Spaces of equivariant maps to toric varieties

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Abstract

The main purpose of this note is to announce the recent result in [13] concerning the homotopy type of spaces of algebraic maps from a real projective space to a compact smooth real toric variety. This note is also based on the joint work with Andrzej Kozlowski and Masahiro Ohno [8].

Toric varieties. An irreducible normal algebraic variety $X$ (over $\mathbb{C}$) is called a toric variety if it has an algebraic action of the complex algebraic torus $\mathbb{T}_{\mathbb{C}} = (\mathbb{C}^*)^r$, such that the orbit $\mathbb{T}_{\mathbb{C}} \cdot *$ of some point $* \in X$ is dense in $X$ and isomorphic to $\mathbb{T}_{\mathbb{C}}$.

A strong convex rational polyhedral cone $\sigma$ in $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ of the form $\sigma = \{ \sum_{k=1}^{s} a_k n_k | a_k \geq 0 \}$, such that the set $\{ n_k \}_{k=1}^{s} \subset \mathbb{Z}^n$ does not contain any line.

A finite collection $\Sigma$ of strongly convex rational polyhedral cones in $\mathbb{R}^n$ is called a fan if every face of element of $\Sigma$ belongs to $\Sigma$ and the intersection of any two elements of $\Sigma$ is a face of each other. It is well-known that a toric variety $X$ is completely characterized up to isomorphism by the fan $\Sigma$. We denote by $X_{\Sigma}$ the corresponding toric variety associated to $\Sigma$. A cone $\sigma$ in $\mathbb{R}^n$ is called smooth (resp. simplicial) if it is generated by a subset of a basis of $\mathbb{Z}^n$ (resp. a subset of a basis of $\mathbb{R}^n$). A fan $\Sigma$ is called complete if $\bigcup_{\sigma \in \Sigma} \sigma = \mathbb{R}^n$. It is known that $X_{\Sigma}$ is compact iff $\Sigma$ is complete, and that $X_{\Sigma}$ is smooth iff every $\sigma \in \Sigma$ is smooth [4, Theorem 1.3.12]. It is also known that $\pi_1(X_{\Sigma})$ is isomorphic to the quotient of $\mathbb{Z}^n$ by the subgroup generated by $\bigcup_{\sigma \in \Sigma} \sigma \cap \mathbb{Z}^n$. [4, Theorem 12.1.10]. In particular, $X_{\Sigma}$ is simply connected if it is compact.

Real toric varieties. For a fan $\Sigma$, let $X_{\Sigma, \mathbb{R}}$ denote the subspace of $X_{\Sigma}$ consisting of all real points of $X_{\Sigma}$. Alternatively the space $X_{\Sigma, \mathbb{R}}$ is given by replacing the complex numbers $\mathbb{C}$ by the real numbers $\mathbb{R}$ everywhere in the given definitions of a toric variety $X_{\Sigma}$ [18, Def. 6.1], and it is called a real toric variety. Note that $X_{\Sigma, \mathbb{R}}$ with the intersection $X_{\Sigma, \mathbb{R}} = X_{\Sigma} \cap \mathbb{R}P^N$ when $X_{\Sigma}$ is a toric variety embedded in $\mathbb{C}P^N$.

Homogenous coordinates of $X_{\Sigma, \mathbb{K}}$. We shall use the symbols $\{ z_k \}_{k=1}^{s}$ to denote variables of polynomials, and we assume that $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For polynomials $f_1, \cdots, f_s \in $
\[ V_{\mathbb{K}}(f_{1}, \ldots, f_{s}) = \{ x \in \mathbb{K}^{r} \mid f_{k}(x) = 0 \text{ for each } 1 \leq k \leq s \}. \]

Let \( \Sigma(1) = \{ \rho_{1}, \ldots, \rho_{r} \} \) denote the set of all one dimensional cones in a fan \( \Sigma \), and let \( n_{k} \in \mathbb{Z}^{n} \) denote the generator of \( \rho_{k} \cap \mathbb{Z}^{n} \) such that \( \rho_{k} \cap \mathbb{Z}^{n} = \mathbb{Z}_{\geq 0} \cdot n_{k} \) (called the \textit{primitive element} of \( \rho_{k} \)) for each \( 1 \leq k \leq r \). Define the affine variety \( Z_{\Sigma, \mathbb{K}} \subset \mathbb{K}^{r} \) by

\[ Z_{\Sigma, \mathbb{K}} = V_{\mathbb{K}}(z^{\hat{\sigma}} \mid \sigma \in \Sigma), \]

where \( z^{\hat{\sigma}} \) denotes the monomial \( z^{\hat{\sigma}} = \prod_{1 \leq k \leq r, n_{k} \not\in \sigma} z_{k} \in \mathbb{Z}[z_{1}, \ldots, z_{r}] \). Let \( T_{\mathbb{K}}^{r} = (\mathbb{K}^{*})^{r} \) and define the subgroup \( G_{\Sigma, \mathbb{K}} \subset T_{\mathbb{K}}^{r} \) by

\[ G_{\Sigma, \mathbb{K}} = \{ (\mu_{1}, \ldots, \mu_{r}) \in T_{\mathbb{K}}^{r} \mid \prod_{k=1}^{r} \mu_{k}(m, n_{k}) = 1 \text{ for all } m \in \mathbb{Z}^{n} \}. \]

It is known that there is an isomorphism \( X_{\Sigma, \mathbb{K}} \cong (\mathbb{K}^{r} \setminus Z_{\Sigma, \mathbb{K}})/G_{\Sigma, \mathbb{K}} \) for \( \mathbb{K} = \mathbb{C} \) if the set \( \{ n_{1}, \ldots, n_{r} \} \) spans \( \mathbb{R}^{n} \), where the group \( G_{\Sigma, \mathbb{K}} \) acts on the complement \( \mathbb{K}^{r} \setminus Z_{\Sigma, \mathbb{K}} \) by the coordinate-wise multiplications and the space \( (\mathbb{K}^{r} \setminus Z_{\Sigma, \mathbb{K}})/G_{\Sigma, \mathbb{K}} \) denotes its orbit space.

It is known that \( G_{\Sigma, \mathbb{K}} \) acts freely on the complement \( \mathbb{K}^{r} \setminus Z_{\Sigma, \mathbb{K}} \) if \( \Sigma \) is smooth and \( \mathbb{K} = \mathbb{C} \). In this case, for \( \mathbb{K} = \mathbb{C} \) there are isomorphisms

\[ X_{\Sigma, \mathbb{K}} \cong (\mathbb{K}^{r} \setminus Z_{\Sigma, \mathbb{K}})/G_{\Sigma, \mathbb{K}} \quad \text{and} \quad G_{\Sigma, \mathbb{K}} \cong T_{\mathbb{K}}^{r-n}. \]

Note that (1.4) also holds for \( \mathbb{K} = \mathbb{R} \) if \( \Sigma \) is smooth and complete [19, Lemma 7.3].

We say that a set of primitive elements \( \{ n_{i_{1}}, \ldots, n_{i_{k}} \} \) is \textit{primitive} if they do not lie in any cone in \( \Sigma \) but every proper subset does. It is known that

\[ Z_{\Sigma, \mathbb{K}} = \bigcup_{\{ n_{i_{1}}, \ldots, n_{i_{k}} \}: \text{primitive}} V_{\mathbb{K}}(z_{i_{1}}, \ldots, z_{i_{k}}). \]

So \( Z_{\Sigma, \mathbb{K}} \) is a closed variety with real dimension \( (r - r_{\min}) \dim_{\mathbb{R}} \mathbb{K} \), where we set

\[ r_{\min} = \min \{ k \in \mathbb{Z}_{\geq 1} \mid \{ n_{i_{1}}, \ldots, n_{i_{k}} \} \text{ is primitive} \}. \]

**Spaces of continuous maps.** For connected spaces \( X \) and \( Y \), let \( \text{Map}(X, Y) \) be the space of all continuous maps \( f : X \to Y \) and \( \text{Map}^{*}(X, Y) \) the corresponding subspace of all based continuous maps. If \( m \geq 2 \) and \( g \in \text{Map}^{*}(\mathbb{R}P^{m-1}, X) \), let \( F(\mathbb{R}P^{m}, X; g) \) denote the subspace of \( \text{Map}^{*}(\mathbb{R}P^{m}, X) \) given by

\[ F(\mathbb{R}P^{m}, X; g) = \{ f \in \text{Map}^{*}(\mathbb{R}P^{m}, X) : f|_{\mathbb{R}P^{m-1}} = g \}, \]

where we identify \( \mathbb{R}P^{m-1} \subset \mathbb{R}P^{m} \) by putting \( x_{m} = 0 \). It is known that there is a homotopy equivalence \( F(\mathbb{R}P^{m}, X; g) \simeq \Omega^{m}X \) if it is not an empty set.
Assumptions. From now on, we assume that the following two conditions are satisfied:

(1.7.1) Let $\Sigma$ be a complete smooth fan in $\mathbb{R}^{n}$, $\Sigma(1) = \{ \rho_1, \cdots, \rho_r \}$ be the set of all one-dimension cones in $\Sigma$, and all primitive elements $\{ n_1, \cdots, n_r \}$ of the fan $\Sigma$ spans $\mathbb{R}^{n}$, where $n_k \in \mathbb{Z}^{n}$ denotes the primitive element of $\rho_k$ for $1 \leq k \leq r$.

(1.7.2) Let $D = (d_1, \cdots, d_r)$ be an $r$-tuple of positive integers such that $\sum_{k=1}^{r} d_k n_k = 0$.

Then we can identify $X_{\Sigma,K} = (\mathbb{K}^r \setminus Z_{\Sigma,K})/G_{\Sigma,K}$ and we denote by $[a_1, \cdots, a_r]$ the corresponding element of $X_{\Sigma,K}$ for each $(a_1, \cdots, a_r) \in \mathbb{K}^r \setminus Z_{\Sigma,K}$.

Spaces of polynomials representing algebraic maps. Let $H_{d_1,m}^{\mathbb{K}} \subset \mathbb{K}[z_0, \cdots, z_m]$ denote the $\mathbb{K}$-vector subspace consisting of all homogeneous polynomials of degree $d$. Let $A_D(m)$ denote the space $A_D^{K}(m) = H_{d_1,m}^{\mathbb{K}} \times H_{d_2,m}^{\mathbb{K}} \times \cdots \times H_{d_r,m}^{\mathbb{K}}$ and let $A_{D,\Sigma}^{K}(m) \subset A_{D}^{K}(m)$ denote the subspace

\[
A_{D,\Sigma}^{K}(m) = \{(f_1, \cdots, f_r) \in A_{D}^{K}(m) | (f_{1}(e_{1}), \cdots, f_{r}(e_{1})) = (1,1, \cdots 1)\},
\]

where we set $F(x) = (f_1(x), \cdots, f_r(x))$. Because $(1,1, \cdots, 1) \in \mathbb{K}^r \setminus Z_{\Sigma,K}$, we choose $x_0 = [1, \cdots, 1] \in X_{\Sigma,K}$ as the base-point of $X_{\Sigma,K}$. Define the subspace $A_D(m, X_{\Sigma,K}) \subset A_{D,\Sigma}^{K}(m)$ by

\[
A_D(m, X_{\Sigma,K}) = \{(f_1, \cdots, f_r) \in A_{D,\Sigma}^{K}(m) | (f_1(x_1), \cdots, f_r(x_1)) = (1,1, \cdots 1)\},
\]

where $x_1 = (1,0, \cdots, 0) \in \mathbb{R}^{m+1}$, and let us choose $[e_1] = [1:0: \cdots : 0]$ as the base-point of $\mathbb{R}P^m$. Define the natural map $j_{D,K} : A_{D,\Sigma}^{K}(m) \to \text{Map}(\mathbb{R}P^m, X_{\Sigma,K})$ by

\[
j_{D,K}(f_1, \cdots, f_r)([x]) = [f_1(x), \cdots, f_r(x)] \quad \text{for} \quad x = (x_0, \cdots, x_m) \in \mathbb{R}^{m+1} \setminus \{0\}.
\]

Since the space $A_{D,\Sigma}^{K}(m)$ is connected, the image of $j_{D,K}$ lies in a connected component of $\text{Map}(\mathbb{R}P^m, X_{\Sigma,K})$, which is denoted by $\text{Map}_D(\mathbb{R}P^m, X_{\Sigma,K})$. This gives the natural map

\[
j_{D,K} : A_{D,\Sigma}^{K}(m) \to \text{Map}_D(\mathbb{R}P^m, X_{\Sigma,K}).
\]

Note that $j_{D,K}(f_1, \cdots, f_r) \in \text{Map}^*(\mathbb{R}P^m, X_{\Sigma,K})$ if $(f_1, \cdots, f_r) \in A_{D}^{K}(m, X_{\Sigma})$. Hence, if we set $\text{Map}^*_D(\mathbb{R}P^m, X_{\Sigma,K}) = \text{Map}^*(\mathbb{R}P^m, X_{\Sigma,K}) \cap \text{Map}_D(\mathbb{R}P^m, X_{\Sigma,K})$, we have the natural map

\[
i_{D,K} = j_{D,K} \mid A_D(m, X_{\Sigma,K}) : A_D(m, X_{\Sigma,K}) \to \text{Map}^*_D(\mathbb{R}P^m, X_{\Sigma,K}).
\]

Suppose that $m \geq 2$ and let us choose a fixed element $(g_1, \cdots, g_r) \in A_{D}(m-1, X_{\Sigma,K})$. For each $1 \leq k \leq r$, let $B_k^{K} = \{ g_k + z_m h : h \in H_{d_k-1,m}^{K} \}$. Then define the subspace $A_D(m, X_{\Sigma,K}; g) \subset A_D(m, X_{\Sigma,K})$ by

\[
A_D(m, X_{\Sigma,K}; g) = A_D(m, X_{\Sigma,K}) \cap (B_1^{K} \times B_2^{K} \times \cdots \times B_r^{K}).
\]
It is easy to see that $i_{D,K}(f_{1}, \cdots, f_{r})|\mathbb{RP}^{m-1} = g$ if $(f_{1}, \cdots, f_{r}) \in A_{D}(m, X_{\Sigma,K}; g)$, where $g$ denotes the map in $Map^{*}_{D}(\mathbb{RP}^{m-1}, X_{\Sigma,K})$ given by

(1.14) $g([x_{0} : \cdots : x_{m-1}]) = [g_{1}(x), \cdots, g_{r}(x)]$ for $x = (x_{0}, \cdots, x_{m-1}) \in \mathbb{R}^{m} \setminus \{0\}$.

Then, one can define the map $i'_{D,K} : A_{D}(m, X_{\Sigma,K}; g) \to F(\mathbb{RP}^{m}, X_{\Sigma,K}; g) \simeq \Omega^{m}X_{\Sigma,K}$ by

(1.15) $i'_{D,K} = i_{D,K}|A_{D}(m, X_{\Sigma,K}; g) : A_{D}(m, X_{\Sigma,K}; g) \to F(\mathbb{RP}^{m}, X_{\Sigma,K}; g) \simeq \Omega^{m}X_{\Sigma,K}$.

Now consider the action of $G_{\Sigma,K}$ on the space $A_{D}^{K}(m)$ given by the coordinate-wise multiplication and define the space $\overline{A}_{D}(m, X_{\Sigma,K})$ by the quotient space

(1.16) $\overline{A}_{D}(m, X_{\Sigma,K}) = A_{D}^{K}(m)/G_{\Sigma,K}$.

It is easy to see that one can define the map $j_{D,K} : \overline{A}_{D}(m, X_{\Sigma,K}) \to Map_{D}(\mathbb{RP}^{m}, X_{\Sigma,K})$ by

(1.17) $j_{D,K}([f_{1}, \cdots, f_{r}])([x_{0}, \cdots, x_{r}]) = [f_{1}(x), \cdots, f_{r}(x)]$ for $x \in \mathbb{R}^{m+1} \setminus \{0\}$.

Let $d_{\min}$ and $D_{\pi}(d_{1}, \cdots, d_{r}; m, r)$ be the positive integer defined by

(1.18) $d_{\min} = \min\{d_{1}, d_{2}, \cdots, d_{r}\}, \quad D(d_{1}, \cdots, d_{r}; m) = (r_{\min} - m - 1)d_{\min} - 2$.

From now on we write $(X_{\Sigma,K}, Z_{\Sigma,K}, G_{\Sigma,K}) = (X_{\Sigma}, Z_{\Sigma}, G_{\Sigma})$ if $K = \mathbb{C}$.

The main results. The main results of this note are stated as follows.

Theorem 1.1 ([13]). Let $\Sigma$ be a complete smooth fan in $\mathbb{R}^{n}$, let $\{d_{k} : 1 \leq k \leq r\}$ be the set of positive integers satisfying the conditions (1.7.1), (1.7.2), and let $X_{\Sigma,R}$ be a smooth compact real toric variety associated to the fan $\Sigma$. Then if $1 \leq m \leq r_{\min} - 2$ and $D = (d_{1}, \cdots, d_{r}) \in (\mathbb{Z}_{\geq 1})^{r}$, the map

$$i'_{D,R} : A_{D}(m, X_{\Sigma,R}; g) \to F(\mathbb{RP}^{m}, X_{\Sigma,R}; g) \simeq \Omega^{m}X_{\Sigma,R}$$

is a homology equivalence through dimension $D(d_{1}, \cdots, d_{r}; m)$.

Theorem 1.2 ([13]). Under the same assumptions as Theorem 1.1, if $1 \leq m \leq r_{\min} - 2$ and $D = (d_{1}, \cdots, d_{r}) \in (\mathbb{Z}_{\geq 1})^{r}$, the maps

$$\begin{align*}
&j_{D,R} : \overline{A}_{D}(m, X_{\Sigma,R}) \to Map_{D}(\mathbb{RP}^{m}, X_{\Sigma,R}) \\
i_{D,R} : A_{D}(m, X_{\Sigma,R}) \to Map^{*}_{D}(\mathbb{RP}^{m}, X_{\Sigma,R})
\end{align*}$$

are homology equivalences through dimension $D(d_{1}, \cdots, d_{r}; m)$.
Remark 1.3. (i) A map $f : X \rightarrow Y$ is called a homology equivalence through dimension $D$ if $f_* : H_k(X, \mathbb{Z}) \xrightarrow{\cong} H_k(Y, \mathbb{Z})$ is an isomorphism for any $k \leq D$.

(ii) Let $G$ be a finite group and let $f : X \rightarrow Y$ be a $G$-equivariant map between $G$-spaces $X$ and $Y$. Then the map $f : X \rightarrow Y$ is called a $G$-equivariant homology equivalence through dimension $D$ if $f^H : H_k(X^H, \mathbb{Z}) \xrightarrow{\cong} H_k(Y^H, \mathbb{Z})$ is an isomorphism for any $k \leq D$ and any subgroup $H \subset G$, where $W^H = \{ x \in W \mid g \cdot x = x \text{ for any } g \in H \}$ for a $G$-space $W$ and $f^H$ denotes the restriction map $f^H = f|X^H$.

(iii) Note that the complex conjugation on $\mathbb{C}$ naturally induces the $\mathbb{Z}/2$-action on the space $X_\Sigma$, and it is easy to see that $(X_\Sigma)^{\mathbb{Z}/2} = X_{\Sigma, R}$. Similarly, it also induces the $\mathbb{Z}/2$-actions on the space $A_D(m, X_\Sigma)$, $\tilde{A}_D(m, X_\Sigma)$, $A_D(m, X_\Sigma; g)$. Moreover, if we consider the space $\mathbb{R}P^m$ as a $\mathbb{Z}/2$-space of the trivial action, the $\mathbb{Z}/2$-action $X_\Sigma$ also induces the $\mathbb{Z}/2$-actions on the spaces $\text{Map}^*_D(\mathbb{R}P^m, X_\Sigma)$, $\text{Map}_D(\mathbb{R}P^m, X_\Sigma)$, $F(\mathbb{R}P^m, X_\Sigma; g)$.

Corollary 1.4 ([8], [13]). Under the same assumptions as Theorem 1.1, if $2 \leq m \leq r_{\min} - 2$ and $D = (d_1, \cdots, d_r) \in (\mathbb{Z}_2)^r$, the maps

\[
\left\{\begin{array}{l}
i_D'D_C : A_D(m, X_\Sigma; g) \rightarrow F(\mathbb{R}P^m, X_\Sigma; g) \simeq \Omega^m X_\Sigma \\
j_D'D_C : A_D(m, X_\Sigma) \rightarrow \text{Map}^*_D(\mathbb{R}P^m, X_\Sigma) \\
i_D'D_C : A_D(m, X_\Sigma) \rightarrow \text{Map}^*_D(\mathbb{R}P^m, X_\Sigma)
\end{array}\right.
\]

are $\mathbb{Z}/2$-equivariant homology equivalences through dimension $D(d_1, \cdots, d_r; m)$. \hfill \Box

References


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