Definable slices in o-minimal structures

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Abstract

Let $G$ be a definably compact definable group and $X$ a definable $G$ set. We prove that there exists a definable slice at every point of $X$ and $X$ is covered by finitely many definable $G$ tubes.

1 Introduction

In this paper we consider definable slices in an o-minimal expansion $\mathcal{N} = (R, +, \cdot, <, ...)$ of a real closed field $R$. It is known that there exist uncountably many o-minimal expansions of the field $\mathbb{R}$ of real numbers([11]).

Definable set and definable maps are studied in [2], [3], and see also [12]. Everything is considered in $\mathcal{N} = (R, +, \cdot, <, ...)$ and definable maps are assumed to be continuous unless otherwise stated.

In this paper we prove the existence of a slice in the definable category.

Theorem 1.1. Let $G$ be a definably compact definable group and $X$ a definable $G$ set.

(1) For every point $x \in X$, there exists a definable slice $S$ at $x$.
(2) $X$ is covered by finitely many definable $G$ tubes.

Theorem 1.1 is a generalization of [5].
2 Preliminaries

Let $G$ be a topological group, $X$ a $G$ space and $x \in X$. A slice at $x$ is a subset $S$ of $X$ containing $x$ such that $G_x S = S$ and the map $\phi : G \times G_x S \to X$ defined by $\phi([g,s]) = gs$ is a $G$ imbedding onto a $G$ invariant open neighborhood $GS$ of $G(x)$ in $X$, and $GS$ is called a $G$ tube. It is known that there exists a slice when $G$ is a compact Lie group and $X$ is a completely regular $G$ space ([4], [8], [9]).

A subset $X$ of $R^n$ is definable (in $\mathcal{N}$) if it is defined by a formula (with parameters). Namely, there exist a formula $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ and elements $b_1, \ldots, b_m \in R$ such that $X = \{(a_1, \ldots, a_n) \in R^n | \phi(a_1, \ldots, a_n, b_1, \ldots, b_m) \}$ is true in $\mathcal{N}$.

For any $-\infty \leq a < b \leq \infty$, an open interval $(a, b)_R$ means $\{x \in R | a < x < b\}$, for any $a, b \in R$ with $a < b$, a closed interval $[a, b]_R$ means $\{x \in R | a \leq x \leq b\}$. We call $\mathcal{N}$ o-minimal (order-minimal) if every definable subset of $R$ is a finite union of points and open intervals.

A real closed field $(R, +, \cdot, <)$ is an o-minimal structure and every definable set is a semialgebraic set [13], and a definable map is a semialgebraic map [13]. In particular, the semialgebraic category is a special case of a definable one.

The topology of $R$ is the interval topology and the topology of $R^n$ is the product topology. Note that $R^n$ is a Hausdorff space.

The field $\mathbb{R}$ of real numbers, $\mathbb{R}_{alg} = \{x \in \mathbb{R} | x$ is algebraic over $\mathbb{Q}\}$ are Archimedean real closed fields.

The Puiseux series $\mathbb{R}[X]^\wedge$, namely $\sum_{i=k}^{\infty} a_i X^{i^q}, k \in \mathbb{Z}, q \in \mathbb{N}, a_i \in \mathbb{R}$ is a non-Archimedean real closed field.

**Fact 2.1.** (1) The characteristic of a real closed field is 0.

(2) For any cardinality $\kappa \geq \aleph_0$, there exist $2^\kappa$ many non-isomorphic real closed fields whose cardinality are $\kappa$.

(3) In a general real closed field, even for a $C^\infty$ function, the intermediate value theorem, existence theorem of maximum and minimum, Rolle’s theorem, the mean value theorem do not hold. Even for a $C^\infty$ function $f$ in one variable, the result that $f' > 0$ implies $f$ is increasing does not hold.

**Definition 2.2.** Let $X \subset R^n$, $Y \subset R^m$ be definable sets.

(1) A continuous map $f : X \to Y$ is a definable map if the graph of $f$ ($\subset R^n \times R^m$) is definable.
(2) A definable map \( f : X \to Y \) is a definable homeomorphism if there exists a definable map \( f' : Y \to X \) such that \( f \circ f' = id_Y \), \( f' \circ f = id_X \).

**Definition 2.3.** A group \( G \) is a definable group if \( G \) is definable and the group operations \( G \times G \to G, G \to G \) are definable.

Let \( G \) be a definable group. A pair \((X, \phi)\) consisting a definable set \( X \) and a \( G \) action \( \phi : G \times X \to X \) is a definable \( G \) set if \( \phi \) is definable. We simply write \( X \) instead of \((X, \phi)\).

**Definition 2.4.** Let \( X, Y \) be definable \( G \) sets.

1. A definable map \( f : X \to Y \) is a definable \( G \) map if for any \( x \in X, g \in G \), \( f(gx) = gf(x) \).
2. A definable \( G \) map \( f : X \to Y \) is a definable \( G \) homeomorphism if there exists a definable \( G \) map \( h : Y \to X \) such that \( f \circ h = id_Y \), \( h \circ f = id_X \).

**Definition 2.5.** (1) A definable set \( X \subset \mathbb{R}^n \) is definably compact if for any definable map \( f : (a, b)_\mathbb{R} \to X \), there exist the limits \( \lim_{x \to a+0} f(x), \lim_{x \to b-0} f(x) \) in \( X \).

2. A definable set \( X \subset \mathbb{R}^n \) is definably connected if there exist no definable open subsets \( U, V \) of \( X \) such that \( X = U \cup V, U \cap V = \emptyset, U \neq \emptyset, V \neq \emptyset \).

A compact (resp. A connected) definable set is definably compact (resp. definably connected). But a definably compact (resp. a definably connected) definable set is not always compact (resp. connected). For example, if \( R = \mathbb{R}_{alg} \), then \( [0, 1]_{\mathbb{R}_{alg}} = \{x \in \mathbb{R}_{alg} | 0 \leq x \leq 1 \} \) is definably compact and definably connected, but it is neither compact nor connected.

**Theorem 2.6 ([10]).** For a definable set \( X \subset \mathbb{R}^n \), \( X \) is definably compact if and only if \( X \) is closed and bounded.

The following is a definable version of the fact that the image of a compact (resp. a connected) set by a continuous map is compact (resp. connected).

**Proposition 2.7.** Let \( X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m \) be definable set, \( f : X \to Y \) a definable map. If \( X \) is definably compact (resp. definably connected), then \( f(X) \) is definably compact (resp. definably connected).
Theorem 2.8. (1) (The intermediate value theorem) For a definable function $f$ on a definably connected set $X$, if $a, b \in X$, $f(a) \neq f(b)$ then $f$ takes all values between $f(a)$ and $f(b)$.

(2) (Existence theorem of maximum and minimum) Every definable function on a definably compact definable set attains maximum and minimum.

(3) (Rolle's theorem) Let $f : [a, b]_R \to R$ be a definable function such that $f$ is differentiable on $(a, b)_R$ and $f(a) = f(b)$. Then there exists $c$ between $a$ and $b$ with $f'(c) = 0$.

(4) (The mean value theorem) Let $f : [a, b]_R \to R$ be a definable function which is differentiable on $(a, b)_R$. Then there exists $c$ between $a$ and $c$ with $f'(c) = \frac{f(b) - f(a)}{b - a}$.

(5) Let $f : (a, b)_R \to R$ be a differentiable definable function. If $f' > 0$ on $(a, b)_R$, then $f$ is increasing.

Example 2.9. (1) Let $\mathcal{N}$ be $((\mathbb{R}_{alg}, +, \cdot, <)$. Then $f : \mathbb{R}_{alg} \to \mathbb{R}_{alg}, f(x) = 2^x$ is not defined ([14]).

(2) Let $\mathcal{N}$ be $((\mathbb{R}, +, \cdot, <)$. Then $f : \mathbb{R} \to \mathbb{R}, f(x) = 2^x$ is defined but not definable in $\mathcal{N}$, and $h : \mathbb{R} \to \mathbb{R}, h(x) = \sin x$ is defined but not definable in $\mathcal{N}$.

3 Idea of proof of Theorem 1.1

We say that two homogeneous definable $G$ sets are equivalent if they are definably $G$ homeomorphic. Let $(G/H)$ be the equivalence class of $G/H$. The set of equivalence classes of homogeneous definable $G$ sets has an defined $(X) \cong (Y)$ if there exists a definable $G$ map $X \to Y$. Then the reflexivity and the transitivity hold and the anti-symmetry is true.

Theorem 3.1 ([6]). Let $G$ be a definably compact definable group. Then every definable $G$ set has only finitely many orbit types.

Theorem 3.2 ([6]). Let $G$ be a definably compact definable group, $X$ a definable $G$ set with transitive action and $x \in X$. Then the map $f : G/G_x \to X$ defined by $f(gG_x) = gx$ is a definable $G$ homeomorphism.

The following is a fundamental facts of o-minimal structures.

Theorem 3.3. (1) (Monotonicity theorem (e.g. 3.1.2, 3.1.6. [2])). Let $f : (a, b)_R \to R$ be a function with the definable graph. Then there exist
finitely many points $a = a_{0} < a_{1} < \cdots < a_{k} = b$ such that on each subinterval $(a_{j}, a_{j+1})_{R}$, the function is either constant, or strictly monotone and continuous. Moreover for any $c \in (a, b)_{R}$, the limits $\lim_{x\to c+0} f(x), \lim_{x\to c-0} f(x)$ exist in $R \cup \{\infty\} \cup \{-\infty\}$.

(2) (Cell decomposition theorem (e.g. 3.2.11. [2])). For any definable subsets $A_{1}, \ldots, A_{k}$ of $R^{n}$, there exists a cell decomposition of $R^{n}$ partitioning each $A_{1}, \ldots, A_{k}$.

Let $A$ be a definable subset of $R^{n}$ and $f : A \to R$ a function with the definable graph. Then there exists a cell decomposition $D$ of $R^{n}$ partitioning $A$ such that each $B \subset A, B \in D$, $f|B : B \to R$ is continuous.

(3) (Triangulation theorem (e.g. 8.2.9. [2])). Let $S \subset R^{n}$ be a definable set and let $S_{1}, S_{2}, \ldots, S_{k}$ be definable subsets of $S$. Then $S$ has a triangulation in $R^{n}$ compatible with $S_{1}, \ldots, S_{k}$.

(4) (Piecewise trivialization theorem (e.g. 8.2.9. [2])). Let $f : S \to A$ be a definable map between definable sets $S$ and $A$. Then there is a finite partition $A_{1}, \ldots, A_{k}$ of $A$ into definable sets such that each $f|f^{-1}(A_{i}) : f^{-1}(A_{i}) \to A_{i}$ is definably trivial.

(5) (Existence of definable quotients (e.g. 10.2.18 [2])). Let $G$ be a definably compact definable group and $X$ a definable $G$ set. Then the orbit space $X/G$ exists as a definable set and the orbit map $\pi : X \to X/G$ is surjective, definable and definably proper.

Let $G$ be a definably compact definable group, $X$ a definable $G$ set and $x \in X$. A definable slice at $x$ is a definable subset $S$ of $X$ containing $x$ such that $G_{x}S = S$ and the map $\phi : G \times_{G_{x}} S \to X$ defined by $\phi([g, s]) = gs$ is a definable $G$ imbedding onto a $G$ invariant definable open neighborhood $GS$ of $G(x)$ in $X$, and $GS$ is called a definable $G$ tube. Remark that $G \times_{G_{x}} S$ exists a definable set because $G_{x}$ is definably compact and Theorem 3.3, and the natural $G$ action $G \times G \times_{G_{x}} S \to G \times_{G_{x}} S, (g, [g', x]) \mapsto [gg', x]$ induced by $G \times G \times S \to G \times S, (g, (g', x)) \mapsto (gg', x)$ is definable.

Proposition 3.4 (e.g. II. 4.2 [1]). Let $G$ be a compact Lie group, $X$ a $G$ set, $S$ a subset of $X$ and $x \in S$. Then the following three conditions are equivalent.

(1) There exists a $G$ imbedding $\phi : G \times_{G_{x}} A \to X$ onto a $G$ invariant open neighborhood of $G(x)$ with $\phi([e, A]) = S$, where $A$ is a $G_{x}$ space.

(2) $S$ is a slice at $x$.

(3) $GS$ is a $G$ invariant open neighborhood of $G(x)$ and there exists a $G$ retraction $f : GS \to G(x)$ such that $f^{-1}(x) = S$. 

By a way similar to the proof of Proposition 3.1, we have the following proposition.

**Proposition 3.5** ([6]). Let $G$ be a definably compact definable group, $X$ a definable $G$ set, $S$ a definable subset of $X$ and $x \in S$. Then the following three conditions are equivalent.

1. There exists a definable $G$ imbedding $\phi : G \times G_x \rightarrow X$ onto a $G$ invariant definable open neighborhood of $G(x)$ with $\phi([e, A]) = S$, where $A$ is a definable $G_x$ set.
2. $S$ is a definable slice at $x$.
3. $GS$ is a $G$ invariant definable open neighborhood of $G(x)$ and there exists a definable $G$ retraction $f : GS \rightarrow G(x)$ such that $f^{-1}(x) = S$.

By a way similar to the proof of 2.5 [7], we have the following theorem.

**Theorem 3.6** ([6]). Let $G$ be a definably compact definable group, $X$ a definable $G$ set, $Y$ a definable set and $f : X \rightarrow Y$ a $G$ invariant surjective definable map. Then there exists a finite partition $\{C_i\}_i$ of $Y$ into definable sets such that each $f|f^{-1}(C_i) : f^{-1}(C_i) \rightarrow C_i$ is definably $G$ trivial.

**Proposition 3.7** ([6]). Let $X$ be a definable set and $A$ a definable closed subset of $X$. Suppose that $A$ is a definable strong deformation retract of $X$. Then for any definable open neighborhood $U$ of $A$ in $X$, there exist a definable closed neighborhood $N$ of $A$ in $U$ and a definable map $\rho : X \rightarrow U$ such that $\rho|N = \text{id}$ and $\rho(X - N) \subset U - N$.

**Idea of Proof of Theorem 1.1.**

We prove the condition (3) in Proposition 3.5.

To do so, we use finiteness of orbit types, piecewise trivialization of restrictions of orbit map, triangulation, Proposition 3.7.

**References**


