数理解析研究所講究録
第1922巻 2014年 1-7

Definable slices in o-minimal structures

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Abstract

Let G be a definably compact definable group and X a definable G set. We prove that there exists a definable slice at every point of X and X is covered by finitely many definable G tubes.

1 Introduction

In this paper we consider definable slices in an o-minimal expansion $\mathcal{N}=(R,+,\cdot,<,...)$ of a real closed field R. It is known that there exist uncountably many o-minimal expansions of the field $\mathbb{R}$ of real numbers([11]).

Definable set and definable maps are studied in [2], [3], and see also [12]. Everything is considered in $\mathcal{N}=(R,+,\cdot,<,...)$ and definable maps are assumed to be continuous unless otherwise stated.

In this paper we prove the existence of a slice in the definable category.

Theorem 1.1. *Let G be a definably compact definable group and X a definable G set.*

(1) For every point $x\in X$, there exists a definable slice S at x.

(2) X is covered by finitely many definable G tubes.

Theorem 1.1 is a generalization of [5].

2010 *Mathematics Subject Classification.* 14P10, 57S99, 03C64.
Key Words and Phrases. Definable slices, o-minimal, real closed fields.

2 Preliminaries

Let G be a topological group, X a G space and $x \in X$. A *slice* at x is a subset S of X containing x such that $G_x S = S$ and the map $\phi : G \times_{G_x} S \to X$ defined by $\phi([g, s]) = gs$ is a G imbedding onto a G invariant open neighborhood GS of $G(x)$ in X, and GS is called a G *tube*. It is known that there exists a slice when G is a compact Lie group and X is a completely regular G space ([4], [8], [9]).

A subset X of R^n is *definable* (in $\mathcal{N}$) if it is defined by a formula (with parameters). Namely, there exist a formula $\phi(x_1, \dots, x_n, y_1, \dots, y_m)$ and elements $b_1, \dots, b_m \in R$ such that $X = \{(a_1, \dots, a_n) \in R^n | \phi(a_1, \dots, a_n, b_1, \dots, b_m)$ is true in $\mathcal{N}\}$.

For any $-\infty \leqq a < b \leqq \infty$, an open interval $(a, b)_R$ means $\{x \in R | a < x < b\}$, for any $a, b \in R$ with $a < b$, a closed interval $[a, b]_R$ means $\{x \in R | a \leqq x \leqq b\}$. We call $\mathcal{N}$ *o-minimal* (*order-minimal*) if every definable subset of R is a finite union of points and open intervals.

A real closed field $(R, +, \cdot, <)$ is an o-minimal structure and every definable set is a semialgebraic set [13], and a definable map is a semialgebraic map [13]. In particular, the semialgebraic category is a special case of a definable one.

The topology of R is the interval topology and the topology of R^n is the product topology. Note that R^n is a Hausdorff space.

The field $\mathbb{R}$ of real nubmers, $\mathbb{R}_{alg} = \{x \in \mathbb{R} | x$ is algeraic over $\mathbb{Q}\}$ are Archimedean real closed fields.

The Puiseux series $\mathbb{R}[X]^\wedge$, namely $\sum_{i=k}^{\infty} a_i X^{\frac{i}{q}}, k \in \mathbb{Z}, q \in \mathbb{N}, a_i \in \mathbb{R}$ is a non-Archimedean real closed field.

Fact 2.1. *(1) The characteristic of a real closed field is* 0.

(2) For any cardinality $\kappa \geqq \aleph_0$, there exist 2^κ many non-isomorphic real closed fields whose cardinality are κ.

(3) In a general real closed field, even for a C^∞ function, the intermediate value theorem, existence theorem of maximum and minimum, Rolle's theorem, the mean value theorem do not hold. Even for a C^∞function f in one varianble, the result that $f' > 0$ implies f is increasing does not hold.

Definition 2.2. Let $X \subset R^n$, $Y \subset R^m$ be definable sets.

(1) A continuous map $f : X \to Y$ is a *definable map* if the graph of f ($\subset R^n \times R^m$) is definable.

(2) A definable map $f : X \to Y$ is a *definable homeomorphism* if there exists a definable map $f' : Y \to X$ such that $f \circ f' = id_Y, f' \circ f = id_X$.

Definition 2.3. A group G is a *definable group* if G is definable and the group operations $G \times G \to G, G \to G$ are definable.

Let G be a definable group. A pair (X, ϕ) consisting a definable set X and a G action $\phi : G \times X \to X$ is a *definable G set* if ϕ is definable. We simply write X instead of (X, ϕ).

Definition 2.4. Let X, Y be definable G sets.

(1) A definable map $f : X \to Y$ is a *definable G map* if for any $x \in X, g \in G$, $f(gx) = gf(x)$.

(2) A definable G map $f : X \to Y$ is a *definable G homeomorphism* if there exists a definable G map $h : Y \to X$ such that $f \circ h = id_Y$, $h \circ f = id_X$.

Definition 2.5. (1) A definable set $X \subset R^n$ is *definably compact* if for any definable map $f : (a, b)_R \to X$, there exist the limits $\lim_{x \to a+0} f(x), \lim_{x \to b-0} f(x)$ in X.

(2) A definable set $X \subset R^n$ is *definably connected* if there exist no definable open subsets U, V of X such that $X = U \cup V, U \cap V = \emptyset, U \neq \emptyset, V \neq \emptyset$.

A compact (resp. A connected) definable set is definably compact (resp. definably connected). But a definably compact (resp. a definably connected) definable set is not always compact (resp. connected). For example, if $R = \mathbb{R}_{alg}$, then $[0, 1]_{\mathbb{R}_{alg}} = \{x \in \mathbb{R}_{alg} | 0 \leqq x \leqq 1\}$ is definably compact and definably connected, but it is neither compact nor connected.

Theorem 2.6 ([10]). *For a definable set $X \subset R^n$, X is definably compact if and only if X is closed and bounded.*

The following is a definable version of the fact that the image of a compact (resp. a connected) set by a continuous map is compact (resp. connected).

Proposition 2.7. *Let $X \subset R^n$, $Y \subset R^m$ be definable set, $f : X \to Y$ a definable map. If X is definably compact (resp. definably connected), then $f(X)$ is definably compact (resp. definably connected).*

Theorem 2.8. *(1) (The intermediate value theorem) For a definable function f on a definably connected set X, if $a, b \in X$, $f(a) \neq f(b)$ then f takes all values between $f(a)$ and $f(b)$.*

(2) (Existence theorem of maximum and minimum) Every definable function on a definably compact definable set attains maximum and minimum.

(3) (Rolle's theorem) Let $f : [a, b]_R \to R$ be a definable function such that f is differentiable on $(a, b)_R$ and $f(a) = f(b)$. Then there exists c between a and c with $f'(c) = 0$.

(4) (The mean value theorem) Let $f : [a, b]_R \to R$ be a definable function which is differentiable on $(a, b)_R$. Then there exists c between a and c with $f'(c) = \frac{f(b)-f(a)}{b-a}$.

(5) Let $f : (a, b)_R \to R$ be a differentiable definable function. If $f' > 0$ on $(a, b)_R$, then f is increasing.

Example 2.9. *(1) Let $\mathcal{N}$ be $(\mathbb{R}_{alg}, +, \cdot, <)$. Then $f : \mathbb{R}_{alg} \to \mathbb{R}_{alg}, f(x) = 2^x$ is not defined([14])。*

(2) Let $\mathcal{N}$ be $(\mathbb{R}, +, \cdot, <)$. Then $f : \mathbb{R} \to \mathbb{R}, f(x) = 2^x$ is defined but not definable in $\mathcal{N}$, and $h : \mathbb{R} \to \mathbb{R}, h(x) = \sin x$ is defined but not definable in $\mathcal{N}$.

3 Idea of proof of Theorem 1.1

We say that two homogeneous definable G sets are *equivalent* if they are definably G homeomorphic. Let (G/H) be the equivalence class of G/H. The set of equivalence classes of homogeneous definable G sets has an defined $(X) \geqq (Y)$ if there exists a definable G map $X \to Y$. Then the reflexivity and the transitivity hold and the anti-symmetry is true.

Theorem 3.1 ([6]). *Let G be a definably compact definable group. Then every definable G set has only finitely many orbit types.*

Theorem 3.2 ([6]). *Let G be a definably compact definable group, X a definable G set with transitive action and $x \in X$. Then the map $f : G/G_x \to X$ defined by $f(gG_x) = gx$ is a definable G homeomorphism.*

The following is a fundamental facts of o-minimal structures.

Theorem 3.3. *(1) (Monotonicity theorem (e.g. 3.1.2, 3.1.6. [2])). Let $f : (a, b)_R \to R$ be a function with the definable graph. Then there exist* *finitely many points* $a = a_0 < a_1 < \cdots < a_k = b$ *such that on each subinterval* $(a_j, a_{j+1})_R$*, the function is either constant, or strictly monotone and continuous. Moreover for any* $c \in (a,b)_R$*, the limits* $\lim_{x\to c+0} f(x)$*,* $\lim_{x\to c-0} f(x)$ *exist in* $R \cup \{\infty\} \cup \{-\infty\}$*.*

(2) (Cell decomposition theorem (e.g. 3.2.11. [2])). For any definable subsets $A_1, \dots, A_k$ *of* R^n*, there exists a cell decomposition of* R^n *partitioning each* $A_1, \dots A_k$*.*

Let A *be a definable subset of* R^n *and* $f : A \to R$ *a function with the definable graph. Then there exists a cell decomposition* $\mathcal{D}$ *of* R^n *partitioning* A *such that each* $B \subset A, B \in \mathcal{D}$*,* $f|B : B \to R$ *is continuous.*

(3) (Triangulation theorem (e.g. 8.2.9. [2])). Let $S \subset R^n$ *be a definable set and let* $S_1, S_2, \dots, S_k$ *be definable subsets of* S*. Then* S *has a triangulation in* R^n *compatible with* $S_1, \dots, S_k$*.*

(4) (Piecewise trivialization theorem (e.g. 8.2.9. [2])). Let $f : S \to A$ *be a definable map between definable sets* S *and* A*. Then there is a finite partition* $A_1, \dots, A_k$ *of* A *into definable sets such that each* $f|f^{-1}(A_i) : f^{-1}(A_i) \to A_i$ *is definably trivial.*

(5) (Existence of definable quotients (e.g. 10.2.18 [2])). Let G *be a definably compact definable group and* X *a definable* G *set. Then the orbit space* X/G *exists as a definable set and the orbit map* $\pi : X \to X/G$ *is surjective, definable and definably proper.*

Let G be a definably compact definable group, X a definable G set and $x \in X$. A *definable slice* at x is a definable subset S of X containing x such that $G_x S = S$ and the map $\phi : G \times_{G_x} S \to X$ defined by $\phi([g,s]) = gs$ is a definable G imbedding onto a G invariant definable open neighborhood GS of $G(x)$ in X, and GS is called a *definable G tube*. Remark that $G \times_{G_x} S$ exists a definable set because G_x is definably compact and Theorem 3.3, and the natural G action $G \times G \times_{G_x} S \to G \times_{G_x} S, (g, [g', x]) \mapsto [gg', x]$ induced by $G \times G \times S \to G \times S, (g, (g', x)) \mapsto (gg', x)$ is definable.

Proposition 3.4 (e.g. II. 4.2 [1]). *Let* G *be a compact Lie group,* X *a* G *set,* S *a subset of* X *and* $x \in S$*. Then the following three conditions are equivalent.*

(1) There exists a G *imbedding* $\phi : G \times_{G_x} A \to X$ *onto a* G *invariant open neighborhood of* $G(x)$ *with* $\phi([e, A]) = S$*, where* A *is a* G_x *space.*

(2) S *is a slice at* x*.*

(3) GS *is a* G *invariant open neighborhood of* $G(x)$ *and there exists a* G *retraction* $f : GS \to G(x)$ *such that* $f^{-1}(x) = S$*.*

By a way similar to the proof of Proposition 3.1, we have the following proposition.

Proposition 3.5 ([6]). *Let G be a definably compact definable group, X a definable G set, S a definable subset of X and $x \in S$. Then the following three conditions are equivalent.*

(1) There exists a definable G imbedding $\phi : G \times_{G_x} A \to X$ onto a G invariant definable open neighborhood of $G(x)$ with $\phi([e, A]) = S$, where A is a definable G_x set.

(2) S is a definable slice at x.

(3) GS is a G invariant definable open neighborhood of $G(x)$ and there exists a definable G retraction $f : GS \to G(x)$ such that $f^{-1}(x) = S$.

By a way similar to the proof of 2.5 [7], we have the following theorem.

Theorem 3.6 ([6]). *Let G be a definably compact definable group, X a definable G set, Y a definable set and $f : X \to Y$ a G invariant surjective definable map. Then there exists a finite partition $\{C_i\}_i$ of Y into definable sets such that each $f|f^{-1}(C_i) : f^{-1}(C_i) \to C_i$ is definably G trivial.*

Proposition 3.7 ([6]). *Let X be a definable set and A a definable closed subset of X. Suppose that A is a definable strong deformation retract of X. Then for any definable open neighborhood U of A in X, there exist a definable closed neighborhood N of A in U and a definable map $\rho : X \to U$ such that $\rho|N = id$ and $\rho(X - N) \subset U - N$.*

Idea of Proof of Theorem 1.1.

We prove the condition (3) in Proposition 3.5.

To do so, we use finiteness of orbit types, piecewise trivialization of restrictions of orbit map, triangulation, Proposition 3.7. ∎

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