq 0, \tag{4.7}$$

For all $n, m \in \mathbb{N}$, we set

$$\alpha_{n,m}(\varepsilon) = \sqrt{\frac{4\varepsilon^2 \lambda_n}{\beta(\varepsilon^2\lambda_n^2 \ + \ (2\pi m)^2)}}, \quad \varepsilon \geq 0.$$

Then, for all $\delta > 0$, there exists a constant $C > 0$ such that

$$|\alpha_{n,m}(\varepsilon)| \leq \frac{C\sqrt{\lambda_n}}{m}, \quad n, m \in \mathbb{N},\ 0 \leq \varepsilon \leq \delta. \tag{4.8}$$

$$|\alpha'_{n,m}(\varepsilon)| \leq \frac{C\sqrt{\lambda_n}}{m}, \quad n, m \in \mathbb{N},\ 0 \leq \varepsilon \leq \delta. \tag{4.9}$$

$$|\alpha''_{n,m}(\varepsilon)| \leq \frac{C\lambda_n^{5/2}}{m}, \quad n, m \in \mathbb{N},\ 0 \leq \varepsilon \leq \delta. \tag{4.10}$$

$$|\alpha'''_{n,m}(\varepsilon)| \leq \frac{C(\lambda_n^{5/2} \ + \ \lambda_n^{9/2})}{m}, \quad n, m \in \mathbb{N},\ 0 \leq \varepsilon \leq \delta. \tag{4.11}$$

We denote by $\mu^{(L)}_{[\,0,1\,]}$ the Lebesgue measure on $[\,0,1\,]$. Then by (4.8),(4.9),(4.10) and (4.11), we can prove the following lemma.

LEMMA 4.1. *$\{F_{N,M}\}_{N,M\in\mathbb{N}}$, $\{F'_{N,M}\}_{N,M\in\mathbb{N}}$, $\{F''_{N,M}\}_{N,M\in\mathbb{N}}$, $\{F'''_{N,M}\}_{N,M\in\mathbb{N}}$ are Cauchy nets in $(\bigcap_{p\in\mathbb{N}} L^p(\Omega \times [\,0,1\,], d(\nu \otimes \mu^{(L)}_{[\,0,1\,]}); E))^{\mathbb{R}_+}_{\text{u.i.}}$.*

For all $N, M \in \mathbb{N}$, we set

$$G_{N,M}(\varepsilon, \omega) = \exp\left(-\beta \int_0^1 V\left(F_{N,M}(\varepsilon, \omega, s)\right) ds \right) \quad \varepsilon \geq 0,\ \omega \in \Omega.$$

Then by Proposition 3.4 and Lemma 4.1, we can prove the following lemma.

LEMMA 4.2. *$\{G_{N,M}\}_{N,M\in\mathbb{N}}$, $\{G'_{N,M}\}_{N,M\in\mathbb{N}}$, $\{G''_{N,M}\}_{N,M\in\mathbb{N}}$, $\{G'''_{N,M}\}_{N,M\in\mathbb{N}}$ are Cauchy nets in $(\bigcap_{p\in\mathbb{N}} L^p(\Omega, d\nu))^{\mathbb{R}_+}_{\text{u.i.}}$.*

By Lemma 4.2 and the fact that $\alpha_{n,m}$ is infinitely differentiable for all $n, m \in \mathbb{N}$, $\int_\Omega H_{N,M}(\varepsilon, \omega)\, d\nu(\omega)$ with $H_{N,M} = G_{N,M}, G'_{N,M}, G''_{N,M}, G'''_{N,M}$ uniformly converges in ε. Hence one can interchange the limit $\lim_{N,M\to\infty}$ with differentiations in ε and see that Z is 3-times continuously differentiable in $\mathbb{R}_+$.

We can prove the following theorem.

THEOREM 4.3. *For all $\beta > 0$,*

$$\begin{aligned}
&\frac{\mathrm{Tr} e^{-\beta\hbar H_\hbar}}{\mathrm{Tr} e^{-\beta\hbar H_0}} \\
= & \int_E \exp\left(-\beta V\left(\sqrt{\frac{2}{\beta}} A^{-1/2}\phi\right)\right) d\mu(\phi) \\
& -\frac{\beta^3\hbar^2}{2}\sum_{m=1}^{\infty}\int_{E^2} d\mu(\phi)d\mu(\psi)\exp\left(-\beta V\left(\sqrt{\frac{2}{\beta}}A^{-1/2}\phi\right)\right) \\
\times\ & V''\left(\sqrt{\frac{2}{\beta}}A^{-1/2}\phi\right)\left(A^{1/2}\left(\frac{1}{\sqrt{\beta}\pi m}\sum_{n=1}^{\infty}\psi_{n,m}e_n\right), A^{1/2}\left(\frac{1}{\sqrt{\beta}\pi m}\sum_{n=1}^{\infty}\psi_{n,m}e_n\right)\right) \\
& +o(\hbar^2)
\end{aligned}$$

as $\hbar \to 0$.

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