数理解析研究所講究録
第 1922 巻 2014 年 135-146

Classifications of homogeneous complexity one GKM manifolds and GKM graphs with symmetric group actions

Dedicated to Professors Mikiya Masuda, Masaharu Morimoto and Kouhei Yamaguchi on their 60th birthday.

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Abstract. In this article, we first give a classification of simply connected, complexity one GKM manifolds with extended transitive G-actions. This is proved by applying the method to classify the homogeneous torus manifolds. Motivated by this result (Theorem 1.2), we next classify the 3-valent complexity one GKM graphs with certain S_3-actions.

1. Introduction

Let (M^{2m}, T^n) be a pair of $2m$-dimensional (compact, connected, simply connected) manifold with (almost) effective n-dimensional torus action, where an *almost effective* means that the T^n-action on M^{2m} has a finite kernel. If there is a fixed point and the one-skelton of its orbit space has the structure of a graph, then we call (M, T) a *(generalized) GKM manifold* (see [**GKM, GuZa, Da, Ku1, MMP**]). Note that this definition is slightly wider than the original definition in [**GuZa**], i.e., we do not assume the existence of an equivariant almost complex structre. By using the differentiable slice theorem, it is easy to check that $n \leq m$. So the extremal class of GKM manifolds would be the class when $m = n$. Such a GKM manifold is known as a *torus manifold* (see [**Ma99, HaMa**]). The torus manifold is defined by Hattori-Mausda in 2003 as the topological generalization of toric manifolds (i.e., non-singular, complete, toric varieties viewed as complex analytic space) in algebraic geometry.

One of the motivations of toric geometry in algebraic geometry is to study the automorphism groups of toric varieties (see [**Co, De, Od**]). Due to the results of Demazure and Cox, the root systems of fans or Cox rings determine the Lie algebras of automorphism groups of toric varieties. On the other hand, in this two decades, motivated by the study of Davis-Januszkiewicz [**DaJa**], the notions in toric geometry have been translated into the notions in topology, and now it is called toric topology (see [**BuPa, ToricTop**]). In toric topology, more general class of manifolds with topological torus T-actions, such as torus manifolds, is studied. Moreover, the problems studied in algebraic geometry inspire topologists to study new topological problems, such as cohomological rigidity problem (see [**CMS**]). In particular, from the topological point of view, the study of automorphism groups may be regarded as the study of extended G-actions of T-actions (see [**Ku3, Ku4**]). Assume that G is a compact Lie group. Motivated by the works of automorphism groups of toric manifolds, the extended G-actions of a torus manifold (and a symplectic toric manifold) are completely classified by several mathematicians in toric topology (and in symplectic geometry), see [**KuMa, Ma10, MT, Wi**].

In algebraic geometry and symplectic geometry, the manifolds with complexity one torus actions (not only toric manifolds) are also studied by several mathematicians (see [**ADHL, KaTo,**

The author was partially supported by Grant-in-Aid for Scientific Research (S)24224002, Japan Society for Promotion of Science.

Mo]). In particular, Arzhantsev-Derenthal-Hausen-Laface study the automorphism groups of such manifolds in [**ADHL**]. The purpose of this article is to study the manifolds with complexity one torus actions from topological point of view. More precisely, we study the extended actions of a *complexity one GKM manifold*, i.e., a GKM manifold (M^{2m}, T^n) with $m = n+1$, and give a partial answer to the following problem:

PROBLEM 1.1. *When does a complexity one GKM manifold admit an extended (M^{2n+2}, G) action? Here, G is a compact (connected) Lie group with maximal torus T^n.*

In this article, we solve Problem 1.1 for the case of simply connected complexity one GKM manifolds with transitive extended actions, called a *homogeneous complexity one GKM manifold.* In order to state our main result, we define some terminology. We call M an *irreducible* if M has the following property: if the manifold has the decomposition $M = M_1 \times M_2$ then $M_2 = \{*\}$. If M is not irreducible, then we call M a *reducible.* The following theorem is the 1st main result:

THEOREM 1.2. *Let (M^{2m}, T^{m-1}) be a simply connected homogeneous GKM manifold with a complexity one torus action. Then, M has the following decomposition:*

$$M^{2m} = M_1 \times \cdots M_k \times M^{2n}$$

such that M_i's are homogeneous irreducible simply connected torus manifolds, i.e., a complex projective space or an even dimensional sphere (see [**Ku3**]*), and M^{2n} is a simply connected, irreducible, homogeneous, complexity one GKM manifold. Furthermore, we have $n = 3$ or 4 and if $n = 3$, M is one of the following manifolds:*

A_2: $\mathcal{F}\ell(\mathbb{C}^3) \cong SU(3)/T^2$;
B_2: $Q_3 \cong SO(5)/SO(3) \times SO(2)$;
B_2: $\mathbb{C}P^3 \cong Spin(5)/Sp(1) \times T^1$;
C_2: $\mathbb{C}P^3 \cong Sp(2)/Sp(1) \times T^1$;
G_2: $S^6 \cong G_2/SU(3)$,

if $n = 4$, M is one of the following manifolds:

A_3: $G_2(\mathbb{C}^4) \cong SU(4)/S(U(2) \times U(2))$;
C_3: $\mathbb{H}P^2 \cong Sp(3)/Sp(2) \times Sp(1)$;
D_3: $Q_4 \cong SO(6)/SO(4) \times SO(2)$,

where $G_2(\mathbb{C}^4) \cong Q_4$.

As a consequence of Massey in [**Ma62**] and well-known results, we can easily check which manifolds in Theorem 1.2 have stably complex structures.

COROLLARY 1.3. Let M be a homogeneous, irreducible, complexity one GKM manifold. If there is a T-invariant stably complex structure on M, i.e., a unitary GKM manifold, then M is one of the followings:

$$\mathcal{F}\ell(\mathbb{C}^3), \quad Q_3, \quad \mathbb{C}P^3, \quad S^6 \quad \text{or} \quad G_2(\mathbb{C}^4).$$

Furthermore, every manifold as above also has a T-invariant almost complex structure.

REMARK 1.4. In Theorem 1.2, the manifolds $\mathcal{F}\ell(\mathbb{C}^3)$, Q_3, $\mathbb{H}P^2$ and $G_2(\mathbb{C}^4)$ are not torus manifolds. This can be proved by using their cohomology rings (see [**BuPa**]). On the other hand, $\mathbb{C}P^3$ and S^6 are torus manifolds; furthermore, they are unitary toric manifolds (see [**Ma99**]).

From the unitary GKM manifold, we can define a labelled graph, so-called *genelraized GKM graph.* Moreover, if a GKM manifold (M, T) has an extension (M, G) such that G preserves the unitary structure, then its Weyl group $W(G) = N_G(T)/T$ acts on the GKM graph induced from (M, T). Motivated by this facts and Theorem 1.2, we may ask the following question:

PROBLEM 1.5. *When does a 3 (resp. 4)-valent GKM graph admit a rank 2 (resp. 3) Weyl group actions?*

In this article, we also give a partial answer to Problem 1.5. More precisely, we also classify the 3-valent GKM graphs (not generalized GKM graph) with certain S_3-actions. The 2nd main result can be stated as follows:

THEOREM 1.6. *Let $(\Gamma, \mathcal{A})$ be a 3-valent complexity one GKM graph with an S_3-action. Assume that for every Weyl subgroup $W' \subset S_3$ and every vertex $p \in V(\Gamma)$, there is a GKM subgraph $(\Gamma', \mathcal{A}|_{E(\Gamma')})$ such that W' acts on it transitively and $p \in V(\Gamma')$. Then, $(\Gamma, \mathcal{A})$ is one of the GKM graphs in Figure 1.*

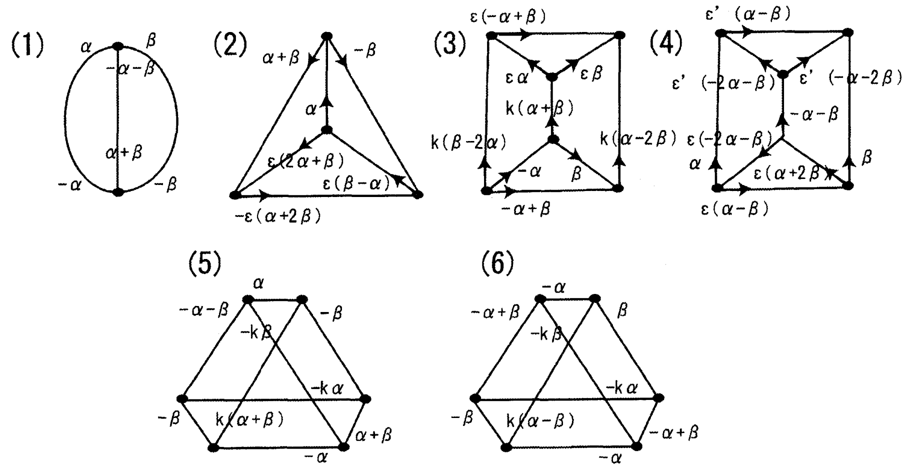

FIGURE 1. The list of complexity one GKM graphs with certain S_3-symmetries.

Furthermore, by using the invariants in [**Ta**] or [**Ku5**], we also have the following corollary:

COROLLARY 1.7. In Theorem 1.6, $(\Gamma, \mathcal{A})$ extends to the torus graph if and only if (2), (3), (4).

The organization of this article is as follows. In Section 2, we prove Theorem 1.2. In Section 3, we quickly recall GKM graphs with Weyl group symmetry. In Section 4, we give a sketch of the proof of Theorem 1.6.

2. Proof of Theorem 1.2 and observations

In this section, we prove Theorem 1.2 with the method similar to that demonstrated in [**Ku3**]. Moreover, we also give the method to construct infinitely many complexity one GKM manifolds with extended G-actions.

2.1. Proof of Theorem 1.2. Let (M^{2n}, T^{n-1}) be a simply connected GKM manifold. Assume (M^{2n}, T^{n-1}) has an extended (M^{2n}, G), where G is a compact (connected) Lie group with maximal torus T^{n-1}. Then, we may put $M = G/H$ such that rank G = rank $H = n-1$ and $\dim G/H = 2n$ for some closed subgroup $H \subset G$. Because M is simply connected, H is a connected subgroup.

Let $(\widetilde{G}, \widetilde{H})$ be the universal covering of (G, H). Then, by using Borel and De Siebenthal's result in [**BoSi**] and the assumption that the T^{n-1}-action is almost effective, we have

$$\widetilde{G} = G_1 \times \cdots \times G_m$$
$$\widetilde{H} = H_1 \times \cdots \times H_m,$$

where G_i is simply connected simple Lie group and H_i is its closed subgroup for $i = 1, \ldots, m$ such that rank G_i = rank $H_i = n_i$ and $\dim G_i - \dim H_i = 2d_i$. Because the T_i-action on G_i/H_i has a fixed point where T_i is a maximal torus of G_i and H_i, we have $n_i \leq d_i$ for all i. Moreover, we have the following lemma:

LEMMA 2.1. *We may assume $d_i = n_i$ for all $i = 1, \ldots, m-1$ and $d_m = n_m + 1$.*

PROOF. By using the assumption, we have that

$$n_1 + \cdots + n_m = d_1 + \cdots + d_m - 1 < d_1 + \cdots + d_m.$$

If $n_i = d_i$ for all i, then it is easy to show the contradiction to this inequality. Therefore, we may assume $n_m < d_m$. If $n_m \leq d_m - 2$, then it follows from $n_i \leq d_i$

$$d_1 + \cdots + d_m - 1 = n_1 + \cdots + n_m \leq d_1 + \cdots + d_{m-1} + (d_m - 2).$$

This also gives a contradiction. Thus, the equation $n_m = d_m - 1$ holds. Hence, by the above equation and $n_i \leq d_i$, we also have $n_i = d_i$ for $i \neq m$. □

By Lemma 2.1, we have $n_i = d_i$ for $i \neq m$. Hence, it follows from the main theorem of [**Ku3**] that we have the following decomposition:

$$\widetilde{G} = \prod_{i=1}^{a} SU(\ell_i + 1) \times \prod_{j=1}^{b} Spin(2m_j + 1) \times G_m$$

$$\widetilde{H} = \prod_{i=1}^{a} S(U(1) \times U(\ell_i)) \times \prod_{n=1}^{b} Spin(2m_j) \times H_m,$$

where $m_j \geq 2$, $a+b = m-1$. Here, rank G_m = rank $H_m = n_m$ and $\dim G_m/H_m = 2d_m = 2n_m+2$.

Consequently, the problem is reduced into the classification of (G_m, H_m). To classify (G_m, H_m) we consider the following two cases.

CASE 1: Assume H_m is a maximal same rank subgroup of a compact connected simply connected Lie group G_m. Then, by using the table [**Ku3**, Table 1, 2], we have that (G_m, H_m) is one of the following:

A_3: $(SU(4), S(U(2) \times U(2)))$, i.e., $M^8 = G_2(\mathbb{C}^4)$;
B_2: $(SO(5), SO(3) \times SO(2))$, i.e., $M^6 = Q_3$;
C_2: $(Sp(2), Sp(1) \times T^1)$, i.e., $M^6 = \mathbb{C}P^3$;
C_3: $(Sp(3), Sp(1) \times Sp(2))$, i.e., $M^8 = \mathbb{H}P^2$;
D_3: $(SO(6), SO(4) \times SO(2))$, i.e., $M^8 = Q_4$;
G_2: $(G_2, SU(3))$, i.e., $M^6 = S^6$.

Note that $Q_4 \cong G_2(\mathbb{C}^4)$.

CASE 2: Assume H_m is a non-maximal same rank subgroup of G_m such that rank G_m = rank $H_m = n$ and $\dim G_m/H_m = 2n + 2$. Then, there is a maximal same rank subgroup K_m such that $H_m \subset K_m \subset G_m$. Because $\dim G_m/H_m = 2n + 2 > \dim G_m/K_m$ (i.e., $\dim G_m/K_m \geq$ 2rank G_m), we have that $\dim G_m/K_m = 2n$. Therefore, G_m/K_m must be the homogeneous torus manifold and $K_m/H_m \cong SU(2)/T^1 \cong S^2$. Hence, by [**Ku3**], (G_m, K_m) is one of the following:

$$(SU(k+1), S(U(1) \times U(k)));$$
$$(Spin(2k+1), Spin(2k)).$$

Because $K_m/H_m \cong S^2$, we also have $k = 2$. Therefore, we have (G_m, K_m, H_m) is one of the followings:

$$(SU(3), S(U(1) \times U(2)), T^2);$$
$$(Spin(5), Spin(4), Sp(1) \times T^1),$$

where $Spin(4) \cong Sp(1)\times Sp(1)$ and $Spin(5) \simeq Sp(2)$. Because $\dim SU(3)/T^2 = 6$ and $\dim Spin(5)/Sp(1)\times T^1 = 6$, we have that (G_m, K_m) is one of the following:

A_2: $(SU(3), T^2)$, i.e., $M^6 = \mathcal{F}\ell(\mathbb{C}^3)$;
B_2: $(Spin(5), Sp(1) \times T^1)$, i.e., $M^6 = \mathbb{C}P^3$.

Consequently, we establish Theorem 1.2.

2.2. Observations. By using the GKM manifolds appearing in Theorem 1.2, we can construct other GKM manifolds; in particular, we can construct infinitely many complexity one GKM manifolds with extended $(SU(3) \times T^n)$-actions. Here, we show the construction.

EXAMPLE 2.2. Let X be a $2n$-dimensional torus manifold with $n \geq 2$ and $\rho : T^2 \to T^n$ be a faithful representation. Define the manifold $M_\rho(X)$ by the twisted product

$$SU(3) \times_{T^2} X$$

by the standard (right) T^2-action on $SU(3)$ and the T^2-action on X via ρ. Because the elements of the image $\rho(T^2)$ commutes with the elements in T^n, the manifold $M_\rho(X)$ has the $(T^2 \times T^n)$-action by the left T^2-action on the $SU(3)$-factor and the T^n-action on the $2n$-dimensional torus manifold X. Because there is a fibre bundle structure $X \to M_\rho(X) \to SU(3)/T^2$, we also have $\dim M_\rho(X) = 2n + 6$. Moreover, it is easy to check that its one-skeleton has the structure of a fibre bundle over the one-skeleton of $SU(3)/T^2$ whose fibre is that of X. Therefore, this is a complexity one GKM manifold. Moreover, there is an extended $SU(3) \times T^n$-action because $SU(3)$ acts naturally on the $SU(3)$-factor in $M_\rho(X)$. Because there are infinitely many torus manifolds X, we can construct infinitely many complexity one GKM manifolds with extended $SU(3) \times T^n$-actions.

EXAMPLE 2.3. Let Y be a $(2n-4)$-dimensional torus manifold for $n > 2$, and $\sigma : T^1 \to T^{n-2}$ be a faithful representation. Define the twisted product $N_\sigma(Y)$ by

$$S^5 \times_{T^1} Y$$

such that T^1 acts on $S^5 \subset \mathbb{C}^3$ diagonally and on a torus manifold Y via σ. Then, $N_\sigma(Y)$ is a $2n$-dimensional torus manifold with extended $(SU(3) \times T^{n-2})$-actions (see [**Ku4**] for details of this construction). Here, $SU(3)$ acts on the S^5-factor transitively. Now we can add the G_2-factor as follows:

$$\begin{aligned} & G_2 \times_{SU(3)} N_\sigma(Y) \\ \simeq\ & G_2 \times_{SU(3)} (S^5 \times_{T^1} Y) \\ \simeq\ & G_2 \times_{SU(3)} ((SU(3)/SU(2)) \times_{T^1} Y) \\ \simeq\ & (G_2/SU(2)) \times_{T^1} Y. \end{aligned}$$

This is an $N_\sigma(Y)$-bundle over S^6 (or a Y-bundle over $G_2/S(U(2) \times U(1))$, also see [**Ku2**, Example 1.4]). Therefore, $\dim(G_2 \times_{SU(3)} N_\sigma(Y)) = 2n + 6$. Moreover, because $\sigma(T^1)$ commutes with T^{n-2}, the structure of a Y-bundle over $G_2/S(U(2) \times U(1))$ induces the $(T^2 \times T^{n-2})$-action on $G_2 \times_{SU(3)} N_\sigma(Y)$. Because both of the fibre and the base space have structures of GKM manifolds, $(G_2 \times_{SU(3)} N_\sigma(Y), T^n)$ is a $(2n+6)$-dimensional GKM manifold, i.e., complexity 3 GKM manifold with an extended $G_2 \times T^{n-2}$-action. However, this might not be a complexity one GKM manifold (see [**Ta**] or [**Ku5**]).

Similarly, we can also construct other GKM manifolds (might not be a complexity one GKM manifold) with extended non-abelian Lie group actions, by using the other irreducible homogeneous complexity one GKM manifolds.

In particular, we can generalize the construction in Example 2.2 as follows.

PROPOSITION 2.4. Let G be a compact, connected, non-abelian Lie group with rank $G = n$, and X be a $2m$-dimensional torus manifold such that $n \leq m$. Then, for any faithful representation $\rho : T^n \to T^m$, the following manifold is a $(\dim G - n + 2m)$-dimensional GKM manifold with the T^{n+m}-action:

$$G \times_{T^n} X$$

where T^n is a maximal torus in G and acts on X via ρ.

Furthermore, this has the extended $(G \times T^m)$-action and a complexity $(\dim G - 3n)/2$.

3. GKM graphs

In this section, we recall GKM graphs introduced in [**GuZa**], and prepare to prove Theorem 1.6.

3.1. Notations. Let $\Gamma = (V(\Gamma), E(\Gamma))$ be an abstract graph, where $V(\Gamma)$ is the set of vertices and $E(\Gamma)$ is the set of oriented edges of Γ. Let $e \in E(\Gamma)$. We denote its initial vertex by $i(e)$, the terminal vertex by $t(e)$, (e.g. $i(pq) = p$ and $t(pq) = q$) and the reversed oriented edge of e by $\bar{e}$. It is easy to check that $i(e) = t(\bar{e})$ and $t(e) = i(\bar{e})$. Put the set of all outgoing edges from the vertex p by $E_p(\Gamma)$, i.e., the set of all edges e such that $i(e) = p$. We say Γ is an *m-valent graph* if $\#E_p(\Gamma) = m$ for all $p \in V(\Gamma)$.

A map between two graphs $\Gamma = (V(\Gamma), E(\Gamma))$ and $\Gamma' = (V(\Gamma'), E(\Gamma'))$ is defined as the pair of maps $f = (f_V, f_E) : \Gamma \to \Gamma'$ such that the following diagram commutes

$$\begin{array}{ccc} E(\Gamma) & \xrightarrow{f_E} & E(\Gamma') \\ \downarrow & & \downarrow \\ V(\Gamma) & \xrightarrow{f_V} & V(\Gamma') \end{array}$$

where two vertical maps are the maps taking the initial vertex, i.e., $e \mapsto i(e)$. In other wards, the map of vertices preserves the edges. An *automorphism group* of Γ, say $\mathrm{Aut}(\Gamma)$, is defined by the set of all maps f on Γ such that both of f_V and f_E are bijective.

Let $H^*(BT^n)$ be the cohomology ring of BT over $\mathbb{Z}$-coefficient, i.e., $H^*(BT^n)$ is isomorphic to the polynomial ring $\mathbb{Z}[\alpha_1, \dots, \alpha_n]$ in the variables $\alpha_i \in H^2(BT)$ $(i = 1, \dots, n)$.

3.2. Abstract GKM graph. Throughout of this paper, Γ is an m-valent (connected) graph, where $n \leq m$. Put

$$\mathcal{A} : E(\Gamma) \to H^2(BT).$$

If $\mathcal{A}$ satisfies the following three conditions:

(1) $\mathcal{A}(e) = -\mathcal{A}(\bar{e})$;
(2) the set $\{\mathcal{A}(E_p(\Gamma))\}$ is pairwise linearly independent for all $p \in V(\Gamma)$;
(3) there is a bijective map $\nabla_e : E_p(\Gamma) \to E_q(\Gamma)$ for $p = i(e)$ and $q = t(e)$ such that
  (a) $\nabla_e = \nabla_{\bar{e}}^{-1}$;
  (b) $\nabla_e(e) = \bar{e}$;
  (c) $\mathcal{A}(\nabla_e(f)) - \mathcal{A}(f) \equiv 0 \mod \mathcal{A}(e)$ for every $f \in E_p(\Gamma)$ (called a *congruence relation*),

then the map $\mathcal{A}$ is called an *axial function* on Γ and the collection $\nabla = \{\nabla_e \mid e \in E(\Gamma)\}$ is called a *connection*. We call the pair $(\Gamma, \mathcal{A})$ a *GKM graph*, where Γ is an m-valent graph and $\mathcal{A}$ is an axial function on Γ.

Assume an axial function $\mathcal{A} : E(\Gamma) \to H^2(BT^n)$ satisfies that the image of $\mathcal{A}$ spans $H^2(BT^n)$. Then, we call the number $m - n$ a *complexity* of $(\Gamma, \mathcal{A})$.

REMARK 3.1. Let (M, T) be a $2m$-dimensional GKM manifold with invariant almost complex structure. Then, the one-skeleton of M/T induces the m-valent graph Γ_M. Moreover, we can define the axial function $\mathcal{A}_M$ by the complex tangential representation on each fixed point (to define this complex structure canonically, we need a complex structure around fixed points). Namely, the m-valent GKM graph $(\Gamma_M, \mathcal{A}_M)$ is define by an almost complex $2m$-dimensional GKM manifold (M, T).

More generally, we can also define the generalized GKM graph, i.e., the condition (1) in the axial function $\mathcal{A}$ is changed to the condition $\mathcal{A}(e) = \pm\mathcal{A}(\bar{e})$, from an unitary GKM manifold (M, T) (i.e., there is an invariant stably complex structure on M) such as an even dimensional sphere. We omit the precise definition of the (generalized) GKM graph induced from a GKM manifold in this article (see **[Da]**, **[GuZa]**, **[Ku1]** or **[MMP]** for precise definition).

Note that a more general labelled graph (i.e., there might not exist any connections) can be defined by an omnioriented GKM manifold (i.e., there is a fixed orientations on M and all invariant 2-spheres) such as a quaternionic projective space whose dimension is greater than four.

3.3. Automorphism group of GKM graph. Let us define the automorphism group of a GKM graph $(\Gamma, \mathcal{A})$ with $\mathcal{A} : E(\Gamma) \to H^2(BT^n) \simeq \mathbb{Z}^n$. Let $f \in \mathrm{Aut}(\Gamma)$ and $\rho \in GL(n; \mathbb{Z})$. We call

(f,ρ) is an *automorphism* on $(\Gamma,\mathcal{A})$ if the following diagram commutes:

$$\begin{array}{ccc} E(\Gamma) & \xrightarrow{f_E} & E(\Gamma) \\ \downarrow & & \downarrow \\ H^2(BT) & \xrightarrow{\rho} & H^2(BT) \end{array}$$

where the vertical maps are both of $\mathcal{A}$. We call the set

$$\{(f,\rho)\in \mathrm{Aut}(\Gamma)\times GL(n;\mathbb{Z}) \mid (f,\rho) \text{ is an automorphism on } (\Gamma,\mathcal{A})\}$$

an *automorphism group* of $(\Gamma,\mathcal{A})$ and denote it by $\mathrm{Aut}(\Gamma,\mathcal{A})$.

The following proposition is one of the motivations to considering the 2nd main result in this paper (Theorem 1.6) (also see [**Kaj**]).

PROPOSITION 3.2. If an almost complex GKM manifold (M,T) extends to an almost complex (M,G), then $W(G)=N_G(T)/T$ acts on $(\Gamma_M,\mathcal{A}_M)$, i.e., $W(G)\subset \mathrm{Aut}(\Gamma_M,\mathcal{A}_M)$.

PROOF. Let (M,T) be a GKM manifold with an almost complex structure J. Assume that there is an extended G-action (M,G) preserving J. Let $g\in W(G)$ and S^2_α be an invariant 2-sphere in (M,T), where $\alpha\in\mathfrak{t}^*$ is the element corresponding to S^2_α (i.e., the element appearing in the tangential representation on S^2_α) Then, gS^2_α satisfies that

$$\begin{aligned} & TgS^2_\alpha \\ = \ & (gTg^{-1})gS^2_\alpha \quad (\text{because } g\in N_G(T)) \\ = \ & gTS^2_\alpha \\ = \ & gS^2_\alpha \quad (\text{because } S^2_\alpha \text{ is } T\text{-invariant}). \end{aligned}$$

This implies that gS^2_α is an invariant 2-sphere; therefore, $W(G)\subset \mathrm{Aut}(\Gamma)$.

By definition, the axial function $\mathcal{A}_M$ is induced by the complex structure around fixed points induced from J. Therefore, it is easy to check that $W(G)\subset \mathrm{Aut}(\Gamma_M,\mathcal{A}_M)$, because G preserves the almost complex structure J of M. □

By the proof described as above, we may think

$$gS^2_\alpha = S^2_\beta$$

where $\beta=g^*\alpha\in\mathfrak{t}^*$ for $g^*\in W(G)$. Therefore, the Weyl group W action on an abstract GKM graph $(\Gamma,\mathcal{A})$ can be defined as follows. It is well known that W is generated by the reflections on $\mathfrak{t}$; in particular, this preserves the weight lattice (see [**Hu**]). Hence, for $g\in W$, the isomorphism g on $\mathfrak{t}_\mathbb{Z}$ induces the isomorphism $g^*:\mathfrak{t}^*_\mathbb{Z}\to\mathfrak{t}^*_\mathbb{Z}$, i.e., the dual of $g\in W$. Therefore, the automorphism $g:(\Gamma,\mathcal{A})\to(\Gamma,\mathcal{A})$ is defined by

$$g^*(\mathcal{A}(e))=\mathcal{A}(g_E(e)), \tag{3.1}$$

where $e\in E(\Gamma)$, $g_E:E(\Gamma)\to E(\Gamma)$ is the bijective map induced from $g\in W\subset \mathrm{Aut}(\Gamma)$ (we often abuse two notations g_E and g).

REMARK 3.3. If the extended action G does not preserve the almost complex structure J (even more generally the stably complex structure) on M, then $W(G)$ do not act on $(\Gamma_M,\mathcal{A}_M)$. However, if M is a unitary GKM manifold and there is an extended G-action (which might not preserve the stably complex structure), then $g\in W(G)$ action changes the relation (3.1) as follows:

$$g^*(\mathcal{A}(e))=\pm\mathcal{A}(g_E(e)).$$

Therefore, $g\in W(G)$ gives the equivalence between $(\Gamma,\mathcal{A})$ and $(\Gamma,\mathcal{A}')$ where $\mathcal{A}'$ satisfies $\mathcal{A}(e)=\pm\mathcal{A}'(e)$ for all edges $e\in E(\Gamma)$. In [**KuMa**], we also study the extended actions for the cases when G does not preserve the stably complex structures (more generally omniorientations) on torus manifolds by introducing root systems on torus graphs.

4. Sketch of a proof of Theorem 1.6

Let $(\Gamma, \mathcal{A})$ be an abstract m-valent GKM graph. Motivated by Theorem 1.2, we classify the following GKM graphs:

(1) Γ is a 3 or 4-valent graph;
(2) $(\Gamma, \mathcal{A})$ is a complexity one, i.e., if $\mathcal{A} : \Gamma \to H^2(BT^n)$ then $n = 2$ (for $m = 3$) or 3 (for $m = 4$), called a $(3,2)$-type GKM graph or $(4,3)$-type GKM graph, respectively;
(3) if $(\Gamma, \mathcal{A})$ is a $(3,2)$-type (resp. $(4,3)$-type), there is the symmetric group S_3, the signed symmetric group $S_2^{\pm}$ or the dihedral group D_6 (resp. S_4 or $S_3^{\pm}$) action on it;
(4) for every Weyl subgroup $W' \subset W$ and every vertex $p \in V(\Gamma)$, there is a GKM subgraph $(\Gamma', \mathcal{A}|_{E(\Gamma')})$ such that W' acts on it effectively and $p \in V(\Gamma')$;

The assumption (4) as above is induced from the fact that there is a connected K-orbit of $p \in M^T$ for all K such that $T \subset K \subset G$.

In this paper, we only prove the case when $(\Gamma, \mathcal{A})$ is a $(3,2)$-type GKM graph with the symmetric group S_3-action. For the other cases, we can prove similarly (it will be shown in somewhere of the future article).

4.1. The property of S_3-orbits. Let $W = S_3$. Then, it is well-known that this is a Coxeter group and there is the following representation:

$$\begin{aligned} W &= \langle \sigma_1, \sigma_2 \mid \sigma_1^2 = \sigma_2^2 = (\sigma_1\sigma_2)^3 = 1 \rangle \\ &= \{1,\ \sigma_1,\ \sigma_2,\ \sigma_1\sigma_2,\ \sigma_2\sigma_1,\ \sigma_1\sigma_2\sigma_1\}. \end{aligned}$$

In this case, there are four Weyl subgroups:

$$\{1\},\ \langle\sigma_1\rangle,\ \langle\sigma_2\rangle,\ \langle\sigma_1\sigma_2\sigma_1\rangle(\simeq \mathbb{Z}_2),$$

and there is another subgroup

$$\mathbb{Z}_3 = \langle\sigma_1\sigma_2\rangle = \langle\sigma_2\sigma_1\rangle.$$

Therefore, if there is a W action on $\Gamma = (V(\Gamma), E(\Gamma))$, then there are the following possible orbit types for $p \in V(\Gamma)$:

$$W(p) \simeq W/W,\ W/\mathbb{Z}_3,\ W/\mathbb{Z}_2,\ \text{or } W/\{e\}. \tag{4.1}$$

By choosing generators of $H^2(BT) = \mathbb{Z}\alpha \oplus \mathbb{Z}\beta$, for $p \in V(\Gamma)$, we may assume that

$$\mathcal{A}_p = \{\mathcal{A}(e) \mid e \in E_p(\Gamma)\} = \{\alpha, \beta, k_1\alpha + k_2\beta\}$$

for some non-zero integers k_1, k_2. From the next subsection, we consider the cases appeared in (4.1) in case by case.

4.1.1. *The case when $W(p) \simeq W/W$.* In this case, W acts on $E_p(\Gamma)$; therefore it also acts on $\mathcal{A}_p$. Because $H^2(BT)^W = \{0\}$, the symmetric group W acts on $\mathcal{A}_p$ transitively; therefore, it also acts on $E_p(\Gamma)$ transitively. Consequently, we may assume that $k_1 = k_2 = -1$ and

$$\sigma_1 : \alpha \mapsto \beta, \quad -\alpha - \beta \mapsto -\alpha - \beta;$$
$$\sigma_2 : \alpha \mapsto -\alpha - \beta, \quad \beta \mapsto \beta.$$

This implies that the neighborhood of this case is Figure 2.

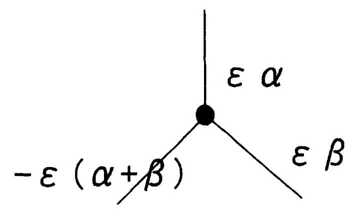

FIGURE 2. This is the axial function around $p \in V(\Gamma)^W$, where $\epsilon = \pm 1$.

4.1.2. *The case when* $W(p) \simeq W/\mathbb{Z}_3$. We claim that this case does not occur. Because $|W(p)| = 2$ and our assumption (4), there is an edge $e \in E(\Gamma)$ such that $V(e) = W(p) = \{p, q\}$ and $\sigma_1 \in W$ such that $\sigma_1(e) = \bar{e}$. Because $\mathcal{A}(e) = -\mathcal{A}(\bar{e})(= \alpha)$, we may assume

$$\sigma_1 : \alpha \mapsto -\alpha. \tag{4.2}$$

On the other hand, the subgroup $\mathbb{Z}_3 = \{1, \sigma_1\sigma_2, \sigma_2\sigma_1\}$ acts on $E_p(\Gamma) = \{e, e', e''\}$. Therefore, because $\sigma_1(e) = \bar{e} \in E_q(\Gamma)$ and the assumption (4), $\sigma_2(\bar{e}) = e$. However, this implies that $\sigma_2 : -\alpha \mapsto \alpha$. This gives a contradiction to (4.2). Consequently, there is no vertex $p \in V(\Gamma)$ such that $W(p) \simeq W/\mathbb{Z}_3$.

4.1.3. *The case when* $W(p) \simeq W/\mathbb{Z}_2$. Because $|W(p)| = 3$ and the assumption (4), in this case there is a triangle GKM subgraph $(\Delta, \mathcal{A}|_\Delta = \mathcal{A}') \subset (\Gamma, \mathcal{A})$ such that $V(\Delta) = W(p) = \{p, q, r\}$. Because the symmetric group W acts on $(\Delta, \mathcal{A}')$ transitively, we may assume that

$$\sigma_1 : p \mapsto p, \quad q \mapsto r;$$
$$\sigma_2 : p \mapsto q, \quad r \mapsto r.$$

Moreover, considering the axial functions around Δ, we can easily check that this case may assume the axial functions appeared in Figure 3 (the case when $\mathcal{A}(pq) = \epsilon\alpha$ and $\mathcal{A}(pr) = \epsilon\beta$) or Figure 4 (otherwise, i.e., the case when $\mathcal{A}(e) = \epsilon\alpha$ and $\mathcal{A}(e') = \epsilon\beta$ for $e \in E_p(\Gamma) \setminus E_p(\Delta)$ and $e' \in E_q(\Gamma) \setminus E_q(\Delta)$).

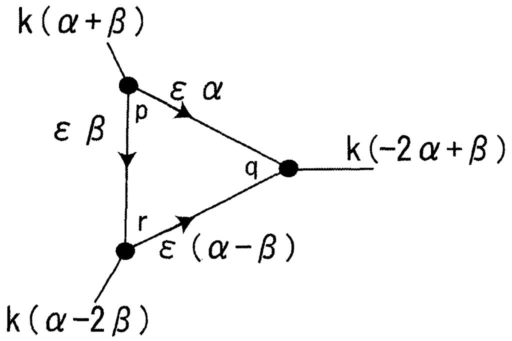

FIGURE 3. The axial function around Δ such that $V(\Delta) = W(p)$, where k is a non-zero integer and $\epsilon = \pm 1$. In this case, $\sigma_1 : \alpha \mapsto \beta$ and $\sigma_2 : \beta \mapsto -\alpha + \beta$.

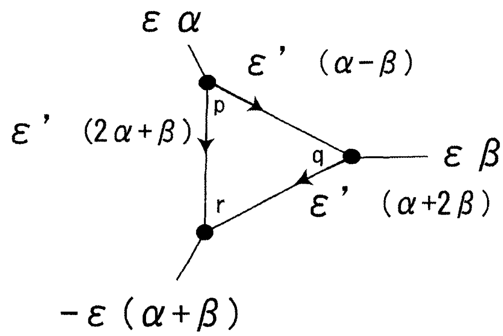

FIGURE 4. The axial function around Δ such that $V(\Delta) = W(p)$, where $\epsilon, \epsilon' = \pm 1$. In this case, $\sigma_1 : \beta \mapsto -\alpha - \beta$ and $\sigma_2 : \alpha \mapsto \beta$.

4.1.4. *The case when* $W(p) \simeq W/\{e\}$. In this case, $|W(p)| = 6$. By using the assumption (4), it is easy to check that W acts on Γ transitively and the axial functions are the labelles appeared in Figure 5 or Figure 6.

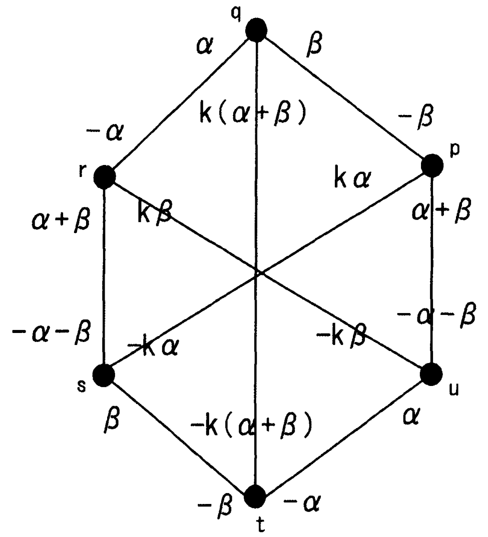

FIGURE 5. The 1st case: $\sigma_1 : \alpha \mapsto -\alpha,\ \beta \mapsto \alpha+\beta$; $\sigma_2 : \beta \mapsto -\beta,\ \alpha \mapsto \alpha+\beta$, where k is a non-zero integer.

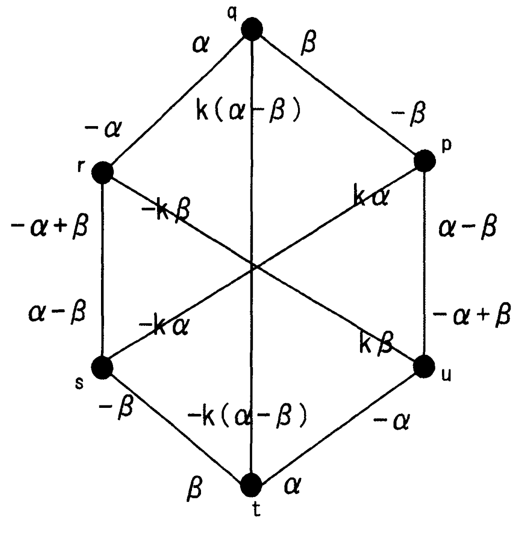

FIGURE 6. The 2nd case: $\sigma_1 : \alpha \mapsto -\alpha,\ \beta \mapsto -\alpha+\beta$; $\sigma_2 : \beta \mapsto -\beta,\ \alpha \mapsto \alpha+\beta$, where k is a non-zero integer.

4.2. The proof of Theorem 1.6 for the case when there is an S_3-action. Finally, in this section, we prove Theorem 1.6 by combining the facts described in Section 4.1. If $W = S_3$ acts on $(\Gamma, \mathcal{A})$ transitively, then this case is one of the cases of Figure 5 and Figure 6. These are the cases (5), (6) in Theorem 1.6.

Assume that W acts on $(\Gamma, \mathcal{A})$ non-transitively. Then, there are three possible orbits, i.e., Figure 2, Figure 3 and Figure 4, say type 1, type 2 and type 3 respectively.

If there is a type 1 orbit, then this orbit must be connecting with one of the following orbits:

(1) the type 1 orbit with distinct ϵ's (this is the case (1) in Theorem 1.6);
(2) the type 3 orbit with distinct ϵ's (this is the case (2) in Theorem 1.6).

Assume that there is no type 1 orbit but there is a type 2 orbit. Then, because there is a W-action on whole Γ, this case is only the connecting two copies of type 2 (with different signs of k's). This is the case (3) in Theorem 1.6. Assume that there is no type 1 orbit but there is a type 3 orbit. Then, similarly, this case is only the connecting two copies of type 3 (with distinct ϵ's). This is the case (4) in Theorem 1.6.

This establishes Theorem 1.6.

REMARK 4.1. In the end of this article, we show some geometric models for GKM graphs appearing in Theorem 1.6:

- The GKM graph (1) is induced from (S^6, T^2), where $S^6 \simeq G_2/SU(3)$ (there is a $SU(3)(\subset G_2)$-extended action);
- The GKM graph (2) is from $(\mathbb{C}P^3, T^2)$, where $\mathbb{C}P^3 \simeq P(V(-\alpha)\oplus V(-\beta)\oplus V(\alpha+\beta)\oplus \underline{\mathbb{C}})$ (there is a $SU(3)$-extended action on the first three coordinates);
- The GKM graph (3) is from $S^5 \times_{T^1} P(\gamma^{\otimes k} \oplus \epsilon)$ (there is a transitive $SU(3)$-action on the S^5-factor);
- The GKM graph (4) is from the connected sum of two copies of $(\mathbb{C}P^3, T^2)$'s which induce the GKM graph (2) (there is an $SU(3)$-extended action because this connected sum is an $SU(3)$-equivariant);
- The GKM graphs (5), (6). These cases are still not known, that is, which GKM manifolds induce these GKM graphs? Some of them must be obtained from the projectivizations of equivariant complex 2-dimensional vector bundles over $\mathbb{C}P^2$, which are not split into line bundles (see [**Kan**]), because the projectivization of Whitney sum of line bundles is isomorphic to $S^5 \times_{T^1} P(\gamma^k \oplus \epsilon)$. In particular, if $k = 1$ for the GKM graph (5), this is obtained from the flag manifold $SU(3)/T^2 \simeq SU(3) \times_{U(2)} \mathbb{C}P^2$, which can be regarded as the projectivization of some complex 2-dimensional vector bundle over $\mathbb{C}P^2$ (see [**KuSu**]).

Acknowledgment

This article is based on author's talk in RIMS Conference "The Topology and the Algebraic Structures of Transformation Groups" from May 26th to May 30th 2014. The author is grateful to the organizer Takao Sato for giving him the chance to give a talk in the conference. He also would like to thank Prof. Takashi Tsuboi for providing him with excellent working conditions.

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