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On association schemes of finite exponent

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1 Introduction

Association schemes are a generalization of finite groups since thin association schemes are essentially equivalent to finite groups. In the present paper, we will consider extensions of finite groups in association schemes.

Definition 1. Let (X, S) be an association scheme and N be a normal closed subset of S. The scheme (X, S) is called an extension of $(X, S)^N$ by N, where $(X, S)^N$ is the factor scheme of (X, S) by N.

Our purpose is to search association schemes which are obtained by repeating an extension by a thin normal closed subset. To do this, we will focus the higher Frobenius-Schur indicators which are studied in many research areas recently.

The classical Frobenius-Schur indicator for association schemes was introduced by Higman [3]. We will define the higher Frobenius-Schur indicators for association schemes as a generalization of this indicator in §3. We will make a conjecture on the connection between the values of the higher Frobenius-Schur indicators of the regular representation and association schemes obtained by repeating an extension by a thin normal closed subset.

In §4, we will define the girth and the strong girth of a relation of an association scheme as candidates correspond to the order of an element of a finite group. Moreover, we will define the exponent of an association scheme by the least common multiple of the strong girths of all relations. We will show that our conjecture holds for a class of association schemes of finite exponent.

2 Notations

For association schemes, refer to [7].

Let X be a finite set and S be a partition of $X \times X$ which does not contain the empty set. For $s \in S$, we define the adjacency matrix σ_s of s by the matrix whose both rows and columns are indexed by X and the (x, y)-entry is 1 if $(x, y) \in s$ and 0 otherwise. The pair (X, S) is called an association scheme if

1. $1 := \{(x, x) \mid x \in X\} \in S$,
2. if $s \in S$, then $s^* := \{(y, x) \mid (x, y) \in s\} \in S$,
3. for $s, t, u \in S$, there exists an integer a_{stu} such that $\sigma_s \sigma_t = \sum_{u \in S} a_{stu} \sigma_u$.

For $s \in S$, we define $n_s := a_{ss^*1}$ and call it the valency of s. For $C \subset S$, we put $\sigma_C := \sum_{c \in C} \sigma_c$ and $n_C := \sum_{c \in C} n_c$. We denote the factor scheme of (X, S) by a closed subset C by $(X, S)^C = (X/C, S/\!/C)$. For any relations $s, \ell, \ell' \in S$ and $n \geq 2$, we set $a_{s^n \ell} = a_{\underbrace{s \cdots s}_{n} \ell}$ and $a_{\ell' s^n \ell} = a_{\ell' \underbrace{s \cdots s}_{n} \ell}$.

We define the adjacency algebra $\mathbb{C}S$ of (X, S) over the complex number field $\mathbb{C}$ by a matrix algebra generated by $\{\sigma_s \mid s \in S\}$. We denote the set of all irreducible characters of $\mathbb{C}S$ by $\mathrm{Irr}(S)$ and the multiplicity of the character χ by m_χ for any $\chi \in \mathrm{Irr}(S)$.

3 The higher Frobenius-Schur indicators

We recall the higher Frobenius-Schur indicators for finite groups (See [4]).

Let G be a finite group and $\mathbb{C}G$ be the group algebra of G over the complex number field $\mathbb{C}$. We denote the set of all irreducible characters of $\mathbb{C}G$ by $\mathrm{Irr}(G)$. We define the n-th Frobenius-Schur indicator $\nu_n(\chi)$ for $\chi \in \mathrm{Irr}(G)$ by

$$\nu_n(\chi) = \frac{1}{|G|} \sum_{g \in G} \chi(g^n),$$

and the n-th indicator $\nu_n(G)$ of G by

$$\nu_n(G) = \sum_{\chi \in \mathrm{Irr}(G)} \chi(1) \nu_n(\chi).$$

It is well-known that $\nu_n(G) = \#\{g \in G \mid o(g) | n\}$, where $o(g)$ is the order of g. Namely, we know that $\nu_n(G)$ is an integer for any n.

Let (X, S) be an association scheme, $\mathbb{C}S$ be the adjacency algebra of (X, S) over $\mathbb{C}$ and $\mathrm{Irr}(S)$ be the set of all irreducible characters of $\mathbb{C}S$. In the paper [3], Higman introduced the second Frobenius-Schur indicator for association schemes as follows: for $\chi \in \mathrm{Irr}(S)$,

$$\nu_2(\chi) = \frac{m_\chi}{n_S \chi(\sigma_1)} \sum_{s\in S} \frac{1}{n_s} \chi({\sigma_s}^2).$$

He showed that

$$\sum_{\chi\in \mathrm{Irr}(S)} \chi(\sigma_1)\nu_2(\chi) = \#\{s \in S \mid s \text{ is symmetric}\}.$$

We define the n-th Frobenius-Schur indicator $\nu_n(\chi)$ for $\chi \in \mathrm{Irr}(S)$ by

$$\nu_n(\chi) = \frac{m_\chi}{n_S \chi(\sigma_1)} \sum_{s\in S} \frac{1}{{n_s}^{n-1}} \chi({\sigma_s}^n),$$

and the n-th indicator $\nu_n(S)$ of (X, S) by

$$\nu_n(S) = \sum_{\chi\in \mathrm{Irr}(S)} \chi(\sigma_1)\nu_n(\chi).$$

By the direct calculation, it follows that

$$\nu_n(S) = \sum_{s\in S} \frac{1}{{n_s}^{n-1}} a_{s^n 1} \in \mathbb{Q}.$$

The following theorem says that there exist association schemes which its n-th indicators for some n are not integers actually.

Theorem 2. *Let (X, S) be an association scheme of rank 2.*

$$\nu_n(S) = \frac{2|X|-1}{|X|} + (-1)^n \frac{1}{|X|(|X|-1)^{n-2}}.$$

Finally, we introduce a conjecture on the n-th indicators.

Conjecture 3. Let (X, S) be an association scheme. If the n-th indicator $\nu_n(S)$ is an integer for any positive integer n, (X, S) is obtained by repeating an extension by a thin normal closed subset.

In §5, we will show that this conjecture holds for a special class of association schemes.

4 The girth, the strong girth and the exponent

We define the *girth*, the *strong girth* and the *exponent* for association schemes.

Definition 4. We define the *girth* $\mathrm{g}(s)$ of a non-identity relation s of S by

$$\mathrm{g}(s) = \min\{e \in \mathbb{N} \setminus \{0, 1\} \mid a_{s^e 1} \neq 0\}.$$

It is well-known that $\mathrm{g}(s)$ is a positive integer for any $s \in S$. We define the *strong girth* $\mathrm{sg}(s)$ of a non-identity relation s of S by

$$\mathrm{sg}(s) = \min\{e \in \mathbb{N} \setminus \{0, 1\} \mid a_{s^e 1} = n_s^{e-1}\},$$

and if there exist no such natural numbers, we define that $\mathrm{sg}(s) = \infty$. We define that $\mathrm{g}(1) = \mathrm{sg}(1) = 1$.

If $\mathrm{sg}(s) < \infty$ for any $s \in S$, we define the *exponent* $\exp(S)$ of S by

$$\exp(S) = \mathrm{l.c.m}\{\mathrm{sg}(s) \mid s \in S\},$$

where l.c.m means the least common multiple. Otherwise, we set $\exp(S) = \infty$.

We list some elementary properties for the girth, the strong girth and the exponent.

Lemma 5. *Let (X, S) be a thin association scheme. We have that $\mathrm{g}(s) = \mathrm{sg}(s)$ for any $s \in S$. Especially, $\mathrm{sg}(s) < \infty$ for any $s \in S$.*

Lemma 6. *Let (X, S) be an association scheme and s be a relation of S.*

1. $\mathrm{g}(s) \leq \mathrm{sg}(s)$.
2. *The followings are equivalent;*
   - *(a)* $\mathrm{g}(s) = 2$,
   - *(b)* $\mathrm{sg}(s) = 2$,
   - *(c)* *s is symmetric.*
3. *(X, S) is a symmetric scheme if and only if* $\exp(S) = 2$.

We can find some association schemes which have a relation s with $\mathrm{sg}(s) = \infty$ and $\mathrm{g}(s) = 3$ (for example, two non-identity relations of as07[2]). It is natural to consider when $\mathrm{sg}(s)$ is equal to $\mathrm{g}(s)$. It is shown in [5] that $\mathrm{sg}(s) = \mathrm{g}(s)$ if $\mathrm{sg}(s) < \infty$.

Theorem 7. *Let $s \in S$ be a relation with* $\mathrm{sg}(s) < \infty$. *Then,* $\mathrm{sg}(s) = \mathrm{g}(s)$.

Corollary 8. *Let $s \in S$ be a relation of S. If there exists a number t such that $a_{s^t 1} = {n_s}^{t-1}$, it follows that* $\mathrm{sg}(s) \mid t$.

Finally, we consider the factor scheme of an association scheme by a normal closed subset.

Theorem 9. *Let (X, S) be an association scheme and N be a normal closed subset of S. Then, it follows that* $\mathrm{sg}(s^N) \mid \mathrm{sg}(s)$ *for any relation s with* $\mathrm{sg}(s) < \infty$.

Corollary 10. *The exponent of an association scheme of finite exponent is divided by the exponent of the factor scheme by any normal closed subset.*

5 Finite exponent association schemes

In this section, we will calculate the n-th indicators of association schemes of finite exponent.

5.1 Association schemes of exponent 3

Theorem 11. *Let (X, S) be an association scheme of exponent 3. For any non-identity relation $s \in S$,*

$$a_{s^n 1} = \begin{cases} {n_s}^{n-1} & \text{if } 3 \mid n, \\ 0 & \text{otherwise.} \end{cases}$$

From Theorem 11, it follows that

$$\nu_n(S) = \begin{cases} |S| & \text{if } 3 \mid n, \\ 1 & \text{otherwise.} \end{cases}$$

Thus we hope that such schemes obtained by repeating an extension by a thin normal closed subset because $\nu_n(S) \in \mathbb{Z}$ for any positive integer n.

Theorem 12. *Let (X, S) be an association scheme of exponent 3. Then, S has a non-identity relation of valency 1.*

From Corollary 10 and this theorem, we have the following theorem.

Theorem 13. *Let (X, S) be a commutative association scheme of exponent 3. Then S has a normal closed subset N isomorphic to the order 3 cyclic group. Moreover the exponent of the factor scheme $(X, S)^N$ of (X, S) by N divides 3.*

5.2 Association schemes of finite exponent

In Corollary 8, it is proved that $\mathrm{sg}(s)|n$ if $a_{s^n 1} = {n_s}^{n-1}$. The converse of this corollary does not hold generally. We will consider a condition providing that the converse of this corollary holds.

Let s be a relation of finite strong girth e. Namely we have $a_{s^e 1} = {n_s}^{e-1}$ and $a_{s^{e-1}\ell} = \delta_{\ell s^*} {n_s}^{e-2}$ for any $\ell \in S$. By the direct calculation, it follows that

$$\begin{aligned} a_{s^{2e}1} &= \sum_{\ell \in S} a_{s^{e-1}\ell} a_{\ell s^{e+1} 1} = {n_s}^{e-2} a_{s^* s^{e+1} 1} \\ &= {n_s}^{e-2} \sum_{\ell \in S} a_{s^* \ell 1} a_{s^{e+1}\ell} = {n_s}^{e-1} a_{s^{e+1} s} \\ &= {n_s}^{e-1} \sum_{\ell \in S} a_{s^{e-1}\ell} a_{\ell s^2 s} = {n_s}^{2e-3} a_{s^* sss}. \end{aligned}$$

Therefore the converse of Corollary 8 holds for $n = 2\mathrm{sg}(s)$ if $a_{s^* sss} = {n_s}^2$.

Proposition 14. *Let s be a relation. The followings are equivalent.*

1. $a_{s^* sss} = {n_s}^2$,
2. $\sigma_{s^*} \sigma_s \sigma_s = {n_s}^2 \sigma_s$,
3. $s^* ss = s$.

If a relation s satisfies one of three conditions, we say the relation s has the property (P).

We investigate relations have the property (P).

Lemma 15. *If a relation has the property (P), the strong girth of the relation is finite.*

The following theorem says that the converse of Corollary 8 holds for relations have the property (P).

Theorem 16. *Let s be a relation has the property (P). We have that*

$$a_{s^n 1} = \begin{cases} {n_s}^{n-1} & \textit{if } \mathrm{sg}(s)|n, \\ 0 & \textit{otherwise.} \end{cases}$$

From Lemma 15, we know that if a relation s has the property (P), its strong girth $\mathrm{sg}(s) < \infty$. However, the converse of this lemma does not always hold. For example, the strong girth of the non-identity relation of any rank two association scheme of order $|X| > 2$ is 2, but the relation does not have the property (P). We know that counterexamples of the converse of this lemma are only symmetric relations.

Theorem 17. *Let s be a relation of finite strong girth $e \neq 2$. The relation s has the property (P).*

Finally we consider the connection on the property (P) between a relation s in an association scheme (X, S) and the relation s^N corresponding to s in the factor scheme $(X, S)^N$ by a normal closed subset N.

Corollary 18. *Let (X, S) be an association scheme and N be a normal closed subset of S. Let $(X, S)^N = (X/N, S//N)$ be the factor scheme of (X, S) by N.*

If a relation $s \in S$ has the property (P), the relation $s^N \in S//N$ has also the property (P).

In the rest of this subsection, we will argue about association schemes all relations of which have the property (P). From Lemma 15, such association schemes are of finite exponent.

Theorem 19. *Any association scheme all relations of which have the property (P) has finite exponent.*

We can calculate the n-th indicator from Theorem 16.

Theorem 20. *Let (X, S) be an association scheme all relations of which have the property (P).*

$$\nu_n(S) = \#\{s \in S \mid \mathrm{sg}(s)|n\}.$$

Thus we hope that the conjecture holds for these schemes.

Theorem 21. *Let (X, S) be an association scheme all relations of which have the property (P). Then, S has a non-identity relation of valency 1.*

From Theorem 21, Corollary 10 and Corollary 18, we have the following theorem.

Theorem 22. *Let (X, S) be a commutative association scheme all relations of which have the property (P). Then S has a normal closed subset N isomorphic to a prime order cyclic group. Moreover the exponent* $\exp(S//N)$ *of the factor scheme $(X, S)^N$ by N divides* $\exp(S)$ *and all relations of $(X, S)^N$ has the property (P).*

Corollary 23. *Any commutative association scheme of odd prime exponent p is a p-scheme.*

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