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READING ENDO-TRIVIAL MODULES FROM THE BRAUER TREE

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ABSTRACT. This note is a survey of a collaboration with G. Malle and E. Schulte [18] and reports on recent results towards a classification of simple endo-trivial modules for finite quasi-simple groups, as well as the use of character theoretic methods for endo-trivial modules. This is the summary of a talk held at the RIMS Symposium *Research on Finite Groups and their Representations, Vertex Operator Algebras, and Algebraic Combinatorics*, March 2014.

1. INTRODUCTION

Throughout, unless otherwise stated, we let p be a prime number, k be an algebraically closed field of characteristic $p > 0$, G be a finite group such that $p \mid |G|$, and let $\mathsf{mod}(kG)$ denote the category of finitely generated left kG-modules. We let $(K, \mathcal{O}, k)$ be a splitting p-modular system for G and its subgroups.

A kG-module $M \in \mathsf{mod}(kG)$ is termed *endo-trivial* if the k-endomorphism ring of M satisfies the condition $\mathrm{End}_k(M) \cong M^* \otimes_k M \cong k \oplus (\mathrm{proj})$ as kG-modules and where (proj) denotes a projective summand. (In other words, $\mathrm{End}_k(M)$ is trivial is the stable module category.)

Basic Facts 1.1. *Let $M \in \mathsf{mod}(kG)$ be an endo-trivial kG-module. Then:*

(a) $\dim_k(M) \equiv \pm 1 \pmod{|G|_p}$ *if* $p > 2$ *and* $\dim_k(M) \equiv \pm 1 \pmod{\frac{1}{2}|G|_p}$ *if* $p = 2$;

(b) $M \cong M_0 \oplus (\mathrm{proj})$, *where M_0 is the unique indecomposable endo-trivial direct summand of M;*

(c) if $H \leq G$, then $M|_H$ is endo-trivial, and if moreover H contains a Sylow p-subgroup of G, then M is endo-trivial if and only if $M|_H$ is endo-trivial.

Thanks to Basic Fact (b), the set $T(G)$ of isomorphism classes of indecomposable endo-trivial modules becomes an abelian group for the law: $[M] + [N] := [(M \otimes_k N)_0]$. The zero element is $[k]$ and $-[M] = [M^*]$.

1.1. Simple endo-trivial modules.

A recent result of Robinson's [23] states that whenever the Sylow p-subgroups of a finite group G are neither cyclic nor quaternion, then any simple endo-trivial kG-module is either simple endo-trivial for a quasi-simple normal subgroup, or induced from a 1-dimensional module of a strongly p-embedded subgroup of G. This leads to the natural problem of classifying simple endo-trivial modules for quasi-simple groups treated hereafter. In fact the existence of such modules seems to restrict even further the structure of the group.

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Conjecture 1.2. *Let G be a finite quasi-simple group with a faithful simple endo-trivial module. Then the Sylow p-subgroups of G have p-rank at most 2.*

As a matter of fact in all examples described below, the Sylow p-subgroups are either homocyclic of rank at most 2, extraspecial of order p^3 with $p \leq 11$, or dihedral. As a consequence of our classifications we obtain:

Theorem 1.3. *Conjecture 1.2 holds in all of the following cases:*
(a) *if $p = 2$,*
(b) *if $G/Z(G)$ is an alternating group,*
(c) *if $G/Z(G)$ is a sporadic group,*
(d) *if $G/Z(G)$ is a group of Lie type and p is its defining characteristic, and*
(e) *if $G/Z(G)$ is an exceptional group of Lie type.*

Remark 1.4. Traditionally endo-trivial modules have been studied through the description of the structure of the group $T(G)$, and, a priori, the classification problem of endo-trivial modules is equivalent to describing the structure of the group $T(G)$. This structure has been computed by Carlson, Mazza, Nakano, Thévenaz and coauthors for several classes of finite groups, such as for examle p-groups, p-soluble groups, symmetric and alternating groups, groups of Lie type in defining characteristic. Nonetheless, we emphasise that the knowledge of the structure of the group $T(G)$ often relies on Green correspondence, hence does not give an explicit description of indecomposable endo-trivial modules in general. This is the main reason that prevented us from using previously known results on endo-trivial modules in the forthcoming classifications. See Section 5.

2. Character theory for endo-trivial modules

The new feature of the approach used in [18, 17, 19] is that it mainly relies on character theoretic methods. This is made possible through the following generalisation of a lifting result due to Alperin [1] from the case of p-groups to arbitrary finite groups, which may be of independent interest:

Theorem 2.1 ([18, Thm. 1.3]). *Let $(K, \mathcal{O}, k)$ be a splitting p-modular system and let V be an endo-trivial kG-module. Then V lifts to an endo-trivial $\mathcal{O}G$-lattice.*

This leads to the following criterion to detect endo-trivial modules from their ordinary characters:

Lemma 2.2. *Let V be a kG-module which is liftable to a simple $\mathbb{C}G$-module with character χ, say. If V is endo-trivial, then $|\chi(g)| = 1$ for all p-singular elements $g \in G$.*

Proof. By assumption $\chi\bar{\chi} \equiv 1 + \psi \pmod{p}$, where ψ is the character of a p-projective module. Thus, $\psi(g) = 0$ for all p-singular elements $g \in G$. The claim follows. □

Now it is well-known that a trivial source kG-module M lifts uniquely to a trivial source $\mathcal{O}G$-lattice $\hat{M}$. Denote by $\chi_{\hat{M}}$ the corresponding ordinary character. So that trivial source endo-trivial modules can be detected from their characters as follows.

Theorem 2.3 ([17], Thm. 2.2). *Let M be an indecomposable trivial source kG-module (with dimension prime to p). Then M is endo-trivial if and only if $\chi_{\hat{M}}(x) = 1$ for all non-trivial p-elements $x \in G$.*

In fact this theorem ensues from a result of Green, Landrock and Scott, stating that the character $\chi_{\hat{M}}$ of a trivial source kG-module takes non-negative integer values on p-elements $x \in G$, which are moreover positive if and only if the element x belongs to a vertex of M.

Remark 2.4. When the normal p-rank of the group G is greater than 1, then the class of an indecomposable endo-trivial kG-module lies in the torsion subgroup $TT(G)$ if and only if M is a trivial source module. Thus assuming the Sylow p-subgroups of G are neither cyclic, nor semi-dihedral, nor generalised quaternion, then Theorem 2.3 tells us the group $TT(G)$ is a function of the character table of the group G.

For groups of Lie type, more character-theoretic criteria for endo-trivial modules can be found in [18, Sec. 6].

3. Cyclic blocks

We start by describing simple endo-trivial modules lying in blocks of cyclic defect. The following result applies to arbitrary finite groups, not only quasi-simple groups.

Let G be a finite group with a non-trivial cyclic Sylow p-subgroup $P \cong C_{p^n}$, $n \in \mathbb{N}$. Let Z denote the unique subgroup of order p of G and let $H := N_G(Z)$. For a p-block $\mathbf{B}$ of kG, let $e_{\mathbf{B}}$ denote its inertial index, and let $e = |N_G(P) : C_G(P)|$ denote the inertial index of the principal block. Call an edge of the Brauer tree $\sigma(\mathbf{B})$ of $\mathbf{B}$ a *leaf* if it is an end edge, and call a leaf *non-exceptional* if the exceptional vertex does not sit at its end.

Theorem 3.1 ([18, Lem. 3.2, Cor. 3.3, Thm 3.7]). *Let G be as above such that $e > 1$. Let $\mathbf{B}$ be a p-block of kG containing an endo-trivial kG-module. Then:*

(a) $e_B = e$;

(b) *the number of p-blocks of kG containing (simple) endo-trivial modules is exactly* $\frac{1}{e}|H/[H:H]|_{p'}$;

(c) *a simple $k\mathbf{B}$-module S is endo-trivial if and only if S labels a non-exceptional leaf of the Brauer tree $\sigma(\mathbf{B})$ of $\mathbf{B}$.*

Proof (Sketch). Part (a) follows by Bessenrodt's theorem [2, Thm. 2.3] on the position of endo-trivial modules in the Auslander-Reiten quiver of kG: they lie at the end. Part (b) is a consequence of (a) together with Mazza and Thévenaz' theorem [21, Thm. 3.2] on the structure of the group $T(G)$ for groups G with cyclic Sylow p-subgroups. For part (c), let S be a simple $k\mathbf{B}$-module and let $f(S)$ denote its kH-Green correspondent. First, again by Bessenrodt's theorem [2, Thm. 2.3], S is endo-trivial if and only if S lies at the end of the stable Auslander-Reiten quiver $\Gamma_s(\mathbf{B}) \cong (\mathbb{Z}/e\mathbb{Z})A_{p^n-1}$ of $\mathbf{B}$. But this happens if and only if $f(S)$ lies at the end of the stable Auslander-Reiten quiver $\Gamma_s(\mathbf{b}) \cong (\mathbb{Z}/e\mathbb{Z})A_{p^n-1}$ of the kH-Brauer correspondent $\mathbf{b}$ of $\mathbf{B}$ if and only if the length of $f(S)$ belongs to $\{1, p^n-1\}$ if and only if S corresponds to a non-exceptional leaf of the Brauer tree of $\mathbf{B}$, where the last equivalence follows from a result of Hiß and Lux [13, Lem. 4.4.12]. □

In the case of cyclic Sylow p-subgroups, the group $T(G)$ is finite. Obviously the subgroup $T_0(G) \leq T(G)$ consisting of the indecomposable endo-trivial modules lying in the principal block is generated by the first syzygy $[\Omega(k)]$ and has order $2e$. Then, picking a simple endo-trivial module in each non-principal block, and tensoring these simple modules with $\Omega(k)$ allows us to recover all indecomposable endo-trivial modules. This yields:

Corollary 3.2. *Let G be a finite group with a non-trivial cyclic Sylow p-subgroup, and assume $e \neq 1$. Then the group of endo-trivial modules $T(G)$ is generated by the classes of syzygy modules of simple kG-modules.*

4. Simple endo-trivial modules for quasi-simple groups.

We now go through several families of quasi-simple groups and give a classification of simple endo-trivial modules in these cases, proving Theorem 1.3 as a by-product.

4.1. **Covering groups of alternating groups.** We start with covering groups of alternating groups, for which the results rely on intrinsic combinatorics, the Murnagham-Nakayam rule and results of Erdmann [9] and Henke [11].

Theorem 4.1 ([18, Thm. 4.9]). *Let V be a faithful simple kG-module for some covering group G of $\mathfrak{A}_n$ with $n \geq \max\{p, 5\}$. Then V is endo-trivial if and only if V is a constituent of the simple module for the corresponding covering group of $\mathfrak{S}_n$ indexed by $\lambda \vdash n$, where one of:*

(1) $G = \mathfrak{A}_n$, $5 \leq p+2 \leq n < 2p$ *and* $\lambda = (p+1, 1^{n-p-1})$ *(cyclic defect);*
(2) $G = \mathfrak{A}_n$, $p > 2$, $n = 2p+1$ *and* $\lambda = (p+1, 1^p)$;
(3) $G = \mathfrak{A}_n$, $p > 2$, $n = 3p-1$ *and* $\lambda = (2p-1, p)$;
(4) $n = 6, 7$, $|Z(G)| \geq 3$ *and* (G, p, V) *are as in Table 1;*
(5) $G = \tilde{\mathfrak{A}}_5 \cong \mathrm{SL}_2(5)$, $p = 3$ *and* $\dim_k V = 2$; *or*
(6) $G = \tilde{\mathfrak{A}}_n$, $5 \leq p \leq n \leq p+3$ *and* λ *is as follows (cyclic defect):*
 (i) $\lambda = ((p+1)/2, (p-1)/2)$ *when* $n = p$;
 (ii) $\lambda = (p+1)$ *or* $\lambda = ((p+1)/2, (p-1)/2, 1)$ *when* $n = p+1$;
 (iii) $\lambda = (p+2)$ *(two non-isomorphic modules) and, for* $p > 5$, $\lambda = ((p+1)/2, (p-1)/2, 2)$ *(two non-isomorphic modules) when* $n = p+2$;
 (iv) $\lambda = (p+2, 1)$ *(two non-isomorphic modules) and, for* $p > 5$, $\lambda = ((p+1)/2, (p-1)/2, 2, 1)$ *(two non-isomorphic modules) when* $n = p+3$.

TABLE 1. Faithful cyclic blocks containing a simple endo-trivial module V in $3.\mathfrak{A}_6$, $3.\mathfrak{A}_7$, $6.\mathfrak{A}_6$ and $6.\mathfrak{A}_7$

G	p	$X(H)$	$X(H)/e$	block	$\dim V$
$3.\mathfrak{A}_6$	2	–	–	4, 5	3, 3, 9
$3.\mathfrak{A}_6$	5	6	3	5, 6	6
$6.\mathfrak{A}_6$	5	12	6	12, 13	6, 6
$3.\mathfrak{A}_7$	5	12	3	6, 7	6, 21
$6.\mathfrak{A}_7$	5	24	6	15, 16	6, 6, 24
$3.\mathfrak{A}_7$	7	9	3	6, 7	6, 15
$6.\mathfrak{A}_7$	7	18	6	15, 16	6, 6

4.2. **Groups of Lie type in defining characteristic.** For groups of Lie type in their defining characteristic, simple endo-trivial modules are also extremely rare.

Theorem 4.2 ([18, Thm. 5.2]). *Let G be a finite quasi-simple group of Lie type in characteristic $p > 0$. Let V be a simple faithful kG-module, where k is algebraically closed of characteristic p. Then V is endo-trivial if and only if one of*

(1) $p \geq 5$, $G = \mathrm{SL}_2(p)$ *and* $\dim V = p-1$; *or*
(2) $p = 2$, $G = \mathrm{SL}_3(2)$ *and* $\dim V = 3$.

If $G = \mathrm{SL}_2(p)$ and $\dim_k V = p-1$, then for $P \in \mathrm{Syl}_p(G)$ (cyclic), $V|_P$ is indecomposable, and up to isomorphism there is a unique indecomposable kP-module of dimension $p-1$, namely $\Omega(k)$. Therefore we must have $V|_P \cong \Omega(k)$, which is endo-trivial, whence so is V. For $G = \mathrm{SL}_3(2) \cong \mathrm{L}_2(7)$ see the next subsection.

Otherwise, we may assume that V is a non-trivial simple kH-module, where H is a group of simply connected type such that $G = H/Z$ for some central subgroup $Z \leq H$. By Steinberg's tensor product theorem the simple kH-modules are tensor products of Frobenius twists of p-restricted highest weight modules, and it is easy to prove that a tensor product of two modules is endo-trivial if and only if both factors are. Now twists of the Steinberg module have dimension divisible by p, hence are not endo-trivial. All other p-restricted highest weight modules have dimension less than $|G|_p - 1$, unless we are in one of the exceptional case, so that they cannot be endo-trivial by the dimensional criteria.

4.3. **The groups** $\mathrm{SL}_2(q)$ **and** $\mathrm{L}_2(q)$ **in cross-characteristic.** For $G = \mathrm{SL}_2(q)$, $q = p^n$, p a prime, k of characteristic $\ell \neq p$ dividing the order of G. If $\ell \neq 2$ the classification of simple endo-trivial modules follows from Theorem 3.1.

Proposition 4.3 ([18, Prop. 3.8]). *Let $G = \mathrm{SL}_2(q)$, $q = p^n$ with p a prime. Let V be a non-trivial simple kG-module, where k is algebraically closed of characteristic $\ell \neq p$. Then V is endo-trivial if and only if one of:*

(1) $2 \neq \ell \mid q-1$ *and V lies in an ℓ-block of full defect and inertial index 2 (cyclic defect);*
(2) $p \neq 2 \neq \ell \mid q+1$ *and V lies in the non-principal ℓ-block of full defect and inertial index 2 (cyclic defect);*
(3) $3 = \ell \mid q+1$, $|G|_\ell = 3$ *and V lies in the principal ℓ-block (cyclic defect).*

Moreover, if $\ell = 2$, $q \equiv -1 \pmod 4$ and V lies in the principal ℓ-block, then V is endo-trivial as a $k\mathrm{L}_2(q)$-module, but not as a kG-module.

For the exceptional covering groups of $\mathrm{L}_2(9) \cong \mathfrak{A}_6$ see Theorem 4.1. For the principal block modules of $\mathrm{L}_2(q)$, it is well-known that they are trivial source modules and easy to check that the corresponding characters take value 1 on 2-elements, hence they are endo-trivial by Theorem 2.3. Clearly these modules are not endo-trivial as $k\mathrm{SL}_2(q)$-modules since $O_{2'}(G) = Z(G)$ acts trivially.

4.4. **Characteristic 2.**

Theorem 4.4 ([18, Thm. 6.7]). *Let G be a finite quasi-simple group. Then G has a non-trivial simple endo-trivial kG-module V over a field of characteristic 2 if and only if one of:*

(a) $G = \mathrm{L}_2(q)$ *with* $7 \leq q \equiv 3 \pmod 4$ *and* $\dim(V) = (q-1)/2$; *or*
(b) $G = 3.\mathfrak{A}_6$ *and* $\dim(V) \in \{3, 9\}$.

In particular, Conjecture 1.2 holds for the prime 2.

The proof of this result goes through the possibilities for G according to the classification of finite simple groups. The exceptions in the statement of the Theorem are given by Theorem 4.3 and 4.1. All other irreducible characters of finite quasi-simple groups are discarded as follows. If G is alternating, the claim is Theorem 4.1, and if G is sporadic it is Theorem 4.8 below. If G is of Lie type in defining characteristic see Theorem 4.2. If G is an exceptional group of Lie type in odd characteristic, the claim will follow from Section 4.5. Thus it remains to deal with classical groups of Lie type in odd characteristic. In this case G has wild representation type, so that the group $T(G)$ has no 2-torsion and it follows that no self-dual simple kG-module can be endo-trivial. If $\chi \neq 1$ is a unipotent character of G whose reduction modulo 2 is irreducible, then V is self-dual. Thus only non-unipotent characters could lead to endo-trivial modules. In all types these characters can be discarded by a closer examination of the Lusztig's series in which they lie.

4.5. **Exceptional groups of Lie type.** Throughout this subsection we assume the groups of Lie type are defined over a field of characteristic p and we assume the field k is of characteristic $\ell \neq p$ dividing the order of the considered group.

To begin with, for the five families of small rank exceptional groups of Lie type, that is, ${}^2B_2(2^{2f+1})$, ${}^2G_2(3^{2f+1})$, ${}^2F_4(2^{2f+1})$, $G_2(q)$ and ${}^3D_4(q)$, complete ordinary character tables are available, making it relatively easy to find the candidate characters for simple endo-trivial modules.

Theorem 4.5 ([18, Thm. 6.8]). *Let G be a covering group of one of the simple groups ${}^2B_2(2^{2f+1})$ (with $f \geq 1$), ${}^2G_2(3^{2f+1})$ (with $f \geq 1$), $G_2(q)$ (with $q \geq 3$), ${}^3D_4(q)$, or ${}^2F_4(2^{2f+1})$ (with $f \geq 1$). Let $\ell \neq p$ denote a prime divisor of $|G|$ and $P \in \mathrm{Syl}_\ell(G)$. If there exists a non-trivial simple endo-trivial kG-module then P is cyclic.*

In the cases of the previous theorem, the simple endo-trivial kG-modules for primes ℓ such that P is cyclic are classified in [18] in Table 2 and Table 3. Again they can be read from the known Brauer trees.

Turning to the exceptional groups of rank at least four (for which no complete generic character tables are available) for unipotent characters we obtain the following:

Proposition 4.6 ([18, Prop. 6.9], [17]). *Let $G = G(q)$ be a finite simple exceptional group of Lie type in characteristic p of rank at least 4 and $\ell \neq p$ a prime for which the Sylow ℓ-subgroups of G are non-cyclic. Then $\chi \in \mathrm{Irr}(G)$ is the character of a simple unipotent endo-trivial kG-module if and only if $G = F_4(2)$, $\ell = 5$ and $\chi = F_4^{II}[1]$ (notation is that of [7, §13]).*

In this case most unipotent characters are discarded by the usual degree criteria or Theorem 2.3. This leads to:

Theorem 4.7 ([18, Prop. 6.9], [17]). *Let G be a quasi-simple exceptional group of Lie type in characteristic p, and $\ell \neq p$ a prime such that Sylow ℓ-subgroups of G have ℓ-rank at least 3. Then G does not have faithful simple endo-trivial kG-modules over a field k of characteristic ℓ.*

4.6. **Covering groups of sporadic simple groups.** For sporadic groups simple endo-trivial modules remain a rare phenomenon, although the following results shows that they do occur for groups whose Sylow p-subgroups are either elementary abelian of rank 2 or extra-special of order p^3 and exponent p.

Theorem 4.8 ([18, Thm. 7.1], [17]). *Let G be a quasi-simple group such that $G/Z(G)$ is sporadic simple. Let V be a faithful simple endo-trivial kG-module, where k is algebraically closed of characteristic p, with p dividing $|G|$. Let P be a Sylow p-subgroup of G. Then one of the following holds:*

(1) *$|P| = p$ and V lies in a p-block* $\mathbf{B}$ *of kG as listed in* [18, Table 7]*; or*
(2) *$(G, P, \dim V)$ are as in Table 2.*

Conversely, all modules listed in Table 2 are endo-trivial except possibly for those marked by a "?" in the last column.

TABLE 2. Candidate characters in sporadic groups

G	P	$\chi(1)$	G	P	$\chi(1)$	
M_{11}	3^2	$10, 10, 10$	Fi_{22}	5^2	1001	
M_{22}	3^2	55	$2.Fi_{22}$	5^2	5824 $(4\times)$	?
$2.M_{22}$	3^2	$10, 10, 154, 154$	$3.Fi_{22}$	5^2	$351, 351$	
M_{23}	3^2	253	$3.Fi_{22}$	5^2	12474 $(4\times)$	?
HS	3^2	$154, 154, 154$	$6.Fi_{22}$	5^2	61776 $(4\times)$	
$3.McL$	5_+^{1+2}	$126, 126, 126, 126$	Th	7^2	$27000, 27000$	?
He	5^2	$51, 51$	Fi_{23}	5^2	111826	
Ru	3_+^{1+2}	406	J_4	11_+^{1+2}	$887778, 887778, 394765284$	?
$2.Ru$	3_+^{1+2}	$28, 28$	B	7^2	$9287037474, 775438738408125$	?
Suz	5^2	1001	${}^2F_4(2)'$	3_+^{1+2}	$26, 26$	
$3.ON$	7_+^{1+2}	$342, 342, 342, 342$	${}^2F_4(2)'$	5^2	$26, 26, 351, 351$	

In the cyclic Sylow case, a complete description of the simple endo-trivial modules can be read directly from the Brauer trees of cyclic blocks using Theorem 3.1 and [13]. (Excepting one block of $2.B$ in characteristic 47 and five blocks of M in various characteristics, for which the Brauer trees are not known completely as of yet.) The results are collected in [18, Table 7].

Now if we assume p^2 divides $|G|$. Using the dimension and character degree criteria (Basic Facts 1.1, Lemma 2.2, Theorem 2.3), the known ordinary character tables and decomposition matrices of the quasi-simple sporadic groups (see [8, 14, 10]) we obtain the list of candidate characters $\chi \in \mathrm{Irr}(G)$ stated in Table 2. Excepting the cases marked by a "?" in the last column, one can prove that the corresponding modules are endo-trivial using one of the following arguments:

(i) $\chi \otimes \chi^*$ has one trivial constituent and one constituent of defect zero;
(ii) there is a subgroup $H \leq G$ containing a Sylow p-subgroup of G and $\chi|_H$ has a unique trivial constituent, and all other constituents are of defect zero;
(iii) χ is the character of a trivial source module and Theorem 2.3 yields the result; or

(iv) for the pairs $(G, \chi(1)) = ({}^2F_4(2)', 351)$ computations with MAGMA [20].

In the case of J_4 and B, it is not known whether these characters remain irreducible modulo p.

Remark 4.9. In fact most of the simple modules listed in Table 2 are trivial source modules. For instance this is obvious as soon as a character satisfies condition (ii) above. As a consequence, these modules provide us with example of *torsion* endo-trivial module, that is modules whose class lies in the torsion subgroup $TT(G) < T(G)$. In [17, 19], it will be shown that any quasi-simple sporadic group admitting a simple endo-trivial module also has *exotic* endo-trivial modules, in the sense that G has torsion endo-trivial modules whose dimension is not one.

Remark 4.10. endo-triviality is in general not preserved by Morita equivalences. Cyclic blocks of sporadic groups provide us with many counter-examples. For instance the Janko group J_1 has four 3-blocks with isomorphic Brauer trees, that is, isomorphic as pointed graphs equipped with a planar embedding. Hence the four blocks are Morita equivalent (see [13, pp. 69–70]). However, only two of these blocks contain simple endo-trivial modules.

5. Discarding the existence of simple endo-trivial modules via Auslander-Reiten Theory

Previous work on endo-trivial modules for different classes of quasi-simple groups G was concerned with the determination of the group $T(G)$ of endo-trivial modules. This includes results on groups with cyclic Sylow subgroups [21], the symmetric and alternating groups [4, 6], and groups of Lie type in their defining characteristic [5].

Nevertheless there are three main obstructions to apply these results to answer the question of finding the simple endo-trivial modules. First, the aforementioned articles do not treat covering groups. Second, they determine the structure of $T(G)$ but not the indecomposable endo-trivial modules themselves, in that their description involves Green correspondence, which is not explicit and notoriously difficult to determine. Third, even in the simplest cases where $T(G) = \langle \Omega(k) \rangle \cong \mathbb{Z}$ (where Ω is the Heller operator), it is not clear whether any of the modules $\Omega^n(k)$ for $n \in \mathbb{Z}$ can be simple.

In the latter case, we give here one possible approach via the stable Auslander-Reiten quiver $\Gamma_s(kG)$ of kG, which could have enabled us to use the structure the known structure of the group $T(G)$ to discard the existence of simple endo-trivial modules. This was suggested a fortiori by S. Koshitani.

Proposition 5.1. *Let G be a finite group and B_0 be the principal block of kG. Assume B_0 has wild representation type and G satisfies the following conditions:*

(i) each AR-component of type $\mathbb{Z}A_\infty$ of $\Gamma_s(B_0)$ contains at most one simple module;
(ii) $T(G) \cong \mathbb{Z}$.

Then G does not have non-trivial simple endo-trivial modules.

Proof. The condition $T(G) \cong \mathbb{Z}$ implies that an indecomposable endo-trivial kG-module is isomorphic to a syzygy module $\Omega^n(k)$ for some $n \in \mathbb{Z}$, and thus lies in B_0. Moreover $\Gamma_s(B_0)$ has only two connected components containing endo-trivial modules: $\Gamma_s(k)$ and

$\Gamma_s(\Omega(k))$, both isomorphic to $\mathbb{Z}A_\infty$ as B_0 is a wild block. By assumption (i), the trivial module k is the unique simple module in $\Gamma_s(k)$. If $\Gamma_s(\Omega(k))$ contains a simple endo-trivial module S, then $S = \Omega^{2n+1}(k)$ for some $n \in \mathbb{Z}$. Therefore $S^* = \Omega^{-2n-1}(k)$ is also simple and lies in $\Gamma_s(\Omega(k))$, which contradicts assumption (i). The claim follows. $\square$

Going back to quasi-simple groups and related groups this yields the following results.

Theorem 5.2. *Let G be a finite group of one of the following types:*

(a) G is a perfect group of Lie type defined over a field of characteristic p and G is not of type $A_1(p)$ $(p > 2)$, ${}^2A_2(p)$, $A_2(p)$, $B_2(p)$ $(p \geq 5)$, $G_2(p)$ $(p \geq 7)$, ${}^2B_2(2^{a+\frac{1}{2}})$ $(a \geq 0)$ or ${}^2G_2(3^{a+\frac{1}{2}})$ $(a \geq 0)$;

(b) $p = 2$ and $G = \mathfrak{S}_n$ is a symmetric group such that $n \geq 6$;

(c) $G = \mathfrak{A}_n$ is an alternating group such that $n \geq 8$ if $p = 2$, or such that $3p \leq n < p^2$ or $p^2 + p \leq n$ if $p \geq 3$.

Then G does not have non-trivial simple endo-trivial modules.

The claim is a direct consequence of Proposition 5.1. Indeed, by [5] and [6, 4] any group of type (a), (b) or (c) is such that $T(G) \cong \mathbb{Z}$. Moreover by [15, 16] and [24, Thm. 6] these groups also satisfy condition (i) of Proposition 5.1.

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