Clifford theory for association schemes

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1 Introduction

Association schemes are regarded as generalizations of finite groups. So it is natural to consider the generalization to association schemes of the theory of representation of finite groups.

Let $K$ be an algebraically closed field. Let $G$ be a finite group, $N$ a normal subgroup of $G$. The usual Clifford theory for finite groups shows that

(CF1) the restriction of an irreducible $KG$-module to $KN$ is a direct sum of $G$-conjugates of an irreducible $KN$-module $L$ with the same multiplicities;

(CF2) there exists a natural bijection between the set of irreducible $KG$-modules over $L$ and the set of $KT$-modules over $L$, where $T$ is the stabilizer of $L$ in $G$;

(CF3) and there exists a natural bijection between the set of irreducible $KT$-modules over $L$ and the set of irreducible modules of a generalized group algebra of $T/N$.

We will generalize them to association schemes. But we only consider module over the complex number field $\mathbb{C}$.

2 Adjacency algebras of association schemes

We fix some notations for association schemes.

Let $(X, S)$ be an association scheme. We denote by $\sigma_s$ the adjacency matrix of $s \in S$. The intersection number is denoted by $p_{st}^u$ for $s, t, u \in S$, namely $\sigma_s \sigma_t = \sum_{u \in S} p_{st}^u \sigma_u$. The valency is denoted by $n_s$ for $s \in S$. An elements in the quotient scheme $S//T$ is denoted by $s^T$.

2.1 Generalized adjacency algebras

In this section we define generalized adjacency algebras based on a definition of generalized group algebra. Details for factor sets and generalized group algebra (ring) are available in the literature [6, Chapter 2, Section 7].
Let $G$ be a group and let $K$ be a field. We say that $\alpha : G \times G \to K^\times$ is a 
\textit{factor set} if it satisfies the following condition:

$$\alpha(xy, z)\alpha(x, y) = \alpha(y, z)\alpha(x, yz)$$

for all $x, y, z \in G$.

Note that in general we can consider the action of $G$ on $K$, but to simplify our arguments, we suppose that the action is trivial. Two factor sets $\alpha$ and $\beta$ are 
\textit{cohomologous} if there exists a map $\gamma : G \to K^\times$ such that

$$\alpha(x, y) = \beta(x, y)\gamma(x)\gamma(y)\gamma(xy)^{-1}$$

and we write $\alpha \sim \beta$ in this case. The relation $\sim$ is an equivalence relation on the 
set of factor sets. A factor set $\alpha$ is said to be \textit{normalized} if $\alpha(x, 1) = \alpha(1, x) = 1$
for all $x \in G$. For a normalized factor set $\alpha$, $\alpha(x, x^{-1}) = \alpha(x^{-1}, x)$ also holds. 
For an arbitrary factor set $\alpha$, there exists a normalized factor set $\beta$ such that $\beta \sim \alpha$.

Let $(X, S)$ be an association scheme and let $T$ be a strongly normal closed subset of $S$. Then the quotient $S//T$ can be regarded as a finite group. Let $\alpha : S//T \times S//T \to K^\times$ be a factor set. We define a $K$-algebra $K^{(\alpha)}S = \bigoplus_{u \in S} K\sigma_u^{(\alpha)}$
with formal basis $\{\sigma_u^{(\alpha)} | u \in S\}$ and multiplication

$$\sigma_u^{(\alpha)}\sigma_v^{(\alpha)} = \sum_{w \in S} p_{uv}^w \alpha(u^T, v^T)\sigma_w^{(\alpha)}.$$

The algebra $K^{(\alpha)}S$ is called the \textit{generalized adjacency algebra} of $(X, S)$ over $K$ 
with factor set $\alpha$. If the strongly normal closed subset $T$ is trivial, then the scheme is thin and the generalized adjacency algebra is just a generalized group algebra.

\subsection{2.2 Graded modules and simple modules}

Let $K$ be a field. Let $(X, S)$ be a scheme and $T$ a strongly normal closed subset 
of $S$. Then $S//T$ is thin and we can regard it as a finite group. Then

$$KS = \bigoplus_{s^T \in S//T} K(TsT)$$

is an $S//T$-graded $K$-algebra, where $K(TsT) = \bigoplus_{u \in TsT} K\sigma_u$. Obviously $(KS)_1^T = KT$. We can apply Dade’s theory for $KS$, but we restrict our attention to the 
case $K = \mathbb{C}$.

\textbf{Theorem 2.1.} [4, Theorem 3.6] For any simple $CT$-module $L$ and $s \in S$, 
$L \otimes \mathbb{C}(TsT)$ is a simple $CT$-module or $0$.

For any simple $CT$-module $L$, the set of $S//T$-conjugates is $\{L \otimes \mathbb{C}(TsT) | s \in S, L \otimes \mathbb{C}(TsT) = 0\}$. We remark that there exist examples such that $L$ and $L'$
are $S//T$-conjugate simple $CT$-modules but their dimensions are different.
3 Clifford Theory

First we define some notations. Let $A$ a finite-dimensional $K$-algebra and let $B$ be a subalgebra of $A$. For a right $B$-module $L$, the induction $L \otimes_B A$ of $L$ to $A$ is denoted by $L \uparrow^A$. For a right $A$-module $M$, we write $M \downarrow_B$ if $M$ is considered as a $B$-module. We denote by $\text{IRR}(A)$ the complete set of representatives of the isomorphism classes of simple $A$-modules. Suppose that both $A$ and $B$ are semisimple. For a simple $B$-module $L$, we define $\text{IRR}(A | L) = \{ M \in \text{IRR}(A) | \text{Hom}_A(L \uparrow^A, M) \neq 0 \}$.

Let $(X, S)$ be an association scheme and let $T$ be a closed subset of $S$. For a right $\mathbb{C}T$-module $L$ and a right $\mathbb{C}S$-module $M$, we write $L \uparrow^S$ and $M \downarrow \tau$ instead of $L \uparrow^{\mathbb{C}S}$ and $M \downarrow \mathbb{C}T$, respectively.

In the rest of this section, we fix a scheme $(X, S)$ and its strongly normal closed subset $T$.

Let $M \in \text{IRR}(\mathbb{C}S)$. Then $M \in \text{IRR}(\mathbb{C}S | L)$ for some $L \in \text{IRR}(\mathbb{C}T)$. Since $M$ is a direct summand of $L \uparrow^S$, any simple submodule of $M \downarrow_T$ is an $S//T$-conjugate of $L$. If $L$ and $L'$ are $S//T$-conjugate, then $L \uparrow^S \cong L' \uparrow^S$ as $\mathbb{C}S$-modules. So

$$\dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}T}(L, M \downarrow_T) = \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}T}(L', M \downarrow_T).$$

This shows the following theorem.

**Theorem 3.1.** [4, Theorem 4.1] Let $M \in \text{IRR}(\mathbb{C}S)$. There exists $L \in \text{IRR}(\mathbb{C}T)$ such that $M \in \text{IRR}(\mathbb{C}S | L)$. Then there exists a positive integer $e$ such that

$$M \downarrow_T \cong e \left( \bigoplus_{L' \in C} L' \right),$$

where $C = \{ L \otimes \mathbb{C}(TsT) | s \in S, L \otimes \mathbb{C}(TsT) \neq 0 \}$.

Fix a simple $\mathbb{C}T$-module $L$. Put $U//T$ the stabilizer of $L$ in $S//T$. Then

$$\bigoplus_{s^T \in S//T} L \otimes \mathbb{C}(TsT) = L \otimes_{\mathbb{C}T} \mathbb{C}S \supset L \otimes_{\mathbb{C}T} \mathbb{C}U = \bigoplus_{u^T \in U//T} L \otimes \mathbb{C}(TuT)$$

and, by Theorem 2.1,

$$\bigoplus_{u^T \in U//T} L \otimes \mathbb{C}(TuT) \cong n_{U//T} L$$

as a $\mathbb{C}T$-module. So $\dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}U}(L \uparrow^U, L \uparrow^U) = \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}T}(L, L \uparrow^U \downarrow_T) = n_{U//T}$. On the other hand, by the Frobenius reciprocity, we have

$$\dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}S}(L \uparrow^S, L \uparrow^S) = \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}T}(L, L \uparrow^S \downarrow_T) = n_{U//T}.$$
So \( \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}S}(L \uparrow^{S}, L \uparrow^{S}) = \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}U}(L \uparrow^{U}, L \uparrow^{U}) \). Let \( L \uparrow^{U} \cong \bigoplus_{i} m_{i} M_{i} \) be the irreducible decomposition of \( L \uparrow^{U} \), with the property that \( M_{i} \cong M_{j} \) if and only if \( i = j \). Then

\[
\dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}U}(L \uparrow^{U}, L \uparrow^{U}) = \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}U}(\bigoplus_{i} m_{i} M_{i}, \bigoplus_{i} m_{i} M_{i}) \\
\leq \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}S}(\bigoplus_{i} m_{i} M_{i} \uparrow^{S}, \bigoplus_{i} m_{i} M_{i} \uparrow^{S}) \\
= \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}S}(L \uparrow^{S}, L \uparrow^{S})
\]

This means that \( \dim_{\mathbb{C}} \text{Hom}_{\mathbb{C}S}(M_{i} \uparrow^{S}, M_{i} \uparrow^{S}) = 1 \) and \( M_{i} \uparrow^{S} \) is a simple \( \mathbb{C}S \)-module for every \( i \). Also \( M_{i} \uparrow^{S} \cong M_{j} \uparrow^{S} \) if and only if \( i = j \). Obviously \( M_{i} \in \text{IRR}(\mathbb{C}U|L) \) and \( M_{i} \uparrow^{S} \in \text{IRR}(\mathbb{C}S|L) \).

Conversely, let \( N \in \text{IRR}(\mathbb{C}S|L) \). Then \( N \) is a direct summand of \( L \uparrow^{S} \). So there exists some \( M_{i} \) such that \( N \) is a direct summand of \( M_{i} \uparrow^{S} \). Since \( M_{i} \uparrow^{S} \) is simple, such \( M_{i} \) is uniquely determined. This shows the following theorem.

**Theorem 3.2.** [4, Theorem 4.2] Fix a simple \( \mathbb{C}T \)-module \( L \). Put \( U//T \) the stabilizer of \( L \) in \( S//T \). Then there exists a bijection \( \tau : \text{IRR}(\mathbb{C}U|L) \rightarrow \text{IRR}(\mathbb{C}S|L) \) such that \( \tau(M) = M \uparrow^{S} \) and \( \tau^{-1}(N) \) is the unique direct summand of \( N \downarrow_{U}^{S} \) contained in \( \text{IRR}(\mathbb{C}U|L) \).

We consider \( \text{End}_{\mathbb{C}U}(L \uparrow^{U}) \). For \( u^{T} \in U//T \), we define \( \rho_{u^{T}} \in \text{End}_{\mathbb{C}U}(L \uparrow^{U}) \) by \( (\rho_{u^{T}}(\ell))_{v^{T}} = \ell_{u^{T}v^{T}} \). Then \( \text{End}_{\mathbb{C}U}(L \uparrow^{U}) = \bigoplus_{u^{T} \in U//T} \mathbb{C} \text{rho}_{u^{T}} \) and this is a \( U//T \)-graded algebra ([3, Section 4]). The multiplication is \( \rho_{u^{T}} \rho_{v^{T}} = \alpha(u^{T}, v^{T}) \rho_{u^{T}v^{T}} \) and this defines a factor set \( \alpha \). Now \( \text{End}_{\mathbb{C}U}(L \uparrow^{U}) \cong \mathbb{C}^{(\alpha)}(U//T) \) is a generalized group algebra with factor set \( \alpha \).

**Proposition 3.3.** [5, Theorem 3.1] Under the above assumptions, the irreducible \( \mathbb{C}T \)-module \( L \) is extensible to a \( \mathbb{C}^{(\alpha^{-1})}U \)-module (\( \mathbb{C}^{(\alpha^{-1})}U \) is the generalized adjacency algebra with factor set \( \alpha^{-1} \)). The action is given by \( \ell \sigma_{u}^{(\alpha^{-1})} = \rho_{(u^{T})^{-1}}(\ell \sigma_{u}) \) for \( \ell \in L \) and \( u \in U \).

We denote by \( \tilde{L} \) the extension of \( L \) to \( \mathbb{C}^{(\alpha^{-1})}U \). Since \( L \) is a simple \( \mathbb{C}T \)-module, \( \tilde{L} \) is a simple \( \mathbb{C}^{(\alpha^{-1})}U \)-module.

If \( M \) is an irreducible \( \mathbb{C}^{(\alpha)}(U//T) \)-module, then \( \tilde{L} \otimes_{\mathbb{C}U} M \) is an irreducible \( \mathbb{C}U \)-module and is in \( \text{IRR}(\mathbb{C}U | L) \). So we can define a map \( \mu : \text{IRR}(\mathbb{C}^{(\alpha)}(U//T)) \rightarrow \text{IRR}(\mathbb{C}U | L) \) by \( \mu(M) = \tilde{L} \otimes_{\mathbb{C}U} M \).

Then \( \mu \) is a bijection. This shows the following theorem.

**Theorem 3.4.** [5, Theorem 3.6] Let \( (X, S) \) be an association scheme, let \( T \) be a strongly normal closed subset, and let \( L \) be an irreducible \( \mathbb{C}T \)-module. Let \( U//T \) be the stabilizer of \( L \) in \( S//T \). Then \( L \) is extensible to a \( \mathbb{C}^{(\alpha^{-1})}U \)-module \( \tilde{L} \) and the map \( \mu : \text{IRR}(\mathbb{C}^{(\alpha)}(U//T)) \rightarrow \text{IRR}(\mathbb{C}U | L) \) defined by \( \mu(M) = \tilde{L} \otimes_{\mathbb{C}U} M \) is a bijection.
References


