Clifford theory of characters in Brauer induction

This is joint work with Britta Späth [4]. In representation theory of finite groups Clifford theory plays a very important role. Here we shall discuss extendibility of ordinary characters of a normal subgroup $N$ of a finite group $G$, by using a subgroup $G[b]$ which is a normal subgroup of $G_b$, where $b$ is a $p$-block of $N$, $G_b$ is the set of all elements in $G$ stabilizing $b$ by the conjugation action, and $p$ is a prime number. The group $G[b]$ is defined by E.C. Dade in his very distinguished paper [1] of early 1970's. Actually $G[b]$ has remarkably nice properties.

The notation used here in this small note is standard. Throughout this note we assume that $G$ is a finite group, $N$ is its normal subgroup, and $b$ is a $p$-block of $N$. We denote by Irr$(N)$ and IBr$(N)$, respectively, the set of all irreducible ordinary and Brauer characters of $N$. Then, we denote by Irr$(b)$ and IBr$(b)$, respectively, those characters belonging to $b$. For a subgroup $H$ of $G$ and a $p$-block $B'$ of $H$, $(B')^G$ means the block induction of $B'$ to $G$ if it is defined. A triple $(\mathcal{K}, \mathcal{O}, k)$ is so-called a $p$-modular system, which is big enough for all finitely many finite groups which we are looking at, including $G$. Namely, $\mathcal{O}$ is a complete discrete valuation ring, $\mathcal{K}$ is the quotient field of $\mathcal{O}$, $\mathcal{K}$ and $\mathcal{O}$ have characteristic zero, and $k$ is the residue field $\mathcal{O}/\text{rad}(\mathcal{O})$ of $\mathcal{O}$ such that $k$ has characteristic $p$. We mean by "big enough" above that $\mathcal{K}$ and $k$ are both splitting fields for the finite groups mentioned above. We denote by $1_b$ the block idempotent of $b$ which is a block algebra of $kN$ (sometimes of $\mathcal{O}N$). We write Bl$(G)$ and Bl$(G|b)$ for the set of all $p$-blocks of $G$ and for the set of all $p$-blocks of $G$ covering $b$,
respectively. When \( \chi \in \text{Irr}(N) \) and \( \phi \in \text{IBr}(N) \), we denote by \( \text{bl}(\chi) \) and \( \text{bl}(\phi) \), respectively, the \( p \)-block of \( N \) to which \( \chi \) and \( \phi \) belong. For \( \phi \in \text{IBr}(N) \), we denote by \( \text{IBr}(G|\phi) \) the set of all characters \( \psi \in \text{IBr}(G) \) such that \( \phi \) is an irreducible constituent of \( \psi \downarrow_N \), see [8, p.155]. For the notation and terminology we shall not explain precisely, see the books of [9].

Let us keep the notation \( G, N \) and \( b \) as above throughout. Then, the group \( G[b] \) is defined by [1] as follows:

\[
G[b] := \{ g \in G \mid (1_bC_g^{-1})(1_bC_g) = 1_bC_1 \}
\]

where \( C_g := C_{OG}(N) \cap ONg \subseteq OG \) for each \( g \in G \). For a \( p \)-block \( B \) of \( G \) we denote by \( \lambda_B \) the central function (central character) \( \lambda_B : Z(kG) \rightarrow k \) associated to \( B \), see [8, p.48]. When \( g \in G \), we denote by \( \text{cc}_G(g) \) the conjugacy class of \( G \) which contains \( g \), and we define \( (\text{cc}_G(g))^+ := \sum_{g \in \text{cc}_G(g)} g \in kG \). Then, we have had several characterizations of \( G[b] \). Namely,

**Proposition.** We have the following three kinds of characterizations of the group \( G[b] \).

(i) (see [5]) \( G[b] = \{ g \in G_b \mid \exists u_g \in b^\times \text{ such that } g^{-1}\beta g = u_g^{-1}\beta u_g \text{ for any } \beta \in b \} \)

(ii) (see [3]) \( G[b] = \{ g \in G_b \mid b \otimes_O g \cong b \text{ as } O[N \times N]-\text{modules} \} \).

(iii) (see [6]) \( G[b] = \{ g \in G_b \mid \exists y \in gN, \exists B' \in \text{Bl}(\langle N, g \rangle) \text{ such that } \lambda_{B'}((\text{cc}_{\langle N, g \rangle}(y))^+) \neq 0 \} \).

The following three theorems are our main results in this note.

First, we obtain a sort of generalization of the Theorem of Harris-Knörr [2].

**Theorem A.** Let \( G \) be a finite group, and let \( N < G, H \leq G \) and \( M := N \cap H \). Let \( b' \in \text{Bl}(M) \) be a block of \( M \) that has a defect group \( D \) with \( C_G(D) \subseteq H \). For \( b := (b')^N \) the map from \( \text{Bl}(H \mid b') \) to \( \text{Bl}(G \mid b) \) given by \( B' \mapsto (B')^G \) is well-defined and surjective.

**Remark.** There is an exmaple where the above map in **Theorem A** is not injective, see [4].
**Theorem B.** Let $b'$ be a block of $M$ that has a defect group $D$ with $C_G(D) \subseteq H$. Assume further that $G = G[b]$ for $b : = (b')^N$. Then for every $\phi \in \text{IBr}(b)$ and every $\phi' \in \text{IBr}(b')$ there is a bijection

$$\Lambda : \text{IBr}(G | \phi) \rightarrow \text{IBr}(H | \phi'),$$

such that $\text{bl}(\Lambda(\rho)) = \text{bl}(\rho)$ for every $\rho \in \text{IBr}(G | \phi)$. Further $\rho \in \text{IBr}(G)$ is an extension of $\phi$ if and only if $\Lambda(\rho)$ is an extension of $\phi'$.

**Theorem C.** Let $G$ be a finite group, and let $N \leq G$, $H \leq G$ and $M := N \cap H$. Let $b' \in \text{Bl}(M)$ be a block of $M$ with defect group $D$ such that $C_G(D) \subseteq H$, and let $b := (b')^N$. Assume further that $G = G[b]$.

(i) **(Ordinary characters)**

1. If $\chi' \in \text{Irr}(b')$ extends to a character $\tilde{\chi}' \in \text{Irr}(H)$, then there exists a character $\chi \in \text{Irr}(b)$ of height zero which extends to a character $\tilde{\chi} \in \text{Irr}(G)$ and which satisfies

   $$\text{bl}((\tilde{\chi})_{J \cap H})^J = \text{bl}(\chi_{J})$$

   for every $J$ with $N \leq J \leq G$.

2. If $\chi \in \text{Irr}(b)$ extends to a character $\tilde{\chi} \in \text{Irr}(G)$, then there exists a character $\chi' \in \text{Irr}(b')$ of height zero which extends to a character $\tilde{\chi}' \in \text{Irr}(H)$ and which satisfies $(\ast)$.

(ii) **(Sylow $p$-subgroups)**

1. If $\chi' \in \text{Irr}(b')$ extends to a character $\tilde{\chi}' \in \text{Irr}(H)$ and if $\chi \in \text{Irr}(b)$ extends to a subgroup $J_0$ of $G$ with $N \leq J_0 \leq G$ and $J_0/N \in \text{Syl}_p(G/N)$, then $\chi$ extends to a character $\tilde{\chi} \in \text{Irr}(G)$ which satisfies $(\ast)$.

2. If $\chi \in \text{Irr}(b)$ extends to a character $\tilde{\chi} \in \text{Irr}(G)$ and if $\chi' \in \text{Irr}(b')$ extends to $J_0 \cap H$ for a subgroup $J_0$ of $G$ with $N \leq J_0 \leq G$ and $J_0/N \in \text{Syl}_p(G/N)$, then $\chi'$ extends to a character $\tilde{\chi}' \in \text{Irr}(H)$ which satisfies $(\ast)$.

(iii) **(Brauer characters)**

1. If $\phi' \in \text{IBr}(b')$ extends to a character $\tilde{\phi}' \in \text{IBr}(H)$, then any $\phi \in \text{IBr}(b)$ extends to a character $\tilde{\phi} \in \text{IBr}(G)$ which satisfies

   $$\text{bl}((\tilde{\phi}')_{J \cap H})^J = \text{bl}(\phi_{J})$$

   for every $J$ with $N \leq J \leq G$. 

(2) If $\phi \in \text{IBr}(b)$ extends to a character $\tilde{\phi} \in \text{IBr}(G)$, then any $\phi' \in \text{IBr}(b')$ extends to a character $\tilde{\phi}' \in \text{IBr}(H)$ which satisfies (**).

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**References**