TENTATIVE STUDY ON
EQUIVARIANT SURGERY OBSTRUCTIONS:
FIXED POINT SETS OF SMOOTH $A_5$-ACTIONS

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Abstract. Let $G$ be the alternating group on 5 letters and let $F$ be a closed smooth manifold diffeomorphic to the fixed point set of a smooth $G$-action on a disk. Marek Kaluba proved that if $F$ is even dimensional then there exists a smooth $G$-action on a closed manifold $X$ being homotopy equivalent to a complex projective space such that the fixed point set of the $G$-action is diffeomorphic to $F$. In this paper we discuss whether series of manifolds diffeomorphic or homotopy equivalent to complex projective spaces, real projective spaces, or lens spaces, admit smooth $G$-actions with fixed point set diffeomorphic to $F$.

1. INTRODUCTION

Let $G$ be a finite group throughout this paper. For a smooth manifold $M$, let $\mathfrak{F}_G(M)$ denote the family of all manifolds $F$ such that $F = M^G$ for some smooth $G$-action on $M$. For a family $\mathfrak{M}$ of smooth manifolds, let $\mathcal{F}_G(\mathfrak{M})$ denote the union of $\mathfrak{F}_G(M)$ with $M \in \mathfrak{M}$. Let $\mathfrak{D}$, $\mathfrak{S}$, and $\mathfrak{P}_\mathbb{C}$ denote the families of disks, spheres, and complex projective spaces, respectively. B. Oliver [19] completely determined the family $\mathfrak{F}_G(\mathfrak{D})$ for $G$ not of prime power order. K. Pawalowski and the author [18, 14] studied $\mathfrak{F}_G(\mathfrak{S})$ for various Oliver groups $G$.

In order to quote a part of Oliver's result on $\mathfrak{F}_G(\mathfrak{D})$, we adopt the notation $\mathcal{G}_\mathbb{R}$, $\mathcal{G}_\mathbb{C}^\sigma$, $\mathcal{G}_\mathbb{C}$ and $\mathcal{E}$ for the families of all finite groups satisfying the following properties, respectively.

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\begin{itemize}
  \item $G \in \mathcal{G}_\mathbb{R}$: $G$ possesses a subquotient $K/H$ isomorphic to a dihedral group of order $2pq$ for some distinct primes $p$ and $q$, where $H \triangleleft K \leq G$.
  \item $G \in \mathcal{G}_\mathbb{C}$: $G$ contains an element $g$ being conjugate to its inverse of order $pq$ for some distinct primes $p$ and $q$.
  \item $G \in \mathcal{G}_\mathbb{C}$: $G$ contains an element $g$ of order $pq$ for some distinct primes $p$ and $q$.
  \item $G \in \mathcal{E}$: A Sylow 2-subgroup of $G$ is not normal in $G$, and any element of $G$ is of prime power order.
\end{itemize}

Note that $\mathcal{G}_\mathbb{R} \subset \mathcal{G}_\mathbb{C}^\sigma \subset \mathcal{G}_\mathbb{C}$.

Let $A_5$ denote the alternating group on 5 letters. Then $A_5$ belongs to $\mathcal{E}$. B. Oliver [19] says that for $G \in \mathcal{F}_\mathbb{C} \cup \mathcal{E}$, a closed manifold $F$ belongs to $\mathfrak{F}_G(\mathcal{D})$ if and only if $\chi(F) \equiv 1 \mod n_G$ and

\begin{align}
\bullet & \quad G \in \mathcal{G}_\mathbb{R} \Rightarrow \text{no restrictions on } T(F), \\
\bullet & \quad G \in \mathcal{G}_\mathbb{C} \setminus \mathcal{G}_\mathbb{R} \Rightarrow c_\mathbb{H}(T(F)) \in c_\mathbb{H}\left(\overline{KSp}(F)\right) + \text{Tor}\left(\overline{KU}(F)\right), \\
\bullet & \quad G \in \mathcal{G}_\mathbb{C} \setminus \mathcal{G}_\mathbb{C}^\sigma \Rightarrow [T(F)] \in r_\mathbb{C}\left(\overline{KU}(F)\right) + \text{Tor}\left(\overline{KO}(F)\right), \\
\bullet & \quad G \in \mathcal{E} \Rightarrow [T(F)] \in \text{Tor}\left(\overline{KO}(F)\right).
\end{align}

If $G \in \mathcal{E}$ and $F \in \mathfrak{F}_G(\mathfrak{D})$ then each connected component of $F$ has same dimension. The Oliver number $n_G$ above is equal to 1 whenever $G$ is nonsolvable.

Marek Kaluba [5] obtained the next two theorems concerned with $\mathfrak{F}_G(\mathcal{P}_\mathbb{C})$.

**Theorem.** [5, Theorem 2.6] Let $G$ be a nontrivial perfect group in the class $\mathcal{G}_\mathbb{C}$ and let $F$ be a closed manifold in $\mathfrak{F}_G(\mathcal{D})$. In the case $G \in \mathcal{G}_\mathbb{C} \setminus \mathcal{G}_\mathbb{R}$, suppose that some connected component of $F$ is even dimensional. Then $F$ belongs to $\mathfrak{F}_G(\mathcal{P}_\mathbb{C})$.

**Theorem.** [5, Theorem 4.11] Let $G$ be $A_5$ and $F$ a closed manifold in $\mathfrak{F}_G(\mathcal{D})$. Suppose that $F$ is even dimensional. Then $F$ is diffeomorphic to the fixed point set of a smooth $G$-action on a closed manifold $X$ which is homotopy equivalent to some complex projective space.

Let $P_\mathbb{C}^k$ (resp. $P_\mathbb{R}^k$) denote the complex (resp. real) projective space of complex (resp. real) dimension $k$, and let $\Gamma$ be a cyclic subgroup of $\mathbb{C}^\times$ of order $\geq 3$. The orbit space $L^{2k+1} = S(\mathbb{C}^{k+1})/\Gamma$ is a lens space of dimension $2k + 1$. Let $\mathfrak{L}$ be the
family of lens spaces $L^{2k+1}, k = 2, 3, 4, \ldots$. By examining and improving the proof of [5, Theorem 4.11] by M. Kaluba, we obtain the next result.

**Theorem 1.1.** Let $G$ be $A_5$ and $F$ a closed manifold in $\mathfrak{F}_G(\mathfrak{D})$. Then there exists an integer $N > 0$ possessing the property that for any $k \geq N,$

1. $F \in \mathfrak{F}_G(D^k),$
2. $F \in \mathfrak{F}_G(S^k),$
3. if $\dim F \equiv 0 \mod 2$ then $F \in \mathfrak{F}_G(P^k),$
4. $F \in \mathfrak{F}_G(X_k)$ such that $X_k$ is a smooth closed manifold homotopy equivalent to $P^k$, 
5. if $\dim F \equiv 1 \mod 2$ then $F \in \mathfrak{F}_G(Y_k)$ such that $Y_k$ is a smooth closed manifold homotopy equivalent to $L^{2k+1}$.

This result follows from Theorem 3.4. In Theorem 1.1, one may conjecture that $P^k$ and $L^{2k+1}$ can be chosen as $X_k$ and $Y_k$ respectively, but the author cannot prove the conjecture so far.

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2. **Dimension conditions of fixed point sets**

Let $G$ be a finite group. Let $U$ be a $G$-manifold and $(H, K)$ a pair of subgroups $H < K \leq G$. We say that $U$ satisfies the gap condition, cobordism gap condition, or strong gap condition for $(H, K)$ if the inequality

\begin{align}
(2.1) & \quad 2\dim(U^H_i)^K < \dim U^H_i, \\
(2.2) & \quad 2(\dim\{(U^H_i)^K \setminus (U^H_i)^{G(H)}\} + 1) \leq \dim U^H_i, \\
\end{align}

or

\begin{align}
(2.3) & \quad 2\dim(U^H_i)^K + 1 < \dim U^H_i,
\end{align}

holds, respectively, for any connected component $U^H_i$ of $U^H$. 

Proposition 2.1. Let $G$ be a perfect group having a cyclic subgroup $C_2$ of order 2, $Y$ the complex projective space associated with the complex $G$-module 

$$V = \mathbb{C}^{m+1} \oplus (\mathbb{C}[G] - \mathbb{C})^n,$$

where $m \geq 0$ and $n \geq 1$, and $U$ the $G$-tubular neighborhood of $Y^G$.

1. $Y$ satisfies the gap condition for $(\{e\}, C_2)$ if and only if $m+1 = n$.
2. $U$ satisfies the gap condition for $(\{e\}, C_2)$ if and only if $m+1 \leq n$.
3. If $m+1 \leq n$ then $U$ satisfies the strong gap condition for $(H, K)$ such that 

$$\{e\} \neq H < K \leq G \text{ and } |K : H| \geq 3.$$ 

Proof. We readily see that $Y^G = P_{\mathbb{C}}(\mathbb{C}^{m+1}) = P_{\mathbb{C}}^m$ and $Y^C_2$ has two connected components 

$$Y_a^C_2 = P_{\mathbb{C}}(\mathbb{C}^{m+1} \oplus ((\mathbb{C}[G] - \mathbb{C})^C)^n) = P_{\mathbb{C}}^{m+n(|G|/2-1)}$$

and 

$$Y_b^C_2 = P_{\mathbb{C}}(((\mathbb{C}[G] - \mathbb{C})^C)^n) = P_{\mathbb{C}}^{n|G|/2-1}.$$

Thus we have 

$$\dim Y = 2m - 2n + 2n|G|, \quad \dim Y_a^C_2 = 2m - 2n + n|G| \quad \text{and} \quad \dim Y_b^C_2 = n|G| - 2.$$ 

Note the equivalences

- $2(2m - 2n + n|G|) < 2m - 2n + 2n|G| \iff m < n$
- $2(n|G| - 2) < 2m - 2n + 2n|G| \iff n - 2 < m.$

Thus $Y$ satisfies the gap condition for $(\{e\}, C_2)$ if and only if $n - 2 < m < n$, namely $m + 1 = n$. Since $\dim U^C_2 = \dim Y_a^C_2$, $U$ satisfies the gap condition for $(\{e\}, C_2)$ if and only if $m < n$, namely $m + 1 \leq n$.

For any $H \leq G$, $U^H$ is connected. Let $y$ denote the point $[1, 0, \ldots, 0]$ in $Y = P_{\mathbb{C}}(\mathbb{C} \oplus \mathbb{C}^{m} \oplus (\mathbb{C}[G] - \mathbb{C})^n)$. The tangential representation $T_y(Y)$ is isomorphic to $\mathbb{C}^m \oplus (\mathbb{C}[G] - \mathbb{C})^n$ as complex $G$-modules. Since $\dim U^H = \dim T_y(Y)^H$, we get

$$\dim U^H = 2\{m + n(|G|/|H| - 1)\} = 2m - 2n + 2n|G|/|H|$$

and

$$\dim U^H - 2(\dim U^K + 1) = 2\{n - (m + 1)\} + 2n|G|(|K| - 2|H|)/(|H||K|).$$

In the case $n \geq m + 1$ and $|K| \geq 3|H|$, we conclude $2(\dim U^K + 1) < \dim U^H.$

Theorem 2.2 (Disk Theorem). Let $G$ be a nontrivial perfect group, $F \in \mathfrak{F}(\mathfrak{D})$, $F_0$ a connected component of $F$ with $x_0 \in F_0$, $m = \dim F_0$, and $W$ a real $G$-module
with \( \dim W^G = m \). Then there exists a smooth \( G \)-action on the disk \( D \) of dimension \( \dim W + N(|G| - 1) \) for some integer \( N \geq 0 \) satisfying the following conditions.

1. \( D^G = F \).
2. \( T_{x_0}(D) \) is isomorphic to \( W \oplus (\mathbb{R}[G] - \mathbb{R})^\oplus N \) as real \( G \)-modules.
3. \( D \) satisfies the strong gap condition for arbitrary pair \((H, K)\) such that \( H < K \leq G \).

**Corollary 2.3.** Let \( G, F \in \mathfrak{F}(\mathfrak{D}) \), \( F_0, x_0 \in F_0 \), \( m = \dim F_0 \), \( W \), \( D \) and \( N \) be as in Theorem 2.2. Let \( n \) be an arbitrary integer \( \geq N \). Then there exists a smooth \( G \)-action on the disk \( S \) of dimension \( \dim W + n(|G| - 1) \) satisfying the following conditions.

1. \( S^G = F' \) and \( F' \) is diffeomorphic to \( F \).
2. \( T_{x_0}(S) \) is isomorphic to \( W \oplus (\mathbb{R}[G] - \mathbb{R})^\oplus n \) as real \( G \)-modules.
3. \( S \) satisfies the strong gap condition for arbitrary pair \((H, K)\) such that \( H < K \leq G \).

**Proof.** We set \( X = D \times D((\mathbb{R}[G] - \mathbb{R})^\oplus(n-N)) \). Let \( S \) be the double of \( X \). Then \( S \) satisfies the desired conditions. \( \square \)

### 3. Deleting theorem and realization theorem

In this section we will give a deleting theorem and a realization theorem. The latter is obtained from the former and Corollary 2.3. Our main result Theorem 1.1 follows from the realization theorem.

Let \( A_5 \) denote the alternating group on 5 letters 1, 2, 3, 4, 5, and let \( A_4 \) denote the alternating group on 4 letters 1, 2, 3, 4. Unless otherwise stated, we use the notation:

- \( C_2 = \langle (1,2)(3,4) \rangle \)
- \( D_4 = \langle (1,2)(3,4), (1,3)(2,4) \rangle \)
- \( C_3 = \langle (1,2,5) \rangle \)
- \( D_6 = \langle (1,2,5), (1,2)(3,4) \rangle \)
- \( C_5 = \langle (1,3,4,2,5) \rangle \)
- \( D_{10} = \langle (1,3,4,2,5), (1,2)(3,4) \rangle \).
These groups and $A_4$ are regarded as subgroups of $A_5$.

Throughout this section, let $G$ be $A_5$. Then $N_G(C_2) = D_4$, $N_G(D_4) = A_4$, $N_G(C_3) = D_6$, $N_G(C_5) = D_{10}$, and any maximal proper subgroup of $G$ is conjugate to one of $A_4$, $D_{10}$, $D_6$. We can readily show

**Proposition 3.1.** Let $H$ be a maximal proper subgroup of $G$. Then any two subgroups of $H$ are conjugate in $G$ if and only if they are conjugate in $H$.

**Proposition 3.2.** Let $\alpha$ be the element of the Burnside ring $\Omega(G)$ given by


Then for any proper subgroup $H < G$, $res_H^G \alpha$ coincides with $[H/H]$ in $\Omega(H)$.

**Theorem 3.3** (Deleting Theorem). (Let $G = A_5$.) Let $Y$ be a compact connected smooth $G$-manifold of dimension $\geq 5$, with $|\pi_1(Y)| < \infty$, and with a decomposition $Y^G = Y_0^G \amalg Y_1^G$ such that $\partial Y_0^G = \emptyset$. Let $U$ be the $G$-tubular neighborhood of $Y_0^G$. Suppose $U$ satisfies the gap condition for $\{(e), C_2\}$, $\{(e), C_3\}$ and $\{(e), C_5\}$, and the cobordism gap condition for $(C_2, D_6)$, $(C_2, D_{10})$ and $(C_3, A_4)$. Then there exists a smooth $G$-manifold $X$ possessing the following properties.

1. $X^G = Y_1^G$.
2. $X$ is homotopy equivalent to $Y$.
3. $X^H$ is diffeomorphic to $Y^H$ for any $H$ such that $\{e\} \neq H < G$.
4. In the case that $\dim Y \equiv 0 \mod 2$ and $\pi_1(Y) = 1$, $X$ is diffeomorphic to $Y$.

**Guideline for Proof.** There exists a compact connected smooth $G$-submanifold $U_1$ of $Y \setminus \partial Y$ with $U \subset U_1$ such that $G$ freely acts on $U_1 \setminus U$ and the inclusion induced homomorphism $\pi_1(U_1) \to \pi_1(Y)$ is an isomorphism. First, we construct a $G$-framed map $f_1 : X_1 \to Y$ rel. $Y \setminus \hat{U}_1$ (i.e., $X_1 \supset Y \setminus \hat{U}_1$,

$$f_1|_{Y \setminus \hat{U}_1} : Y \setminus \hat{U}_1 \to Y \setminus \hat{U}_1$$

is the identity map, and $f_1(X_2) \subset U_1$, where

$$X_2 = X_1 \setminus (Y \setminus \hat{U}_1)^C \setminus \partial Y).$$

Next we convert $f_2 = f_1|_{X_2} : X_2 \to U_1$, to a $G$-framed map $f_3 : X_3 \to U_1$ such that $f_3$ is a homotopy equivalence by $G$-surgeries rel. $\partial U_1$ of isotropy types $(H)$.
for $H < G$. The construction of a $G$-framed map is discussed in Section 4. We perform the $G$-surgeries of isotropy types $(H)$ with $\{e\} < H < G$ by means of the reflection method in [8]. We do the $G$-surgeries of isotropy type $(\{e\})$ by showing triviality of the algebraic $G$-surgery obstruction in the relevant Bak group described in [9] with the $G$-cobordism invariance property given in [10] and the induction-restriction property presented in [13, 4].

**Theorem 3.4** (Realization Theorem). *(Let $G = A_5$.)* Let $W$ a real $G$-module with $\dim W_G = m$, $N$ an integer possessing the property described in Theorem 2.2, and $Z$ a compact connected smooth $G$-manifold of dimension $\geq 5$ such that $\partial Z^G = \emptyset$, $V$ the $G$-tubular neighborhood of $Z^G$, $z_0$ a point in $Z^G$, and $n$ an integer $\geq N$. Suppose $|\pi_1(Z)| < \infty$ and $T_{x_0}(Z)$ is isomorphic to $W \oplus (\mathbb{R}[G] - \mathbb{R})^\oplus n$. Further suppose $V$ satisfies the gap condition for $(\{e\}, C_2)$, $(\{e\}, C_3)$ and $(\{e\}, C_5)$, and the cobordism gap condition for $(C_2, D_6)$, $(C_2, D_{10})$, and $(C_3, A_4)$. Let $F, F_0, x_0 \in F_0$ be as in Theorem 2.2. Then there exists a compact $G$-manifold $X$ satisfying the following conditions.

1. $X^G$ is diffeomorphic to $F$.
2. $X$ is homotopy equivalent to $Z$.
3. In the case that $\dim Z \equiv 0 \mod 2$ and $\pi_1(Z) = 0$, $X$ is diffeomorphic to $Z$.

**Proof.** Take the smooth $G$-action on the sphere $S$ described in Corollary 2.3 with $x_0 \in F_0 \subset F$ and $S^G = F \amalg F'$ such that $T_{x_0}(S) \cong W \oplus (\mathbb{R}[G] - \mathbb{R})^\oplus n$ and $F \cong F'$. Let $Y$ be the connected sum of $S$ and $Z$ at points $x_0$ and $z_0$. By setting $Y^G_0 = F \# Z^G$ and $Y^G_1 = F'$ we have $Y^G = Y^G_0 \amalg Y^G_1$. Note $Y^G_0$ is without boundary. The tubular neighborhood $U$ of $Y^G_0$ satisfies the gap condition for $(\{e\}, C_2)$, $(\{e\}, C_3)$ and $(\{e\}, C_5)$, and the cobordism gap condition for $(C_2, D_6)$, $(C_2, D_{10})$, and $(C_3, A_4)$. Deleting the fixed point submanifold $F^G_0$ from $Y$ by means of Theorem 3.3, there exists a $G$-manifold $X$ satisfying the desired conditions in the theorem, where $X^G = F' \cong F$. □

4. EQUIVARIANT COHOMOLOGY THEORY $\omega(\bullet)_G^*$ AND $G$-FRAMED MAPS

Equivariant surgeries are operated on smooth $G$-manifolds, but more precisely on $G$-framed maps. T. Petrie gave an idea to construct $G$-framed maps by using
the equivariant cohomology theory $\omega^*_G(\bullet)$ and the Burnside ring $\Omega(G)$. In order to construct our $G$-framed maps, we employ a modification given in [11] of Petrie's construction.

Let $\Omega(G)$ denote the Burnside ring, i.e.

$$\Omega(G) = \{ [X] \mid X \text{ is a finite } G\text{-CW complex} \},$$

where $[X] = [Y]$ if and only if $\chi(X^H) = \chi(Y^H)$ for all $H \leq G$. The next proposition is well known, see [2, 20, 1].

**Proposition 4.1.** Let $G$ be a nontrivial perfect group. Then there exists an idempotent $\beta$ in the Burnside ring $\Omega(G)$ such that $\chi_G(\beta) = 1$ and $\chi_H(\beta) = 0$ for all $H \neq G$.

Let $S(G)$ and $S(G)_{\text{max}}$ denote the set of all subgroups and all maximal proper subgroups of $G$, respectively. For a subgroup $H$ of $G$, $(H)$ stands for the $G$-conjugacy class containing $H$, i.e.

$$(H) = \{ gHg^{-1} \mid g \in G \}.$$

The $\beta$ in Proposition 4.1 has the form

$$\beta = [G/G] - \sum_{(K) \subset S(G)_{\text{max}}} [G/K] - \sum_{(H) \subset \mathcal{F}} a_H [G/H],$$

for some $G$-invariant lower closed $\mathcal{F} \subset S(G) \setminus (S(G)_{\text{max}} \cup (G))$ and $a_H \in \mathbb{Z}$.

**Proposition 4.2.** Let $G = A_5$ and let $\alpha$ be the element of the Burnside ring $\Omega(G)$ given by


Then for any proper subgroup $H < G$, $\text{res}^G_H \alpha$ coincides with $[H/H]$ in $\Omega(H)$.

**Proposition 4.3.** Let $G$ be $A_5$. Then there exists a finite $G$-CW complex $Z$ fulfilling the following conditions.

1. $Z^G = \{ z_0, z_1 \}$.
2. $\bigcup_{H \in S(G)_{\text{max}}} Z^H = \{ z_0, z_1 \} \ast S(G)_{\text{max}}$, where each subgroup in $S(G)_{\text{max}}$ is regarded as a point. Thus $Z^H$ is homeomorphic to the 1-dimensional disk $[-1, 1]$ for each $H \in S(G)_{\text{max}}$. 

(3) \( Z^{D_4} = Z^{A_4} \) and \( Z^{C_5} = Z^{D_{10}} \).

(4) \( Z^H \) is contractible for any subgroup \( H < G \).

The \( G \)-CW complex \( Z \) in this lemma is constructed using Oliver-Petrie's \( G \)-CW-surgery theory [21] and the wedge sum technique [16].

Let \( M_n = \mathbb{C}[G]^{\oplus n} \) and let \( M_n^* \) be the one-point compactification of \( M_n \), hence we write \( M_n^* = M_n \cup \{ \infty \} \). For a finite \( G \)-CW complex \( X \) with base point in \( X^G \),

\[
\overline{\omega}_G^0(X) = \lim_{n \to \infty} [X \wedge M^*, M^*]^G_0,
\]

where \([-,-]^G_0\] stands for the set of all homotopy classes of maps in the category of pointed \( G \)-spaces. For a finite \( G \)-CW complex \( Z \), \( \omega_G^0(Z) \) is defined to be \( \overline{\omega}_G^0(Z^+) \), where

\[
Z^+ = Z \amalg (G/G)
\]

and \( G/G \) is regarded as the base point of \( Z^+ \).

For the set \( S \) of all powers \( \beta^k, k \in \mathbb{N} \), the restriction \( j^*: S^{-1}\omega_G^0(Z) \to S^{-1}\omega_G^0(Z^G) \) induced by the inclusion map \( j: Z^G \to Z \) is an isomorphism. It is obvious that for any element \( \gamma \in \omega_H^0(Z) \) and any proper subgroup \( H \) of \( G \), \( \text{res}_H^G \beta \cdot \gamma = 0_{Z} \) in \( \omega_H^0(Z) \).

Lemma 4.4. Let \( G \) be a nontrivial perfect group, \( \beta \) the element in Proposition 4.1, and \( Z \) a finite \( G \)-CW complex with

\[
Z^G = \{ z_0, z_1 \}.
\]

Then there exists an element \( \gamma \in \omega_G^0(Z) \) such that \( \gamma|_{z_0} = \beta \) and \( \gamma|_{z_1} = 0_{z_1} \) in \( \Omega(G) \) and \( \beta \gamma = \gamma \). In addition, for any proper subgroup \( H < G \), there exists a 'homotopy' \( \Gamma_H \in \omega_H^0(Z \times I) \) from \( \text{res}_H^G \gamma \) to \( 0_Z \), rel. \( z_1 \times I \), i.e. \( \Gamma_H|_{Z \times \{0\}} = \text{res}_H^G \gamma, \Gamma_H|_{Z \times \{1\}} = 0_Z, \) and \( \Gamma_H|_{z_1 \times I} = 0_{z_1 \times I} \), such that \( \text{res}_H^G \beta \cdot \Gamma_H = \Gamma_H \). Moreover, for any pair of distinct proper subgroups \( H \) and \( K \) of \( G \), there exists a 'homotopy' \( \overline{\Gamma}_{H,K} \in \omega_{H \cap K}^0(Z \times I \times I) \) from \( \text{res}_{H \cap K}^H \Gamma_H \) to \( \text{res}_{H \cap K}^K \Gamma_K \), rel. \( z_1 \times I \times I \) and \( Z \times \partial I \times I \).

As a next step, consider the elements \( 1_{Z} - \gamma \in \omega_G^0(Z), 1_{Z \times I} - \Gamma_H \in \omega_H^0(Z \times I), \) and \( 1_{Z \times I \times I} - \overline{\Gamma}_{H,K} \in \omega_{H \cap K}^0(Z \times I \times I) \). Recall that an element \( \alpha \in \omega_G^0(Z) \) is represented by a \( G \)-map

\[
Z^+ \wedge M^* \to M^*
\]

preserving the base point \( \infty \), where \( M = \mathbb{C}[G]^{\oplus n} \) for some \( n \).
Lemma 4.5. Let $G$ be a nontrivial perfect group, $\beta$ the element in Proposition 4.1, and $Z$ a finite $G$-CW complex with $Z^G = \{z_0, z_1\}$. Then there exist maps $\alpha$, $A_H$, and $\overline{A}_{H,K}$ satisfying the following conditions (1)–(3), where $H$ and $K$ range all proper subgroups of $G$ such that $H \neq K$.

1. $\alpha$ is a map of pointed $G$-spaces $Z^+ \times M^* \to M^*$ such that $[\alpha] = 1 - \gamma$ for the $\gamma$ above, and $\alpha|_{\{z_1\}^+ \wedge M^*} = id_{\{z_1\}^+ \wedge M^*}$.

2. $A_H$ is a homotopy of pointed $H$-spaces $(Z \times I)^+ \wedge M^* \to M^*$ from $\alpha$ to $1_Z$, where $1_Z : Z^+ \wedge M^* \to M^*$ and $1_Z(z, v) = v$ for all $z \in Z$ and $v \in M$, rel. $\{z_1\}^+ \wedge M^*$.

3. $\overline{A}_{H,K}$ is a homotopy of pointed $H \cap K$-spaces $((Z \times I) \times I)^+ \wedge M^* \to M^*$ from $A_H$ to $A_j$ rel. $(z_1 \times I)^+ \wedge M^*$ and $(Z \times \partial)^+ \wedge M^*$.

Proposition 4.6. Let $G = A_5$ and $\mathcal{K} = (A_4) \cup (D_{10}) \cup (D_6) \cup (D_4) \cup (C_5)$. Let $Z$ be the finite $G$-CW complex in Proposition 4.3. Then there exist maps $\alpha$, $A_H$, and $\overline{A}_{H,K}$ of Lemma 4.5 satisfying the additional conditions:

1. $\alpha|_{z_0}^{-1}(0)^G = \emptyset$ and $|((\alpha|_{z_0}^{-1}(0))^H| = 1$ for $H \in \mathcal{K}$.

2. For each $H \in \mathcal{K}$, there exists a connected component $X(H)$ of $\alpha^{-1}(0)$ containing both $(\alpha|_{z_0}^{-1}(0))^H$ and $(z_1, 0)$ such that $\alpha$ is transversal on $X(H)$ to $0 \subset M$, the normal derivative of $\alpha$ on $X(H)$ is the identity, and the projection $Z \times M \to Z$ diffeomorphically maps $X(H)$ to $Z^H$.

3. For each pair of $H \in S(G)_{\text{max}}$ and $L \in \mathcal{K}$ with $L \leq H$, there exists a connected component $W(H, L)$ of $A_H^{-1}(0)$ containing $X(H)^L \times \{0\}$ ($\subset Z \times M \times I$) and $Z^L \times \partial \times \{0\}$ ($\subset Z \times M \times I$) such that $A_H$ is transversal on $W(H, L)$ to $0 \subset M$, the normal derivative of $A_H$ on $W(H, L)$ is the identity, and the projection $Z \times I \times M \to Z \times I$ diffeomorphically maps $W(H, L)$ to $Z^L \times I$.

A $G$-framed map $f = (f, b)$ consists of a $G$-map $f : X \to Y$ such that $X$ and $Y$ are compact smooth $G$-manifold and $f(\partial X) \subset \partial Y$, and an isomorphism $b : T(X) \oplus \mathcal{E}_X(\mathbb{R}^m) \to f^*T(Y) \oplus \mathcal{E}_X(\mathbb{R}^m)$ of real $G$-vector bundles for some integer $m \geq 0$. In the following we suppose $Y$ is connected and $f : (X, \partial X) \to (Y, \partial Y)$ is of degree 1.
Lemma 4.7. Let $Y$ be a compact smooth $G$-manifold with a decomposition $Y^G = Y_0^G \amalg Y_1^G$ such that $\partial Y_0^G = \emptyset$. Let $U$ be the $G$-tubular neighborhood of $Y_0^G$. Then there exist a $G$-framed map $f = (f, b)$, $H$-framed cobordisms $F_H = (F_H, B_H) : f \sim id_Y$, rel. $Y \setminus \hat{U}$ for $H < G$, and $H \cap K$-framed cobordisms $\overline{F}_{H,K} = (\overline{F}_{H,K}, \overline{B}_{H,K}) : F_H \sim F_K$ rel. $((Y \setminus \mathring{U}) \times I) \cup (Y \times \partial I)$, for $H, K < G$ such that $H \neq K$, where

\[ f : X \rightarrow Y, \]
\[ b : T(X) \oplus \varepsilon_X(\mathbb{R}^m) \rightarrow f^*(Y) \oplus \varepsilon_X(\mathbb{R}^m), \]
\[ F_H : W_H \rightarrow Y \times I, \]
\[ B_H : T(W_H) \oplus \varepsilon_{W_H}(\mathbb{R}^m) \rightarrow F_H^*T(Y \times I) \oplus \varepsilon_{W_H}(\mathbb{R}^m), \]
\[ \overline{F}_{H,K} : \overline{W}_{H,K} \rightarrow Y \times I \times I, \]
\[ \overline{B}_{H,K} : T(\overline{W}_{H,K}) \oplus \varepsilon_{\overline{W}_{H,K}}(\mathbb{R}^m) \rightarrow \overline{F}_{H,K}^*T(Y \times I \times I) \oplus \varepsilon_{\overline{W}_{H,K}}(\mathbb{R}^m), \]

for some integer $m > 0$.

This lemma is obtained by the arguments in [11].

Lemma 4.8. Let $G = A_5$ and $\mathcal{K} = (A_4) \cup (D_{10}) \cup (D_6) \cup (D_4) \cup (C_5)$. Then the framed maps $f, F_H$ and $\overline{F}_{H,K}$ in Lemma 4.7 can be chosen so that $X^L$ and $W_H^L$ are $N_H(L)$-diffeomorphic to $Y^L$ and $Y^L \times I$, respectively, for all $H, K \in S(G)_{\text{max}}$ and all $L \in \mathcal{K}$ with $L \leq H$.

This modification is achieved by using Proposition 4.6 and the reflection method in [8].

REFERENCES


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