On a Characterization of the Bilinear Forms Graphs
\[ Bil_q(d \times d) \]

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June 10, 2014

1 Introduction

Much attention has been paid to a problem of classification of all \( Q \)-polynomial distance-regular graphs with large diameter [1] (for the definitions, we refer the reader to Section 2). One of the steps towards solution of this problem is a characterization of known distance-regular graphs by their intersection arrays. For the current status of the classification of the \( Q \)-polynomial distance-regular graphs, we refer the reader to the survey paper [3] by Van Dam, Koolen and Tanaka.

The bilinear forms graph denoted here by \( Bil_q(d \times n) \) is a graph defined on the set of \( d \times n \)-matrices over \( \mathbb{F}_q \) with two matrices being adjacent if and only if the rank of their difference is 1. We refer to [2, Chapter 9.5.A] for the detailed description of these graphs.

In 1999, K. Metsch [5] obtained the following result.

Result 1.1 The bilinear forms graph \( Bil_q(d \times n) \) is characterized by its intersection array if:

- \( q = 2 \) and \( n \geq d + 4 \),
- \( q \geq 3 \) and \( n \geq d + 3 \).

Thus, the open cases are:

- \( q = 2 \) and \( n \in \{d, d + 1, d + 2, d + 3\} \),
- \( q \geq 3 \) and \( n \in \{d, d + 1, d + 2\} \).

In this paper, we discuss a problem of characterization of the bilinear forms graphs \( Bil_q(d, d) \), \( d \geq 3 \), by their intersection arrays.
2 Definitions and preliminaries

All the graphs considered in this paper are finite, undirected and simple. Suppose that \( \Gamma \) is a connected graph with vertex set \( V(\Gamma) \) and edge set \( E(\Gamma) \), where \( E(\Gamma) \) consists of unordered pairs of adjacent vertices. The distance \( d(x, y) \) between any two vertices \( x, y \) of \( \Gamma \) is the length of a shortest path connecting \( x \) and \( y \) in \( \Gamma \).

For a subset \( X \) of the vertex set of \( \Gamma \), we will also write \( X \) for the subgraph of \( \Gamma \) induced by \( X \). For a vertex \( x \in V(\Gamma) \), define \( \Gamma_i(x) \) to be the set of vertices which are at distance precisely \( i \) from \( x \) (\( 0 \leq i \leq D \)), where \( D := \max\{d(x, y) \mid x, y \in V(\Gamma)\} \) is the diameter of \( \Gamma \). In addition, define \( \Gamma_{-1}(x) = \Gamma_{D+1}(x) = \emptyset \). The subgraph induced by \( \Gamma_1(x) \) is called the neighborhood or the local graph of a vertex \( x \). The ball of radius 1 around \( x \) is denoted by \( x^1 \), i.e. \( x^1 = \{x\} \cup \Gamma_1(x) \). We write \( \Gamma(x) \) instead of \( \Gamma_1(x) \) for short, and we denote \( x \sim y \) for notational convenience and note that \( x \sim y \) if two vertices \( x \) and \( y \) are adjacent in \( \Gamma \). For a graph \( G \), a graph \( \Gamma \) is called locally \( G \) if any local graph of \( \Gamma \) is isomorphic to \( G \).

For a set of vertices \( x_1, \ldots, x_n \), let \( \Gamma(x_1, \ldots, x_n) \) denote \( \cap_{i=1}^{n} \Gamma_1(x_i) \). Moreover, if \( x \) and \( y \) are at distance 2 in \( \Gamma \), we call \( \Gamma(x, y) \) the \( \mu \)-graph of \( x, y \).

The eigenvalues of a graph are the eigenvalues of its adjacency matrix (recall that they are algebraic integers). If, for some eigenvalue \( \eta \) of \( \Gamma \), its eigenspace contains a vector orthogonal to the all ones vector, we say the eigenvalue \( \eta \) is non-principal. If \( \Gamma \) is regular with valency \( k \) then all its eigenvalues are non-principal unless the graph is connected and then the only eigenvalue that is principal is its valency \( k \).

For a graph \( \Gamma \) and its vertex \( x \), we say that \( \eta \) is a local eigenvalue at \( x \), if \( \eta \) is an eigenvalue of \( \Gamma_1(x) \).

A connected graph \( \Gamma \) with diameter \( D \) is called distance-regular if there exist integers \( b_{i-1}, c_i \) (\( 1 \leq i \leq D \)) such that, for any two vertices \( x, y \in V(\Gamma) \) with \( d(x, y) = i \), there are precisely \( c_i \) neighbors of \( y \) in \( \Gamma_{i-1}(x) \) and \( b_i \) neighbors of \( y \) in \( \Gamma_{i+1}(x) \). In particular, any distance-regular graph is regular with valency \( k := b_0 \). We define \( a_i := k - b_i - c_i \) for notational convenience and note that \( a_i = |\Gamma(y) \cap \Gamma_i(x)| \) holds for any two vertices \( x, y \) with \( d(x, y) = i \) (\( 1 \leq i \leq D \)). The array \( \{b_0, b_1, \ldots, b_{D-1}; c_1, c_2, \ldots, c_D\} \) is called the intersection array of the distance-regular graph \( \Gamma \).

A distance-regular graph with diameter 2 is called a strongly regular graph. We say that a strongly regular graph \( \Gamma \) has parameters \( (v, k, \lambda, \mu) \), if \( v = |V(\Gamma)| \), \( k \) is its valency, \( \lambda := a_1 \), and \( \mu := c_2 \).

If a graph \( \Gamma \) is distance-regular then, for all integers \( h, i, j \) (\( 0 \leq h, i, j \leq D \)), and all vertices \( x, y \in V(\Gamma) \) with \( d(x, y) = h \), the number

\[
p_{i,j}^h := |\{z \in V(\Gamma) \mid d(x, z) = i, \ d(y, z) = j\}|
\]
does not depend on the choice of \( x, y \). The numbers \( p_{ij}^h \) are called the \textit{intersection numbers} of \( \Gamma \).

Note that \( c_i = p_{i-1}^i, a_i = p_i^i, \) and \( b_i = p_{i+1}^i \).

For each integer \( i \) \((0 \leq i \leq D)\), the \( i \)th distance matrix \( A_i \) of \( \Gamma \) has rows and columns indexed by the vertex of \( \Gamma \), and, for any \( x, y \in V(\Gamma) \),

\[
(A_i)_{x,y} = \begin{cases} 
1 & \text{if } d(x, y) = i, \\
0 & \text{if } d(x, y) \neq i.
\end{cases}
\]

Then \( A := A_1 \) is just the \textit{adjacency matrix} of \( \Gamma \), \( A_0 = I, A_i^T = A_i \) \((0 \leq i \leq D)\), and

\[
A_i A_j = \sum_{h=0}^{D} p_{ij}^h A_h \quad (0 \leq i, j \leq D),
\]

in particular,

\[
A_1 A_i = b_{i-1} A_{i-1} + a_i A_i + c_{i+1} A_{i+1} \quad (1 \leq i \leq D - 1),
\]

\[
A_1 A_D = b_{D-1} A_{D-1} + a_D A_D,
\]

and this implies that \( A_i = p_i(A_1) \) for certain polynomial \( p_i \) of degree \( i \).

The \textit{Bose-Mesner algebra} \( M \) of \( \Gamma \) is a matrix algebra generated by \( A_1 \) over \( \mathbb{C} \). It follows that \( M \) has dimension \( D + 1 \), and it is spanned by the set of matrices \( A_0 = I, A_1, \ldots, A_D \), which form a basis of \( M \).

Since the algebra \( M \) is semi-simple and commutative, \( M \) also has a basis of pairwise orthogonal idempotents \( E_0 := \frac{1}{|V(\Gamma)|} J, E_1, \ldots, E_D \) (the so-called \textit{primitive idempotents} of \( M \)):

\[
E_i E_j = \delta_{ij} E_i \quad (0 \leq i, j \leq D), \quad E_i = E_i^T \quad (0 \leq i, j \leq D),
\]

\[
E_0 + E_1 + \ldots + E_D = I,
\]

where \( J \) is the all ones matrix.

In fact, \( E_j \) \((0 \leq j \leq D)\) is the matrix representing orthogonal projection onto the eigenspace of \( A_1 \) corresponding to some eigenvalue of \( \Gamma \). In other words, one can write

\[
A_1 = \sum_{j=0}^{D} \theta_j E_j,
\]

where \( \theta_j \) \((0 \leq j \leq D)\) are the real and pairwise distinct scalars, known as the \textit{eigenvalues} of \( \Gamma \). We say that the eigenvalues are in \textit{natural} order if \( b_0 = \theta_0 > \theta_1 > \ldots > \theta_D \). We denote \( \hat{\theta}_i = -1 - \frac{b_i}{\theta_i + 1} \) for \( i \in \{1, D\} \).

The Bose-Mesner algebra \( M \) is also closed under entrywise (Hadamard or Schur) matrix multiplication, denoted by \( \circ \). Then the matrices \( A_0, A_1, \ldots, A_D \) are the primitive idempotents of \( M \) with respect to \( \circ \), i.e., \( A_i \circ A_j = \delta_{ij} A_i \), and \( \sum_{i=0}^{D} A_i = J \). This implies that

\[
E_i \circ E_j = \sum_{h=0}^{D} q_{ij}^h E_h \quad (0 \leq i, j \leq D)
\]
holds for some real numbers $q_{ij}^h$, known as the Krein parameters of $\Gamma$.

Let $\Gamma$ be a distance-regular graph, and $E$ be a primitive idempotent of its Bose-Mesner algebra. The graph $\Gamma$ is called $Q$-polynomial (with respect to $E$) if there exist real numbers $c_i^*, a_i^*, b_i^{*-1}$ ($1 \leq i \leq D$) and an ordering of primitive idempotents such that $E_0 = \frac{1}{|V(\Gamma)|}J$ and $E_1 = E$, and

$$E_1 \circ E_i = b_{i-1}^* E_{i-1} + a_i^* E_i + c_{i+1}^* E_{i+1} \quad (1 \leq i \leq D-1),$$

$$E_1 \circ E_D = b_{D-1}^* E_{D-1} + a_D^* E_D.$$ 

Note that a $Q$-polynomial ordering of the eigenvalues/idempotents does not have to be the natural ordering.

Further, the dual eigenvalues of $\Gamma$ associated with $E$ are the real scalars $\theta_i^*$ ($0 \leq i \leq D$) defined by

$$E = \frac{1}{|V(\Gamma)|} \sum_{i=0}^{D} \theta_i^* A_i.$$

We say that a distance-regular graph $\Gamma$ has classical parameters $(D, b, \alpha, \beta)$ if the diameter of $\Gamma$ is $D$, and the intersection numbers of $\Gamma$ satisfy

$$c_i = \begin{bmatrix} i \\ 1 \end{bmatrix} \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}\right), \quad (1)$$

so that, in particular, $c_2 = (b + 1)(\alpha + 1)$,

$$b_i = \left( \begin{bmatrix} D \\ 1 \end{bmatrix} - \begin{bmatrix} i \\ 1 \end{bmatrix} \right) \left( \beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \right), \quad (2)$$

where

$$\begin{bmatrix} j \\ 1 \end{bmatrix} := 1 + b + b^2 + \ldots + b^{j-1}.$$

The following important fact about $Q$-polynomial distance-regular graphs was proven in [7].

**Result 2.1** Let $\Gamma$ be a $Q$-polynomial distance-regular graph with diameter $D \geq 3$. Then, for any $i = 2, \ldots, D - 1$, there exists a polynomial $T_i$ of degree 4 such that, for any vertex $x \in V(\Gamma)$ and any non-principal eigenvalue $\eta$ of the local graph of $x$, $T_i(\eta) \geq 0$ holds. The polynomials $T_i$, $i = 2, \ldots, D - 1$, differ only in a scalar multiple.

We call these polynomials the Terwilliger polynomials of $\Gamma$. The existence of these polynomials was established in [7]. In [4], the polynomial $T_2$ was calculated explicitly.
**Result 2.2** Suppose that $\Gamma$ has classical parameters $(D, b, \alpha, \beta)$. Then the Terwilliger polynomial $T_2(\lambda)$ of $\Gamma$ is

$$T_2(\lambda) = \frac{b_2}{\alpha + 1} \left( - \lambda^2 + \lambda \left( \alpha \binom{D}{1} + \beta - \alpha - 1 - (\alpha + 1)(b + 1) \right) + \beta \binom{D}{1} - (\alpha + 1)(b + 1) \right) \times \left( \lambda^2 + \lambda(2 - \alpha b) - \alpha b + 1 \right) - b_2^2(\lambda + 1)^2. \quad (3)$$

Furthermore, the roots of $T_2(\lambda)$ are

$$\beta - \alpha - 1, \ -1, \ -b - 1, \ alpha \frac{b^{D-1} - 1}{b - 1} - 1.$$

Note that the bilinear forms graph $Bil_q(d \times n)$, $n \geq d$, has classical parameters $(D, b, \alpha, \beta) = (d, q, q - 1, q^n - 1)$. In particular, if $\Gamma$ is a distance-regular graph with the same intersection array as $Bil_q(d \times d)$, $d \geq 3$, then, for any vertex $x \in V(\Gamma)$ and any non-principal eigenvalue $\eta$ of the local graph of $x$, one has:

$$\eta \in [-q - 1, -1] \text{ or } \eta = q^n - q - 1, \quad (4)$$

### 3 Main result

In this section, we suppose that $\Gamma$ is a distance-regular graph with the same intersection array as $Bil_2(d \times d)$, $d \geq 3$.

**Proposition 3.1** The local graph of any vertex $x$ of $\Gamma$ is the $(2^d - 3) \times (2^d - 3)$-grid.

**Proof:** By (4), for $q = 2$, a local non-principal eigenvalue $\eta$ at any vertex $x \in \Gamma$ satisfies:

$$\eta \in [-3, -1] \text{ or } \eta = 2^d - 3.$$

**Claim 3.2** $\Gamma_1(x)$ has only integral eigenvalues, i.e., $-3$, $-2$, $-1$, or $2^d - 3$.

**Proof:** Recall that the eigenvalues of a graph are algebraic integers, and their product is an integer. Let $\eta_1, \ldots, \eta_s$ be all *irrational* eigenvalues of $\Gamma_1(x)$. Then $\eta_i \in (-3, -1)$ and $\Pi_{i=1}^s \eta_i$ is an integer, and thus $\Pi_{i=1}^s (\eta_i + 2)$ is an integer. Now $\eta_i \in (-3, -1) \Rightarrow |\eta_i + 2| < 1 \Rightarrow \Pi_{i=1}^s (\eta_i + 2) = 0$. The claim is proved.

**Claim 3.3** $\Gamma_1(x)$ has spectrum $2(2^n - 2)^1, (2^n - 3)^2(2^n - 2), (-2)^{(2^n-1)^2}$. 
Proof: Recall the following basic fact from algebraic graph theory. Let $\theta_m^0, \theta_m^1, \ldots, \theta_m^s$ be the spectrum of a regular (with valency $k$) graph on $v$ vertices, and $A$ be its adjacency matrix. Then:

$$\sum_{i=0}^{s} m_i v, \quad tr(A) = \sum_{i=0}^{s} m_i \theta_i = 0, \quad tr(A^2) = \sum_{i=0}^{s} m_i \theta_i^2 = vk,$$

where we may put $\theta_0 = k$ and, moreover, $m_0 = 1$ if the graph is connected.

Apply this fact to $\Gamma_1(x)$. In our notation:

$$b_0 = v = (2^n - 1)^2, \quad \theta_0 = k = a_1 = 2(2^n - 2),$$

$$\theta_1 = 2^n - 3, \quad \theta_2 = -1, \quad \theta_3 = -2, \quad \theta_4 = -3,$$

and $m_1, m_2, m_3, m_4$ are unknown multiplicities of $\theta_1, \theta_2, \theta_3, \theta_4$, respectively, while $m_0 = 1$ (as $\Gamma_1(x)$ is connected).

Then (5) gives a system of (three) linear equations with respect to (four) unknowns $m_1, \ldots, m_4$. One can show that this system has the only non-negative integral solution:

$$m_1 = 2(2^n - 2), \quad m_2 = 0, \quad m_3 = (2^n - 1)^2, \quad m_4 = 0,$$

which shows the claim.

We now see that $\Gamma_1(x)$ is a regular graph with exactly 3 distinct eigenvalues. This yields that $\Gamma_1(x)$ is a strongly regular graph with smallest eigenvalue $-2$. It now easily follows from Seidel’s classification of strongly regular graphs with smallest eigenvalue $-2$, see [9], that $\Gamma_1(x)$ is a $(2^d - 3) \times (2^d - 3)$-grid.

Lemma 3.4 For every pair of vertices $x, y \in \Gamma$ with $d(x,y) = 2$, the induced subgraph $\Gamma(x) \cap \Gamma(y)$ is a 6-gon.

Proof: The lemma easily follows from Proposition 3.1 and the fact that $c_2 = 6$.

We now see that $\Gamma$ has the same local graphs as $Bil_2(d \times d)$.

Let $\mathcal{H}$ denote the bilinear forms graph $Bil_2(d \times d)$. For vertices $x \in \mathcal{H}, x \in \Gamma$, an isomorphism $\varphi : x^\perp \rightarrow x^\perp$ is called extendable if there is a bijection $\varphi' : x^\perp \cup \mathcal{H}_2(x) \rightarrow x^\perp \cup \Gamma_2(x)$, mapping edges to edges, such that $\varphi'|_{x^\perp} = \varphi$ (in this case $\varphi'$ is called the extension of $\varphi$). We say that $\Gamma$ has distinct $\mu$-graphs if $\Gamma(x,y) = \Gamma(x,z)$ for $y, z \in \Gamma_2(x)$ implies $y = z$. This property yields that the extension $\varphi'$ above is unique.

A graph $\Delta$ is called triangulable if every cycle in it can be decomposed into a product of triangles (see [6, Section 6]).

For the following result, see [6, Theorem 7.1].
**Result 3.5** Assume:

1. $\Gamma$ has distinct $\mu$-graphs.
2. There exist a vertex $x$ of $\mathcal{H}$ and a vertex $x$ of $\Gamma$, and an extendable isomorphism $\varphi : x^\perp \to x^\perp$.
3. If $x, x$ are vertices of $\mathcal{H}$, $\Gamma$, respectively, $\varphi : x^\perp \to x^\perp$ is an extendable isomorphism, $\varphi'$ is its extension, and $w \in \mathcal{H}(x)$, then $\varphi'|_{w^\perp} : w^\perp \to \varphi(w)^\perp$ is extendable.
4. $\mathcal{H}$ is triangulable.

Then $\Gamma$ is covered by $\mathcal{H}$.

Indeed, since $\Gamma$ and $\mathcal{H}$ have the same intersection arrays, Result 3.5 implies that $\Gamma \cong \mathcal{H}$.

It is not difficult to see that $\Gamma$ satisfies Conditions (1) and (4) of Result 3.5.

Let $\Gamma(x) := \{w_{ij}\}_{i,j}$, and, as usually, for distinct pairs $(i, j)$ and $(i', j')$, $w_{ij} \sim w_{i'j'}$ holds if and only if $i = i'$ or $j = j'$. Denote by $L_i$ the maximal clique of $\Gamma(x)$ that contains the vertices $w_{ij}$ for all $j$, and by $L_j^\top$ the maximal clique of $\Gamma(x)$ that contains the vertices $w_{ij}$ for all $i$. For a vertex $x \in \Gamma$, $x^\perp$ denotes $\{x\} \cup \Gamma(x)$.

Without loss of generality, we may assume that there is a vertex $z \in \Gamma_2(x)$ such that $\Gamma(x, z) \subset L_1 \cup L_2 \cup L_3$. Define a subgraph $\Sigma$ induced in $\Gamma$ by the vertex subset

$$\{x\} \cup L_1 \cup L_2 \cup L_3 \cup \{z' \in \Gamma_2(x) \mid \Gamma(x, z') \subset L_1 \cup L_2 \cup L_3\},$$

so that $\Sigma(x) = L_1 \cup L_2 \cup L_3$.

In order to show that $\Gamma$ satisfies Conditions (2) and (3) of Result 3.5, one has to show the following.

**Lemma 3.6** $\Sigma$ is isomorphic to $\text{Bil}_2(2, d)$.

The main result of this work is the following theorem.

**Theorem 3.7** The bilinear forms graphs $\text{Bil}_2(d, d)$, $d \geq 3$, are uniquely determined by their intersection arrays.

**Acknowledgements.** Part of this work was done while the first author was visiting Tohoku University as a JSPS Postdoctoral Fellow.
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