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Categories of Elements

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1 Elements of a set-valued functor

References: [ML98], [Bo94]

1.1 Elements of a functor

Definition 1.1 An **element** of a set valued functor $F : \mathscr{C} \to \mathbf{Set}$ is a pair (X, x) of an object $X \in \mathscr{C}$ and $x \in F(X)$. A **morphism** $f : (X, x) \to (Y, y)$ between elements is a morphism $f : X \to Y$ in $\mathscr{C}$ such that

$$F(f) : F(X) \to F(Y); x \mapsto y$$

The elements of F form the **category of elements**, which is denoted by

$$\mathbf{Elts}(F) \text{ or } \mathbf{Elts}(\mathscr{C}, F)$$

with **projection functor**

$$\pi_F : \mathbf{Elts}(F) \to \mathscr{C}; (X, x) \mapsto X.$$

For a contravariant functor, the category of elements is similarly defined.
See[Yo60], [Bo94, I.p37]. ■

Lemma 1.1 In $\mathbf{Elts}(\mathscr{C}, F)$, the following hold:
(i) $(X, x) \cong (Y, y)$ if and only if there exists $f : X \cong Y$ in $\mathscr{C}$ such that $y = f(x)$.
(ii) There is a bijection

$$\mathrm{Obj}(\mathbf{Elts}(\mathscr{C}, F))/\cong \longleftrightarrow \coprod_{X \in \mathscr{C}}{}' \mathrm{Aut}(X)\backslash F(X)$$

Here $\coprod'$ is the coproduct over the isomorphisms classes $\mathrm{Obj}(\mathscr{C})/\cong$

1.2 comma categories and slice categories

Definition 1.2 The **comma category** $(S \downarrow T)$ of a pair of functors $\mathscr{D} \xrightarrow{S} \mathscr{C} \xleftarrow{T} \mathscr{E}$ has as objects all triplets $(X, Y, S(X) \xrightarrow{f} T(Y))$.and as morphisms $(X, Y, S(X) \xrightarrow{f} T(Y)) \to (X', Y', S(X') \xrightarrow{f'} T(Y'))$ all pairs $(X \xrightarrow{u} X', Y \xrightarrow{v} Y')$ such that

$$\begin{tikzcd} X \arrow[d, "u"'] & Y \arrow[d, "v"'] & SX \arrow[r, "f"] \arrow[d, "Su"'] \arrow[dr, phantom, "\circlearrowleft"] & TY \arrow[d, "Tv"] \\ X' & Y' & SX' \arrow[r, "f'"] & TY' \end{tikzcd}$$

The compositions are given by those of $\mathscr{D}$ and $\mathscr{E}$.
[ML98], [Bo94] ■

Definition 1.3 The **slice category** $\mathscr{C}/X$ over an object $X \in \mathscr{C}$ is the category of morphisms into X. A morphism from $(A \xrightarrow{\alpha} X)$ to $(B \xrightarrow{\beta} X)$ is a morphism $f : A \to B$ in $\mathscr{C}$ such that $\alpha = f\beta$.

Similarly, the **coslice category** $X\backslash\mathscr{C}$ is defined ass the category of morphisms from X.

Let $S = \mathrm{Id}_{\mathscr{C}}$ be an identity functor of $\mathscr{C}$, and $T : * := \{*, \mathrm{id}_*\} \to \mathscr{C}; * \mapsto X$. Then there are equivalences of categories

$$(S \downarrow T) \approx \mathscr{C}/X \text{ and } (T \downarrow S) \approx X\backslash\mathscr{C}$$

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$F : \mathscr{C} \to \mathbf{Set}$, $S : \{*\} \hookrightarrow \mathbf{Set}$. Then the category of elements of F is presented by a comma category:

$$\mathbf{Elts}(\mathscr{C}, F) \cong S \downarrow F$$

■

1.3 Examples

Example 1.1 A monoid M can be identified with a category $\boldsymbol{M}$ with a single object $*$ and with $\mathrm{Hom}(*,*) = M$. Let X be an M-set with left M-action $M \times X \to X; (a, x) \mapsto ax$.

Such an M-set X can be viewed as
(i) a functor $X : \boldsymbol{M} \to \mathbf{Set}; * \mapsto X$;
and also as
(ii) a category $\boldsymbol{X}$ with $\mathrm{Obj}(\boldsymbol{X}) = X$ and with

$$\mathrm{Hom}_{\boldsymbol{X}}(x, y) = \{a \in M \mid ax = y\}$$

Then the category of elements of the functor X is equivalent to $\boldsymbol{X}$:

$$\mathbf{Elts}(\boldsymbol{M}, X) \approx \boldsymbol{X}; (*, x) \longleftrightarrow x$$

■

Example 1.2 Let $X \in \mathscr{C}$. Then

(1) Let $H_X : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}; A \mapsto \mathrm{Hom}(A, X)$ denote the contravariant Hom-functor. Then an element of H_X has the form $(A \xrightarrow{\alpha} X)$, i.e., an object over X, and so the category of elements of H_X is equivalent to the slice category:

$$\mathbf{Elts}(\mathscr{C}, H_X) \approx \mathscr{C}/X$$

(2) Similarly, for the covariant Hom-functor.$H^X : \mathscr{C} \to \mathbf{Set}; A \mapsto \mathrm{Hom}(X, A)$, the category of elements is equivalent to the coslice category:

$$\mathbf{Elts}(\mathscr{C}, H^X) \approx X\backslash\mathscr{C}$$

■

Example 1.3 Let G be a finite group. Let $\mathbf{set}^G$ denote the category of finite (left) G-sets and G-maps and $\mathbf{trans}^G$ the subcategory of $\mathbf{set}^G$ consisting of transitive G-sets. Then a G-map $f : G/H \to G/K$ is decided by the image of $H \in G/H$:

$$\mathrm{Map}_G(G/H, G/K) = \{xK \in G/K \mid H \subset {}^xK\}$$

The **subgroup category** $\mathbf{sub}(G)$ has all subgroups of G as objects. A morphism $H \to K$ is a coset xK such that $H \subset {}^xK := xKx^{-1}$; and the composition is defined by $yL \circ xK = xyL$. Then $\mathbf{sub}(G)$ is equivalent to $\mathbf{trans}^G$ by $H \mapsto G/H$. Two subgroups are isomorphic in $\mathbf{set}(G)$ if and only if they are conjugate, and so $C(G) := \mathbf{sub}(G)/\cong$ is the set of conjugacy classes of subgroups.

Let $\mathrm{Sub}(G)$ be the **subgroup lattice** of G. Note that any poset can be viewed as a category. Let $\mathbf{hom}(1, -) : H \mapsto G/H$ be the Hom-functor from the trivial subgroup $1 \in \mathbf{set}(G)$. Then

$$\begin{aligned}
&\mathbf{Elts}(\mathbf{sub}(G), \mathbf{hom}(1, -)),\\
&1\backslash\mathbf{sub}(G),\\
&\mathbf{Elts}(\mathbf{trans}^G, \mathrm{Map}_G(G/1, -)),\\
&(G/1)\backslash\mathbf{trans}^G
\end{aligned}$$

are all equivalent to $\mathrm{Sub}(G)$ as categories. In particular, the isomorphism classes of these categories are all are bijectively corresponding to the set of subgroups of G.

As a conclusion the subgroup lattice $\mathrm{Sub}(G)$ is categorically viewed as the category of elements of a functor!! ■

Categories of elements are used to prove the following two important theorems. Refer to [Ri14].

Example 1.4 Yoneda's density theorem: Let $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set}$ and let $\widehat{\mathscr{C}} := [\mathscr{C}^{\mathrm{op}}, \mathbf{Set}]$. Then

$$F \cong \varinjlim \left(\mathbf{Elts}(F) \xrightarrow{\pi_F} \mathscr{C} \xrightarrow{\boldsymbol{y}} \widehat{\mathscr{C}} \right),$$

where $\boldsymbol{y} : X \mapsto \mathrm{Hom}(-, X)$ denotes the Yoneda embedding.

Example 1.5 Kan extension:
Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Then $\widehat{F} : \widehat{\mathscr{D}} \to \widehat{\mathscr{C}}; Y \mapsto Y \circ F$ has a left adjoint functor and a right adjoint functor:

$$\mathrm{Lan}(F) \dashv \widehat{F} \dashv \mathrm{Ran}(F)$$

The value of $\mathrm{Lan}(F)$ at $X \in \widehat{\mathscr{C}}$ is given by

$$\mathrm{Lan}(F)(X) = \varinjlim \left(F \downarrow J \xrightarrow{\pi} \mathscr{C} \xrightarrow{X} \mathbf{Set}\right)$$

$$\cong \varinjlim \left(\mathbf{Elts}(H_J \circ F) \xrightarrow{\pi} \mathscr{C} \xrightarrow{X} \mathbf{Set}\right)$$

Similarly $\mathrm{Ran}(F)(X)$ is obtained by replacing the limit instead of the colimit. [ML98, X.3]

1.4 Operations on set-valued functors

There are some arithmetical o perations on categories and functors. We study what categories of the elements of set-valued functors play in such operations. Refer to [Yo01]

Let $\mathscr{C}, \mathscr{D}, \mathscr{E}$ be categories, and $F : \mathscr{C} \to \mathbf{Set}$, $G : \mathscr{D} \to \mathbf{Set}$, $H : \mathscr{E} \to \mathbf{Set}$ set-valued functors. Then we define additions and products as follows :

(i) $\mathscr{C} + \mathscr{D}$: the disjoint union of categories.
(ii) $\mathscr{C} \times \mathscr{D}$: the Cartesian product of categories.
(iii) $F + G$: the summation of functors.

$$F + G : \mathscr{C} + \mathscr{D} \to \mathbf{Set}; Z \mapsto \begin{cases} F(Z) & (Z \in \mathscr{C}) \\ G(Z) & (Z \in \mathscr{D}) \end{cases}$$

(iv)$F \times G$: the product of functors.

$$F \times G : \mathscr{C} \times \mathscr{D} \to \mathbf{Set}; (X, Y) \mapsto F(X) + G(Y)$$

Here $F(X) + G(Y)$ denotes the disjoint union of sets $F(X)$ and $G(Y)$.
(v) F^n: the power of a functor.

$$F^n : \mathscr{C}^n \to \mathbf{Set}; (X_k)_{k=1}^n \mapsto \coprod_{k=1}^{n} F(X_k)$$

Then the 2-category $\mathfrak{Cat}$ has a commutative semi-ring structure by $+$ and $\times$.. Furthermore, so is the 2-category $\mathfrak{Cat}/\mathbf{Set}$ of set valued functors. For example, the following distributive law holds

$$(F + G) \times H \cong F \times H + G \times H$$

"Zero" and "One" in $\mathfrak{Cat}/\mathbf{Set}$ is

$$\boldsymbol{O} : \emptyset \to \mathbf{Set},$$
$$\boldsymbol{I} : \mathbf{1} = \{*, \mathrm{id}_*\} \to \mathbf{Set}; * \mapsto \{*\}$$

respectively.

For a functor $F : \mathscr{C} \to \mathbf{Set}$, define a functor

$$\partial F : \mathbf{Elts}(\mathscr{C}, F) \xrightarrow{\pi_F} \mathscr{C} \xrightarrow{F} \mathbf{Set}$$

Then the following hold:

$$\mathbf{Elts}(F \times G) \approx \mathbf{Elts}(F) \times \mathscr{D} + \mathscr{C} \times \mathbf{Elts}(G)$$
$$\partial(F \times G) \cong F \times \partial(G) + \partial(F) \times G$$
$$\mathbf{Elts}(F^n) \approx n\mathscr{C}^{n-1} \times \mathbf{Elts}(F)$$
$$\partial(F^n) \cong n\mathbb{F}^{n-1} \times \partial(F)$$

These formulas look like Leibniz's product rule for differentiation. This is the reason why we used ∂F for the functor from the category of elements.

Remark. In some literature (e.g., [ML98]), $\mathbf{Elts}(\mathscr{C}, F)$ is often denoted by the symbol

$$\int_{\mathscr{C}} F \quad \text{or} \quad \int F.$$

This symbol is not suitable for the category of elements because of Leibniz rule.

2 Generating functions

Refrence: [Yo13], [Yo01], [Jo81].

2.1 Universal zeta functions (UZF)

The reason why the category of elements of a functor works like derivation becomes clear by considering generating functions of categories and functors.

Let $\mathscr{C}$ be a essentially small and locally finite category, and so $\mathscr{C}$ is equivalent to a small category and each hom-set $\mathrm{Hom}(X,Y)$ is a finite set for any $X, Y \in \mathscr{C}$. Then the **universal zeta function** (or **exponential generating function** of $\mathscr{C}$ is defined as a formal series

$$\mathscr{C}(t) := \sum_{M\in\mathscr{C}}{}' \frac{1}{|\mathrm{Aut}(M)|} t^M$$

where $\sum'$ takes over isomorphism classes of objects of $\mathscr{C}$. The symbols t^M $(M \in \mathscr{C})$ are assumed to satisfy the relations

(i) $M \cong M' \Rightarrow t^M = t^{M'}$
(ii) $t^{\emptyset} = 1$, $t^{M+M'} = t^M \cdot t^{M'}$ if there exist any finite coproducts, where $\emptyset$ is an initial object.

The **universal zeta function** (or **exponential generating function** of a functor $F : \mathscr{C} \to \mathscr{D}$ is

$$F(t) := \sum_{M\in\mathscr{C}}{}' \frac{1}{|\mathrm{Aut}(M)|} t^{F(M)}$$

Here the summation is well-defined only if the fibers of F are all finite sets, that is, for any $N \in \mathscr{D}$,

$$\sharp\{M \in \mathscr{C}/\cong \mid F(M) \cong N\} < \infty.$$

Such a functor F is said to have **finite fibers**.

Let **set** be the category of finite sets. We identify the symbol t^N with the monomial polynomial $t^{|N|}$. Thus if $F : \mathscr{C} \to \mathbf{set}$ is a faithful functor with finite fibers, then the UZF $F(t)$ is the usual formal power series. For example,

$$\mathbf{set}(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} = \exp(t) \in \mathbb{Q}[[t]]$$

2.2 $\mathscr{C}$-structures

Let $F : \mathscr{C} \to \mathscr{D}$ a faithful functor.

Definition 2.1 An **$\mathscr{C}$-structure** on $N(\in \mathscr{D})$ is (X, σ), where $X \in \mathscr{C}$ and $\sigma : F(X) \xrightarrow{\cong} N$. The isomorphism σ is called a **labeling**. We denote by $\mathbf{Str}(\mathscr{C}/N) \subset F \downarrow N$ the category of $\mathscr{C}$-structures on N.

The isomorphism of two $\mathscr{C}$-structures on N is defined by

$$(X, \sigma) \cong (Y, \tau) \Leftrightarrow \exists f : X \cong Y \text{ s.t. } \tau \circ F(f) = \sigma$$

■

Lemma 2.1 The UZF of F satisfying the following:

$$F(t) = \sum_{N\in\mathscr{D}}{}' \frac{|\mathbf{Str}(\mathscr{C}/N)/\cong|}{|\mathrm{Aut}(N)|} t^N$$

Furthermore, $|\mathbf{Str}(\mathscr{C}/N)/\cong|$, the number of isomorphism classes of $\mathscr{C}$-structures on N, is equal to

$$\sum_{F(X)\cong N}{}' \left(\mathrm{Aut}\left(F(X)\right) : F\left(\mathrm{Aut}(X)\right)\right),$$

where the summation is taken over isomorphism classes of $\mathscr{C}$-structures on N.

2.3 Operations on UZF

The definitions of operations on faithful functors match those on power series, that is, for any faithful functors $F : \mathscr{C} \to \mathbf{set}$ and $G : \mathscr{D} \to \mathbf{set}$ into the category of finite sets with finite fibers, we have the equations of formal power series:

$$(F + G)(t) = F(t) + G(t)$$
$$(FG)(t) = F(t)G(t)$$
$$\emptyset(t) = 0, \quad \mathbf{1}(t) = 1.$$

As before, let

$$\partial F : \mathbf{Elts}(\mathscr{C}, F) \xrightarrow{\pi_F} \mathscr{C} \xrightarrow{F} \mathbf{set}; (X, x) \mapsto X \mapsto F(X)$$

Then its UZF is

$$(\partial F)(t) = \sum_{M\in\mathscr{C}}{}' \frac{|F(M)|}{|\mathrm{Aut}(M)|} t^{F(M)} = t\frac{dF(t)}{dt}$$

Remark.

$$F' : \mathbf{Elts}(\mathscr{C}, F) \to \mathbf{set}; (X, x) \mapsto F(X) - \{x\},$$

gives the usual derivation $F'(t) = dF(t)/dt$. Unfortunately, unless all $F(f)$ are monic, F' is not a functor.

Let $F : \mathscr{C} \to \mathscr{D}$ be a functor. Let $H^I := \mathrm{Hom}(I, -) : \mathscr{D} \to \mathbf{set}$ be a Hom-functor associated to $I \in \mathscr{D}$. Then a **partial derivation** of F is defined by

$$\begin{aligned}\partial_I(F) := \partial(H^I \circ F) : \mathbf{Elts}(H^I \circ F) \xrightarrow{\pi} \mathscr{C} \xrightarrow{H^I} \mathbf{set} \\ ; (X, x) \mapsto \mathrm{Hom}(I, F(X))\end{aligned}$$

It is possible to define a so-called plethysm compositions of categories (or functors). Here we only give exponential of categories.

Definition 2.2 For a category $\mathscr{C}$, the **fibred category** $\mathbf{Exp}(\mathscr{C})$(or often $\mathbf{set}(\mathscr{C})$) is the category with objects all indexed $\mathscr{C}$-objects $(X_i)_{i\in I}$, where I is a finite set and X_i is an object of $\mathscr{C}$, and with morphisms $(\pi, (f_i)_{i\in I}) : (X_i)_{i\in I} \to (Y_j)_{j\in J}$, where $\pi : I \to J$ and $f_i : X_i \to Y_{\pi(i)}$. The category $\mathbf{Exp}(\mathscr{C})$ has any finite coproduts.

For any functor $F : \mathscr{C} \to \mathbf{Set}$ can be uniquely extended to

$$\mathbf{Exp}(F) : \mathbf{Exp}(\mathscr{C}) \to \mathbf{Set}; (X_i)_{i\in I} \mapsto \coprod_{i\in I} F(X_i)$$

which preserves finite coproducts. ■

Let $\mathbf{1}$ be the category with only one object $*$ and only one morphism id_*. Then $\mathbf{Exp(1)} \approx \mathbf{set}$, the category of finite sets.

Lemma 2.2 (1) $\mathbf{Exp}(\mathscr{C})(t) = \exp(\mathscr{C}(t))$.
(2) $\mathbf{Exp}(F)(t) = \exp(F(t))$.
(3) $\mathbf{Exp}(\mathscr{C} + \mathscr{D}) \approx \mathbf{Exp}(\mathscr{C}) \times \mathbf{Exp}(\mathscr{D})$
(4) $\mathbf{Exp}(F + G) \cong \mathbf{Exp}(F) \times \mathbf{Exp}(G)$
(5) $\partial(\mathbf{Exp}(F)) = (\partial F) \cdot \mathbf{Exp}(F)$.

Example 2.1 Tree

2.4 Wohlfahrt formula

Theorem 2.3 Let G be a finitely generated group.. Then the following hold:

(1) $\mathbf{set}^G \approx \mathbf{Exp}(\mathbf{trans}^G)$.
(2) $\mathbf{set}^G(t) = \exp\left(\mathbf{trans}^G(t)\right)$.
(3) $\mathbf{trans}^G(t) = \sum_{H\leq_f G} \frac{t^{G/H}}{(G:H)},$
where H runs over all subgroups of G of finite index.
(4) Let $F : \mathbf{set}^G \to \mathbf{set}$ be the forgetful functor. Then the following identity holds:

$$\begin{aligned}F(t) &= 1 + \sum_{n=1}^{\infty} \frac{|\mathrm{Hom}(G, S_n)|}{n!} t^n \\ &= \exp\left(\sum_{H\leq_f G} \frac{t^{(G:H)}}{(G:H)}\right)\end{aligned}$$

(1) follows from the unique decomposition of any finite G-set into the disjoint union of its orbits. (3) follows from the fact that a transitive G-set is G-isomorphic to a homogeneous G-set of the form G/H and that (i) $G/H \cong_G G/K$ iff H and K are G-conjugate; (ii) $\mathrm{Aut}(G/H) \cong WH := N_G(H)/H$; (iii) the number of subgroups of G conjugate to H is equal to $(G : N_G(H))$. (4) follows from the existence of a bijection:

$$\mathbf{Str}(\mathbf{set}^G/[n])/\cong \longleftrightarrow \mathrm{Hom}(G, S_n)$$

remark. The identity in (4) is first published be Wohlfahrt (1977).

Example 2.2 Let $C = \langle\alpha\rangle$ be an infinite cyclic group. For $n \geq 1$, we put $C^n := \langle\alpha^n\rangle \leq C$ and $C(n) := C/C^n$. Then a finite C-set, that is, a finite dynamical system, is uniquely decomposed into a disjoint union of some transitive (connected) C-sets. Thus $\mathbf{set}^C \approx \mathbf{Exp}(\mathbf{trans}^C)$ and so

$$\mathbf{set}^C(t) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} t^{C(n)}\right)$$

For any finite C-set X, the substitution $t^N \leftarrow |\mathrm{Hom}_C(N,X)|u^{|N|}$ gives

$$\sum_{N\in\mathbf{set}^C}' \frac{|\mathrm{Hom}(N,X)|}{|\mathrm{Aut}(N)|}u^{|N|} = \exp\left(\sum_{n=1}^{\infty}\frac{|\mathrm{Fix}_X(\alpha^n)|}{n}u^n\right)$$

where the right hand side is the **Artin-Mazur zeta function** of X.

Furthermore, the UZF of the Hom-functor

$$\mathrm{Hom}(C(l),-) : X \mapsto \mathrm{Hom}(C(l),X) \cong \mathrm{Fix}_X(\alpha^l)$$

is the generating function for the numbers of finite C-sets in which α^l fixes exactly l-points.

$$\exp\left(\sum_{n|l} t^n\right)$$

Refer to [DS89]. ■

2.5 Theory of species

There is another categorical theory of generating functions introduced and developed by Joyal([Jo81]).

Definition 2.3 Let **bij** be the cat of finite sets and bijections and let S_n be the symmetric group of degree n. Then a (set valued) **species** is a functor **bij** $\rightarrow$ **set**. Thus a species $\boldsymbol{A}$ is nothing but a series $(\boldsymbol{A}[n])_{n=0,1,\ldots}$ of finite S_n-sets.

The **generating function (series)** of a species $\boldsymbol{A}$ is

$$\boldsymbol{A}(t) := \sum_{n=0}^{\infty} |\boldsymbol{A}[n]| \frac{t^n}{n!}$$

■

Combinatorially, $\boldsymbol{A}[I]$ means "the set of $\boldsymbol{A}$-structures on a finite set I".

As in the case of Set-valued functors, species also have arithmetical operations, for example, the derivation of $\boldsymbol{A}$ is defined by

$$\boldsymbol{A}'[I] := \boldsymbol{A}[I \cup \{I\}]$$

Then $\boldsymbol{A}'(t)$ is the derivation of $\boldsymbol{A}(t)$.

The theory of species is included in those of faithful functors with finite fibers. In fact, given a species $\boldsymbol{A} : \mathbf{bij} \rightarrow \mathbf{set}$,

$$A : \mathbf{Elts}(\boldsymbol{A}) \xrightarrow{\pi} \mathbf{bij} \subset \mathbf{set}; (I,i) \mapsto I$$

is a faithful functor with finite fibers and with the same generating functions $A(t) = \boldsymbol{A}(t)$. Note that $\mathbf{Elts}(\boldsymbol{A})$ is a groupoid, that is, a category in which all morphisms are isomorphisms. Conversely, given a faithful functor $F : \mathscr{C} \rightarrow \mathbf{set}$ with finite fibers,

$$\boldsymbol{F} : \mathbf{bij} \rightarrow \mathbf{set}; N \mapsto \mathbf{Str}(\mathscr{C}/N)/\cong$$

is a species.

Theorem 2.4 The notion of species is equivalent to those of faithful functors from a groupoid to **set** with finite fibers.

Problem. Rewrite the theory of species by using the notion of faithful functors with finite fibers.

3 Abstract Burnside rings (ABR)

References: Yoshida [Yo87], [Yo90]

3.1 Burnside homomorphisms

Let Γ be an essentially finite and locally finite category. $\mathrm{Obj}(\Gamma)/\cong$ or simply $\Gamma/\cong$ denote the finite set of isomorphisms of objects; $[X]$ or often X denotes the isomorphism class of an object $X \in \Gamma$. Define two abelian groups as follows:

$$\Omega(\Gamma) := \mathbb{Z}\Gamma := \text{free abelian group on} \Gamma/\cong,$$
$$\widetilde{\Omega}(\Gamma) := \mathbb{Z}^{\Gamma} := \mathrm{Map}(\Gamma/\cong, \mathbb{Z}) \cong \prod_{I\in\Gamma}' \mathbb{Z},$$

where the product $\prod'$ is taken over isomorphism classes of objects of Γ. The product ring $\mathbb{Z}^{\Gamma}$ (often wrote as $\mathrm{gh}(\Gamma)$) is called the **ghost ring**.

The linear map

$$\varphi = (\varphi_I) : \mathbb{Z}\Gamma \to \mathbb{Z}^\Gamma; [X] \mapsto (|\mathrm{Hom}(I,X)|)_{I\in\Gamma/\cong}$$

is called the **Burnside homomorphism**, whose representation matrix is the **Hom-set matrix**:

$$H := (|\Gamma(I,J)|)_{I,J\in\Gamma/\cong}.$$

Definition 3.1 $\mathbb{Z}\Gamma$ $(= \Omega(\Gamma))$ is called an **abstract Burnside ring** if $\mathbb{Z}\Gamma$ has a ring structure with 1 and if φ is an injective ring homomorphisms. The abstract Burnside rings with other coefficient rings, for example $\mathbb{Q}, \mathbb{Z}_{(p)}$, etc. can be similarly defined.

Example 3.1 Let $\Gamma := (\mathbf{set}_{\leq n})^{op}$ be the dual category of the category of finite sets of size at most n. We put $[i] := \{1, 2, \ldots, in\}$ and $[0] := \emptyset$.

$$\varphi : \Omega(\Gamma) \to \widetilde{\Omega}(\Gamma); \sum_{i=0}^{n} a_i[i] \mapsto \left(\sum_{i=0}^{n} a_i x^i\right)_{0\leq x\leq n}$$

Thus $\Omega(\Gamma)$ is the module of integral polynomials of degree $\leq n$ and φ is the evaluation map $f(X) \mapsto (f(x))_{0\leq x\leq n}$

$\Omega(\Gamma) \cong \mathbb{Z}[X]/(X(X-1)\cdots(X-n))$.

Example 3.2 Let $\Gamma := \mathbf{set}^*_{\leq n}$ be the category of nonempty sets of size at most n..

$$\varphi : \Omega(\Gamma) \to \widetilde{\Omega}(\Gamma); \sum_{i=1}^{n} a_i[i] \mapsto \left(\sum_{i=0}^{n} a_i i^x\right)_{1\leq x\leq n}$$

Thus $\Omega(\Gamma)$ is the "ring" of finite Dirichlet polynomials of "degree" $\leq n$.

3.2 Möbius rings

Let P be a finite poset, which can be viewed as a finite category such that fro any $x, y \in P$, there exists at most one morphism from x to y. Thus the hom-set matrix is a $P \times P$-matrix $H = (\zeta(x,y))_{x,y\in P}$, where

$$\zeta(x,y) = \begin{cases} 1 & \text{if } x \leq y \\ 0 & \text{else} \end{cases}$$

As is well-known, H is invertible, and so

$$\varphi : \mathbb{Z}P \xrightarrow{\cong} \mathbb{Z}^P; [x] \mapsto (\zeta(i,x))_i$$

is isomorphic. Thus $\mathbb{Z}P$ becomes an abstract Burnside ring, which is called a **Möbius ring**.

The inverse matrix of H is presented by the Möbius function:

$$H^{-1} = (\mu(x,y))_{x,y\in P}.$$

Thus we have and inversion formula and and idempotent formula:

$$\varphi^{-1} : \mathbb{Z}^P \to \mathbb{Z}P; (\chi(i))_i \mapsto \sum_{x,j\in P} \mu(x,j)\chi(j)[x],$$

$$e_t := \sum_{x\in P} \mu(x,t)[x].$$

3.3 Fundamental Theorem for ABR

We assume that two conditions for Γ hold:

(E) All the morphisms of Γ are epimorphic.

(C) For any object I and $\sigma \in \mathrm{Aut}(I)$, there exists a coequalizer diagram:

$$\begin{tikzcd} I \arrow[r, shift left, "1"] \arrow[r, shift right, "\sigma"'] & I \arrow[r, "c_\sigma"] & I/\sigma \end{tikzcd}$$

Definition 3.2 Define an abelian group and homomorphism

$$\mathrm{Obs}(\Gamma) := \prod_{I\in\Gamma}{}' (\mathbb{Z}/|\mathrm{Aut}(I)|\mathbb{Z})$$

$$\psi : (\chi : \Gamma \to \mathbb{Z}) \mapsto \left(\sum_{\sigma\in\mathrm{Aut}(I)} \chi(I/\sigma) \bmod |\mathrm{Aut}(I)|\right)_I$$

$\mathrm{Obs}(\Gamma)$ is called the **group of obstructions** and ψ is called the **Cauchy-Frobenius map**.

Theorem 3.1 The following sequence is exact:

$$0 \to \mathbb{Z}\Gamma \xrightarrow{\varphi} \mathbb{Z}^\Gamma \xrightarrow{\psi} \mathrm{Obs}(\Gamma) \to 0.$$

Theorem 3.2 $\mathbb{Z}\Gamma$ is an abstract Burnside ring.

Remark. (1) The condition F can be replaced by (F) the existence of the unique (E, M)-factorization system such that $E \subset \mathrm{Epi}(\Gamma)$. But then ABR $\mathbb{Z}\Gamma$ is ring isomorphic to another ABR$\mathbb{Z}\Gamma_e$, where Γ_e is the subcategory of Γ consisting of all epimorphisms of Γ. Thus we may assume that (E) holds at first.
(2) $\mathbb{Q}\Gamma$ is always an ABR isomorphic to $\mathbb{Q}^{\Gamma}$ via φ under the condition (F) without C.
(3) For a prime p, $\mathbb{Z}_{(p)}\Gamma$ is an ABR under the condition (F) and the following condition

(C_p) For any $I \in \Gamma$ and any p-element $\sigma \in \mathrm{Aut}(I)$, there exists a coequalizer of $1, \sigma$ similarly as (C).
(4) We may assume that Γ is skeletal, i.e., $X \cong Y \Rightarrow X = Y$.

Let $H := (|\mathrm{Hom}(I, J)|)_{[I],[J]}$ the Hom-set matrix of Γ. Then the inversion formula and the idempotent formula are given by

$$\varphi^{-1} : \mathbb{Q}^{\Gamma} \to \mathbb{Q}\Gamma; \theta \mapsto \sum_{I \in \Gamma}{}' H^{-1}_{IK}\theta(K)\,[I]$$
$$e_K := \sum_{I \in \Gamma}{}' H^{-1}_{IK}[I]$$

We need to calculate the inverse matrix H^{-1} to obtain an explicit idempotent formula.

Example 3.3 Let G be a finite group. The **Burnside ring** $\Omega(G)$ of G is the Grothendieck ring of $\mathbf{set}^G$. It is canonically isomorphic to the ABR $\mathbb{Z}\mathbf{trans}^G$. The Burnside homomorphism is defined by

$$\varphi : \Omega(G) \to \widetilde{\Omega}(G) := \prod_{(S) \in C(G)} \mathbb{Z}$$
$$; [X] \mapsto (|X^S|)_{(S)}$$

Note that there is a bijection

$$X^S := \mathrm{Fix}_S(X) \leftrightarrow \mathrm{Map}_G(G/S, X); x_0 \mapsto (gS \mapsto x_0)$$

The primitive idempotent of $\mathbb{Q}\Omega(G)$ associated to $H \leq G$ is give by

$$e_H = \frac{1}{|N_G(H)|} \sum_{D \leq H} |D|\mu(D, H)[G/D],$$

where μ is the Möbius function of the subgroup lattice of G.

3.4 Discrete cofibration (DCF)

In order to obtain the inverse matrix H^{-1} of the Hom-set matrix $H = (|\mathrm{Hom}(I, J)|)_{I,J \in \Gamma/\cong}$, we have to construct a poset like the subgroup lattice.
We may assume that all the morphisms of Γ are epimorphic. In this case, H is decomposed as $H = LD$, and so $H^{-1} = D^{-1}L^{-1}$, where

$$L = (|\mathrm{Hom}(I, J)|/|\mathrm{Aut}(J)|)_{I,J \in \Gamma},$$
$$D := (|\mathrm{Aut}(I)|\delta(I, J)) = \begin{cases} |\mathrm{Aut}(I)| & \text{if } I \cong J \\ 0 & \text{otherwise.} \end{cases}$$

$L_{I,J}$ is equal to the number of quotient objects of I isomorphic to J. When Γ is the category of set of size at most n, the number $L(I, J) = S(|I|, |J|)$ is the Stirling number of second kind and $L^{-1}(I, J) = s(|I|, |J|)$ is the Stirling number of first kind.
Now, in the case of $\mathbf{trans}^G$, the subgroup lattice is categorically constructed as follows:

$$\mathrm{Sub}(G) \approx \mathbf{Elts}(\mathbf{trans}^G, \mathrm{Hom}(G/1, -))$$
$$\cong (G/1)\backslash\mathbf{trans}^G.$$

Thus if th category Γ has a "generator" like $G/1$ using the notion of categories of elements (or coslice categories), we can construct a poset we need.

Definition 3.3 A functor $\pi : \widetilde{\Gamma} \to \Gamma$ is called a **discrete cofibration** (DCF) if

$$\begin{array}{ccc} \mathrm{Mor}(\widetilde{\Gamma}) & \xrightarrow{\mathrm{dom}} & \mathrm{Obj}(\widetilde{\Gamma}) \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle\pi} \\ \mathrm{Mor}(\Gamma) & \xrightarrow{\mathrm{dom}} & \mathrm{Obj}(\Gamma) \end{array}$$

is a fibre product diagram. See [Yo87]. More precisely, this means that for any $\widetilde{X} \in \widetilde{\Gamma}$, π induces an equivalence between slice categories:

$$\begin{aligned}\widetilde{X}\backslash\pi : \widetilde{X}\backslash\widetilde{\Gamma} &\xrightarrow{\cong} \pi(\widetilde{X})\backslash\Gamma \\ &; (\widetilde{X} \to \widetilde{Y}) \mapsto (\pi(\widetilde{X}) \to \pi(\widetilde{Y}))\end{aligned}$$

Note (1) DCF $\pi : \widetilde{\Gamma} \to \Gamma$ is faithful.
(2) Any functor which has a right adjoint is a DCF.

Example 3.4 Let G be a finite group. Let $\mathbf{trans}^G$ be the cat of transitive G-sets and $\mathrm{Sub}(G)$ the subgroup lattice viewed as a category. Then

$$\pi : \mathrm{Sub}(G) \to \mathbf{trans}^G; I \mapsto G/I$$

is a DCF. The bijection

$$I\backslash\pi : I\backslash\mathrm{Sub}(G) \xrightarrow{\cong} (G/I)\backslash\mathbf{trans}^G$$

is given by

$$\begin{aligned}&K(\supset I) \mapsto (G/I \to G/K; gI \mapsto gK),\\ &(G/I \xrightarrow{\alpha} X) \mapsto G_{\alpha(I)}(\supset I),\end{aligned}$$

where $G_{\alpha(I)}$ is the stabilizer of $\alpha(I) \in X$.

Let $\mathbf{sub}(G)$ be the subgroup category, which is equivalent to $\mathbf{trans}^G$ by $I \mapsto G/I$. Then $\mathrm{Sub}(G) \to \mathbf{sub}(G); I \mapsto I$ gives a DCF.

Note that the inverse matrix of the Hom-set matrix $\widetilde{H}$ of $\mathrm{Sub}(G)$ is given by the Möbius function:

$$\widetilde{H}^{-1} = (\mu(I,J))_{I,J \leq G}$$

3.5 The inverse of the Hom-set matrix

We continue assuming that the morphisms of Γ are all epimorphic. We consider the following conditions for a discrete cofibration $\pi : \widetilde{\Gamma} \to \Gamma$:

(S) $\pi : \widetilde{\Gamma}/\cong \longrightarrow \Gamma/\cong$ is surjective on objects.
(P) $\widetilde{\Gamma}/\cong$ is a poset, i.e., $|\mathrm{Hom}(\widetilde{X},\widetilde{Y})| \leq 1$ for any $\widetilde{X},\widetilde{Y} \in \widetilde{\Gamma}$.

For any $G \in \Gamma$, let $G\backslash\Gamma$ be the coslice category, which is equivalent to $\mathbf{Elts}(\mathrm{Hom}_\Gamma(G,-))$.

Example 3.5 (1) For any $G \in \Gamma$,

$$\pi_G : G\backslash\Gamma \to \Gamma; (G \xrightarrow{x} X) \mapsto X$$

is a DCF satisfying (P). It satisfies (S) if and only if any $X \in \Gamma$ has a morphism from G. Such a G exists uniquely up to isomorphism if it exists.
(2) Let $\mathbf{G}$ be a set of objects of Γ such that any $X \in \Gamma$ has a morphisms from some $G \in \mathbf{G}$. Then

$$\pi_{\mathbf{G}} := \coprod_{G\in\mathbf{G}} \pi_G : \mathbf{G}\backslash\Gamma := \coprod_{G\in\mathbf{G}} G\backslash\Gamma \to \Gamma$$

is a DCF satisfying (S) and (P).
(3) For finite group G, $\pi : \mathrm{Sub}(G) \to \mathbf{trans}^G; I \mapsto G/I$ and $\pi\mathrm{Sub}(G) \to \mathbf{sub}(G); I \mapsto I$ are both DCF satisfying (S) and (P).

Let $\pi : \widetilde{\Gamma} \to \Gamma$ be a DCF satisfying (S) and (P). Let μ be the Möbius function of the poset $\widetilde{\Gamma}/\cong$, which value at the isomorphism classes $[\widetilde{I}], [\widetilde{J}])$ is simply wrote as $\mu(\widetilde{I},\widetilde{J})$. For any $I \in \Gamma$, we define

$$\begin{aligned}N_I &:= \sharp\{[\widetilde{I}] \in \widetilde{\Gamma}/\cong \mid \pi(\widetilde{I}) \cong I\},\\ \mathrm{ind}(I) &:= N_I|\mathrm{Aut}(I)|\end{aligned}$$

Example 3.6 When

$$\pi : \widetilde{\Gamma} = \mathrm{Sub}(G) \to \Gamma = \mathbf{sub}(G); I \mapsto I,$$

we have

$$\begin{aligned}&N_I = \sharp\{\widetilde{I} \leq G \mid \widetilde{I} \sim_G I\} = (G : N_G(I)),\\ &\mathrm{Aut}(I) \cong N_G(I)/I,\end{aligned}$$

and so $\mathrm{ind}(I) = (G : I)$. ■

Theorem 3.3 The inverse of the Hom-set marix $H := (|\mathrm{Hom}(I,J)|)_{I,J\in\Gamma/\cong}$ of Γ is given by

$$H^{-1}_{IJ} = \frac{1}{\mathrm{ind}(I)} \sum'_{\pi(\widetilde{I})\cong I} \sum'_{\pi(\widetilde{J})\cong J} \mu(\widetilde{I},\widetilde{J})$$

Theorem 3.4 The primitive idempotent associated to $J \in \Gamma$ is given by

$$e_J = \sum'_{\widetilde{I}} \frac{1}{\mathrm{ind}(\pi(\widetilde{I}))} \sum'_{\pi(\widetilde{J})\cong J} \mu(\widetilde{I},\widetilde{J})\,[\pi(\widetilde{I})]$$

Theorem 3.5 Let $\theta \in \mathbb{Q}^{\Gamma}$. Then

$$\varphi^{-1}(\theta) = \sum_{\widetilde{I},\widetilde{J}}' \frac{\mu(\widetilde{I},\widetilde{J})\,\theta(\pi(\widetilde{J}))}{\mathrm{ind}(\pi(\widetilde{I}))}[\pi(\widetilde{I})] \in \mathbb{Q}\Gamma$$

4 Abstract monomial Burnside rings

Refer to [Dr71], [Sn88], [Sn94], [Ta10].

4.1 Definition of AMBR

Definition 4.1 As before, let Γ denote an essentially finite and locally finite category Let

$$\wedge : \Gamma^{\mathrm{op}} \to \mathbf{mon}; I \mapsto \widehat{I}$$

be a functor to the category of finite monoids Thus an $f : I \to J$ induces a monoid homomorphism $\widehat{f} : \widehat{J} \to \widehat{I}$, which we often extend to a ring homomorphism $\widehat{f} : \mathbb{Z}[\widehat{J}] \to \mathbb{Z}[\widehat{I}]$ between monoid rings. In particular, $\widehat{I}$ is a right $\mathrm{Aut}(I)$-set, and so $\mathrm{Aut}(I)$ acts the monoid algebra $\mathbb{Z}[\widehat{I}]$. We can consider the centralizer algebra $\mathbb{Z}[\widehat{I}]^{\mathrm{Aut}(I)}$ under this action. Then the **monomial ghost ring** is defined as the product algebra

$$\widetilde{\Omega}(\Gamma,\wedge) := \prod_{I\in\Gamma/\cong} \mathbb{Z}[\widehat{I}]^{\mathrm{Aut}(I)}$$

Let $\Omega(\Gamma,\wedge) := \mathbb{Z}[\mathbf{Elts}(\Gamma,\wedge)]$ be the free abelian group generated by $\mathbf{Elts}(\Gamma,\wedge)/\cong$. ■

Definition 4.2 The monomial Burnside homomorphism is the linear map defined by

$$\varphi : \Omega(\Gamma,\wedge) \to \widetilde{\Omega}(\Gamma,\wedge); [X,x] \mapsto \left(\sum_{f:I\to X} \widehat{f}(x)\right)_I$$

$\Omega(\Gamma,\wedge)$ is called an **abstract monomial Burnside ring** (AMBR) if

(a) $\Omega(\Gamma,\wedge)$ has a ring structure, and

(b) φ is an injective ring homomorphism. ■

Example 4.1 (1) Let G be a finite group. and $\Gamma = \mathbf{sub}(G)$, the subgroup category, and $\wedge : H \mapsto \widehat{H} := \mathrm{Hom}(H,\mathbb{C}^*)$. the linear character functor. Then as the AMBR, we have a classical monomial Burnside ring $\Omega(G,\wedge)$ which is an abelian group generated by the symbols $[H,\lambda]$, where $H \le G$ and $\lambda \in \widehat{H}$, a linear character, and with relation $[H^g,\lambda^g] = [H,\lambda]$. The multiplication is defined by

$$[H,\lambda]\cdot[K,\mu] = \sum_{HgK} [H^g \cap K, \lambda^g{}_{H^g\cap K}\cdot \mu_{H^g\cap K}]$$

There is a ring homomorphism into the character ring:

$$\Omega(G,\wedge) \to R(G); [H,\lambda] \mapsto \mathrm{ind}^G(\lambda)$$

(2) Let G be a finite group and S a monoid with right G-action. Take the centralizer functor $C_S : \mathbf{sub}(G) \to \mathbf{mon}; H \mapsto C_S(H)$. Then the AMBR $\Omega(\mathbf{sub}(G), C_S)$ is the crossed Burnside ring $\Omega(G,S)$., In general, this ring is not commutative, but when $S = G^c$, the group G with G-action by G-conjugation, $\Omega(G,G^c)$ is commutative.

(3) Let A be a finite abelian group with G-action. Then $\Omega(\mathbf{sub}(G), H^1(-,A)) = \Omega(G,A)$ is the Dress monomial BR.

4.2 The fundamental theorems for AMBR

As before, we assume that Γ satisfies the following two conditions:

(E) All the morphisms of Γ are epimorphic.

(C) For any object I and $\sigma \in \mathrm{Aut}(I)$, there exists a coequalizer diagram:

$$I \underset{\sigma}{\overset{1}{\rightrightarrows}} I \xrightarrow{c_\sigma} I/\sigma$$

By (C), we have a monoid homomorphism

$$\widehat{c_\sigma} : \widehat{I/\sigma} \to \widehat{I}^{\langle\sigma\rangle} := \{i \in \widehat{I} \mid \widehat{\sigma}(i) = i\}$$

Furthermore, if $f : I \to X$ satisfies $f \circ \sigma = f$, then there exists a unique $g : I/\sigma \to X$ such that $g \circ c_\sigma = f$, and so $\widehat{c_\sigma} \circ \widehat{g} = \widehat{f}$.

By (C), the coequalizer $c_\sigma : I \to I/\sigma$ of $1, \sigma \in \mathrm{Aut}(I)$ induces a monoid homomorphism $\widehat{c}_\sigma : \widehat{I/\sigma} \to \widehat{I}^{\langle\sigma\rangle} \hookrightarrow \widehat{I}$, which furthermore induces

$$\widehat{c}_\sigma : \mathbb{Z}[\widehat{I/\sigma}] \to \mathbb{Z}[\widehat{I}^{\langle\sigma\rangle}] \to \mathbb{Z}[\widehat{I}]$$

Define the **group of obstructions** by

$$\mathrm{Obs}(\Gamma, \wedge) := \prod_{I \in \Gamma/\cong} ((\mathbb{Z}/|\mathrm{Aut}(I)|)[\widehat{I}])^{\mathrm{Aut}(I)}$$

and define a module endmorphism $\widetilde{\psi} = (\widetilde{\psi}_I)$ of $\widetilde{\Omega}(\Gamma, \wedge)$ by

$$\widetilde{\psi}_I(\theta) := \sum_{\sigma \in \mathrm{Aut}(I)} \widehat{c}_\sigma \theta(I/\sigma).$$

Finally define the **Cauchy-Frobenius map** by

$$\psi : \widetilde{\Omega}(\Gamma, \wedge) \xrightarrow{\widetilde{\psi}} \widetilde{\Omega}(\Gamma, \wedge) \xrightarrow{\mathrm{pr}} \mathrm{Obs}(\Gamma, \wedge).$$

Theorem 4.1 The following is an exact sequence of modules:

$$0 \to \Omega(\Gamma, \wedge) \xrightarrow{\varphi} \widetilde{\Omega}(\Gamma, \wedge) \xrightarrow{\psi} \mathrm{Obs}(\Gamma, \wedge) \to 0$$

Theorem 4.2 $\Omega(\Gamma, \wedge)$ is an AMBR.

4.3 Monomial G-sets

It is often more convenient to use the notion of monomial G-set than of $\mathbf{sub}(G)$. The category of monomial G-sets is equivalent to $\mathbf{Exp}(\mathbf{Elts}(\mathbf{sub}(G)))$.

Then any functor $\wedge : \mathbf{sub}(G)^{\mathrm{op}} \to \mathbf{mon}$ can be extend to $\mathbf{set}^G$. In fact, the monoid $\widehat{X}$ for any G-set X is defined by the set of X-indexed family $(\lambda_x)_{x \in X}$ such that $\lambda_x \in \widehat{G_x}$ and λ_{gx} ${}^g\lambda_x$ for any $x \in X$ and $g \in G$.

Then the AMBR $\Omega(\mathbf{sub}(G), \wedge)$ is isomorphic to the Grothendieck ring of monomial G-sets with respect to disjoint union and multiplication defined by

$$(X, (\lambda_x)) \otimes (Y, (\mu_y)) = (X \times Y, (\lambda_x \downarrow_{G_{xy}} \cdot \mu_y \downarrow_{G_{xy}})_{(x,y)})$$

In this notation, the monomial Burnside homomorphism $\varphi = (\varphi_I)$ $(I \leq G)$ is given by

$$\varphi_I : [X, (\lambda_x)] \mapsto \sum_{x \in X^I} \lambda_{x|I} \in (\mathbb{Z}[\widehat{I}])^{N_G(I)}$$

4.4 Idempotent formula

Theorem 4.3 (Takegahara) The primitive idempotent of the complex coefficient MBR $\mathbb{C}\Omega(G, \wedge)$ associated to (H, t) is given by

$$\begin{aligned} e_{H,t} &= \frac{1}{|N_G(H)| \cdot |H|} \sum_{D \leq H} \sum_{\lambda \in \widehat{H}} |D| \mu(D, H) \overline{\lambda(t)} [D, \lambda_{|D}] \\ &= \epsilon_t \otimes e_H, \end{aligned}$$

where

$$\epsilon_t := \frac{1}{|H|} \sum_{\lambda \in \widehat{H}} \overline{\lambda(t)} \lambda$$

is the primitive idempotent of the complex coefficient character ring $\mathbb{C}R(H)$ associated to $t \in H$, and

$$e_H := \frac{1}{|N_G(H)|} \sum_{D \leq H} |D| \mu(D, H)[D],$$

is the primitive idempotent of the Burnside ring $\mathbb{C} \otimes \Omega(G)$. Furthermore, we used the notation $\lambda \otimes [D] := [D, \lambda_{|D}]$.

Corollary 4.4 (Snaith, Boltje) Explicit Brauer induction theorem!

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