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The classification of TD-pairs

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1 Introduction

Tridiagonal pairs (TD-pairs) over $\mathbb{C}$ are classified by constructing all of them explicitly as certain sort of tensor product of Leonard pairs (L-pairs). We give a summary of the classification. Details will be published in a joint paper of Makoto Tagami, Paul Terwilliger and myself.

The classification problem of TD-pairs [3] comes from an attempt to establish representation theory of the Terwilliger algebras [12] for P- and Q-polynomial association schemes [1]. A TD-pair arises from each irreducible representation of the Terwilliger algebra of a P- and Q-polynomial association scheme. Strictly speaking, not all TD-pairs appear in this way. The category of TD-pairs is wider than that of irreducible representations of the Terwilliger algebras for P- and Q-polynomial association schemes, yet TD-pairs are the right target of the classification since they capture the essence of irreducible modules for the Terwilliger algebras.

First, we recall the definition of TD-pairs and some basic properties of them, following [3]. Let V be a finite-dimensional vector space over the complex number field $\mathbb{C}$. Let A, A^* be diagonalizable linear transformations of V. By V_i, $0 \leq i \leq d$ (resp. V_i^*, $0 \leq i \leq d^*$), we denote the eigenspaces of A (resp. A^*) and by θ_i, $0 \leq i \leq d$ (resp. θ_i^*, $0 \leq i \leq d^*$), the eigenvalues of A on V_i (resp. A^* on V_i^*). The diagonalizable linear transformations A, A^* of V are called a *TD-pair* if (i) there exists an ordering $V_0^*, V_1^*, \ldots, V_{d^*}^*$ of the eigenspaces of A^* such that

$$AV_i^* \subseteq V_{i-1}^* + V_i^* + V_{i+1}^* \tag{1}$$

for $0 \leq i \leq d^*$, where $V_{-1}^* = V_{d^*+1}^* = 0$, (ii) there exists an ordering $V_0, V_1, \ldots, V_d$ of the eigenspaces of A such that

$$A^*V_i \subseteq V_{i-1} + V_i + V_{i+1} \tag{2}$$

for $0 \leq i \leq d$, where $V_{-1} = V_{d+1} = 0$, and (iii) V contains no proper subspace that is invariant under the actions of both A and A^*.

A TD-pair A, A^* is isomorphic to a TD-pair B, B^* if there exists a vector space isomorphism φ from the underlying vector space of A, A^* to that of B, B^* such that $\varphi A = B\varphi$ and $\varphi A^* = B^*\varphi$.

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For a TD-pair A, A^*, it is known that A, A^* have the same number of eigenspaces, i.e., $d+1 = d^*+1$ and d is called the *diameter* of the TD-pair. A TD-pair is called *trivial* if $d=0$. In what follows, we assume TD-pairs are non-trivial unless otherwise stated.

Let A, A^* be a TD-pair. An ordering $\{V_i\}_{i=0}^d$ (resp. $\{V_i^*\}_{i=0}^d$) of the eigenspaces of A (resp. A^*) is called *standard* if it satisfies (2) (resp. (1)). If $\{V_i\}_{i=0}^d$ (resp. $\{V_i^*\}_{i=0}^d$) is a standard ordering of the eigenspaces of A (resp. A^*), then the reversed ordering $\{V_{d-i}\}_{i=0}^d$ (resp. $\{V_{d-i}^*\}_{i=0}^d$) is standard and A (resp. A^*) has no other standard orderings of the eigenspaces. A TD-pair A, A^* together with a pair $(\{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ of standard orderings for the eigenspaces of A, A^* is called a *TD-system* and denoted by $(A, A^*; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$. If a TD-pair A, A^* is given in advance, a pair $(\{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ of standard orderings for the eigenspaces of A, A^* is called a *TD-system for* A, A^*, allowing abuse of terminology. Thus a TD-pair A, A^* has exactly four TD-systems: if $(\{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ is one of them, then the other three are $(\{V_{d-i}\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$, $(\{V_i\}_{i=0}^d, \{V_{d-i}^*\}_{i=0}^d)$, $(\{V_{d-i}\}_{i=0}^d, \{V_{d-i}^*\}_{i=0}^d)$.

In what follows, A, A^* denote a TD-pair and we fix a TD-system $(\{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ for A, A^*. Define the *weight space* U_i, $0 \le i \le d$, by

$$U_i = (V_0^* + \cdots + V_i^*) \cap (V_i + \cdots + V_d). \tag{3}$$

Then it holds that

$$\dim U_i = \dim V_i = \dim V_i^*, \quad 0 \le i \le d, \tag{4}$$

and V is decomposed into the direct sum of U_i, $0 \le i \le d$. Let us denote the projection onto U_i by $F_i \colon V = \bigoplus_{j=0}^d U_j \longrightarrow U_i$ and set

$$R = A - \sum_{i=0}^d \theta_i F_i, \tag{5}$$

$$L = A^* - \sum_{i=0}^d \theta_i^* F_i, \tag{6}$$

where θ_i (resp. θ_i^*) is the eigenvalue of A (resp. A^*) on V_i (resp. V_i^*). The linear transformation R (resp. L) of V is called the *raising* (resp. *lowering*) map. In fact, it holds that

$$RU_i \subseteq U_{i+1}, \quad LU_i \subseteq U_{i-1}, \quad 0 \le i \le d, \tag{7}$$

where $U_{-1} = U_{d+1} = 0$. Moreover R^{d-2i} (resp. L^{d-2i}) maps U_i onto U_{d-i} (resp. U_{d-i} onto U_i) bijectively. Thus, we have two bijections

$$R^{d-2i} \colon U_i \longrightarrow U_{d-i}, \tag{8}$$

$$L^{d-2i} \colon U_{d-i} \longrightarrow U_i \tag{9}$$

for $0 \le i \le \frac{d}{2}$, and $\dim U_i = \dim U_{d-i}$ holds for $0 \le i \le d$. We set $\rho_i = \dim U_i$, $0 \le i \le d$, and call the sequence $\rho_0, \rho_1, \ldots, \rho_d$ the *shape* of the TD-pair A, A^*. By (4), (7), (8), (9), the shape satisfies $\rho_0 \le \rho_1 \le \cdots \le \rho_{[\frac{d}{2}]}$ and $\rho_i = \rho_{d-i}$, $0 \le i \le d$.

It is conjectured in [3, Conjecture 13.5], and proved in [4], [8, Theorem 1.8] [10, Theorem 1.3], [11, Corollary 1.4] that the shape satisfies

$$\rho_i \le \binom{d}{i}, \quad 0 \le i \le d. \tag{10}$$

In particular, $\rho_0 = 1$ holds, namely the weight space U_0 from (3) has dimension 1. (The fact is that $\rho_0 = 1$ is shown before (10). For more about (10), see [11].) If $\rho_i = 1$, $0 \le i \le d$, the TD-pair is called an *L-pair* (Leonard pair) [13]. The isomorphism classes of L-pairs are in one-to-one correspondence with the Askey-Wilson polynomials with finite support [1], [13]. Define a polynomial $ch(\lambda)$ in λ of degree d by

$$ch(\lambda) = \sum_{i=0}^{d} \rho_i \lambda^i. \tag{11}$$

The polynomial $ch(\lambda)$ is called the *character* of the TD-pair A, A^*. Then A, A^* are an L-pair if and only if $ch(\lambda) = (1-\lambda^{d+1})/(1-\lambda)$. It is conjectured in [3, Conjecture 13.7] that there exist integers $\ell_1, \ell_2, \cdots, \ell_n$ such that

$$ch(\lambda) = \prod_{i=1}^{n} \frac{(1-\lambda^{\ell_i+1})}{(1-\lambda)}. \tag{12}$$

Note that (12) is stronger than (10) and that the equality holds in (10) for every i if and only if $\ell_1 = \ell_2 = \cdots = \ell_d = 1$ with $n = d$ holds in (12). The equation (12) is called the *character formula.* It suggests that a TD-pair is certain sort of tensor product of n L-pairs [3, Conjecture 13.8]. We shall come back to this point later in this section.

For a TD-pair A, A^*, there exist scalars β, γ, γ^*, δ, $\delta^* \in \mathbb{C}$ such that

$$A^3A^* - (\beta+1)(A^2A^*A - AA^*A^2) - A^*A^3 = \gamma(A^2A^* - A^*A^2) + \delta(AA^* - A^*A), \tag{13}$$
$$A^{*3}A - (\beta+1)(A^{*2}AA^* - A^*AA^{*2}) - AA^{*3} = \gamma^*(A^{*2}A - AA^{*2}) + \delta^*(A^*A - AA^*). \tag{14}$$

If $d \ge 3$, the scalar β is uniquely determined. If $d \le 3$, we can choose β arbitrarily. The identities (13), (14) are called *TD-relations* (tridiagonal relations). The eigenvalues θ_i of A on V_i are forced to satisfy

$$\delta = \theta_{i+1}^2 - \beta\theta_{i+1}\theta_i + \theta_i^2 - \gamma(\theta_{i+1} + \theta_i), \quad 0 \le i \le d-1, \tag{15}$$

and the eigenvalues θ_i^* of A^* on V_i^* to satisfy

$$\delta^* = \theta_{i+1}^{*2} - \beta\theta_{i+1}^*\theta_i^* + \theta_i^{*2} - \gamma^*(\theta_{i+1}^* + \theta_i^*), \quad 0 \le i \le d-1. \tag{16}$$

One can easily check that under the identities (15), (16), the TD-relations (13), (14) are rewritten in terms of the raising map R and the lowering map L from (5), (6) as follows:

$$R^3L - (\beta+1)(R^2LR - RLR^2) - LR^3 = \alpha_i R^2 \quad \text{on } U_i, \tag{17}$$
$$L^3R - (\beta+1)(L^2RL - LRL^2) - RL^3 = -\alpha_i L^2 \quad \text{on } U_{i+2} \tag{18}$$

for $0 \le i \le d-2$, where

$$\alpha_i = (\beta+1)\{\theta_i\theta_i^* - \theta_{i+2}\theta_{i+2}^* + (\theta_{i+1}\theta_{i+2}^* + \theta_{i+2}\theta_{i+1}^*) - (\theta_i\theta_{i+1}^* + \theta_{i+1}\theta_i^*)\}. \tag{19}$$

A TD-pair A, A^* is called of *type I*, *type II*, *type III* according to $\beta \ne \pm 2$, $\beta = 2$, $\beta = -2$ in the TD-relations.

Next, we summarize what has been done about the classification of TD-pairs and left open to further studies. Let $(A, A^*; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ be a TD-system. Recall $dim\, U_0 = 1$ by (10). So by (7), L^iR^i acts on U_0 as a scalar $\sigma_i \in \mathbb{C}$:

$$L^iR^iu = \sigma_i u \quad (u \in U_0), \quad 0 \leq i \leq d. \tag{20}$$

The sequence $\{\sigma_i\}_{i=0}^d$ is known in [5, Theorem 1.6, Theorem 4.4], [7, Theorem 3.3], [8, Remark 1.10, Corollary 1.12], [2, Theorem 3.1(iii)] to satisfy

$$\sigma_0 = 1, \quad \sigma_d \neq 0, \tag{21}$$

$$\sum_{i=0}^{d} \frac{\sigma_i}{(\theta_0 - \theta_1)\cdots(\theta_0 - \theta_i)(\theta_0^* - \theta_1^*)\cdots(\theta_0^* - \theta_i^*)} \neq 0, \tag{22}$$

where θ_i (resp. θ_i^*) is the eigenvalue of A on V_i (resp. A^* on V_i^*). Thus from a TD-system $(A, A^*; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$, we get a trio $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\sigma_i\}_{i=0}^d)$ that satisfies (15), (16), (21), (22). The trio determines the isomorphism class of the TD-system. Precisely speaking, if a trio $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\sigma_i\}_{i=0}^d)$ is derived from a TD-system $(A, A^*; \{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ and also from a TD-system $(B, B^*; \{W_i\}_{i=0}^d, \{W_i^*\}_{i=0}^d)$, then there exists a vector space isomorphism φ from the underlying vector space of A, A^* to that of B, B^* such that $\varphi A = B\varphi$, $\varphi A^* = B^*\varphi$ and $\varphi V_i = W_i$ $(0 \leq i \leq d)$. Moreover a trio $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\sigma_i\}_{i=0}^d)$ of sequences of scalars comes from a TD-system if and only if (i) $\theta_i \neq \theta_j$, $\theta_i^* \neq \theta_j^*$ for distinct $i, j \in \{0, 1, \cdots, d\}$, (ii) there exist scalars β, γ, δ, γ^*, δ^* such that $\{\theta_i\}_{i=0}^d$ (resp. $(\{\theta_i^*\}_{i=0}^d)$ satisfies the identities (15) (resp.(16)), and (iii) $\{\sigma_i\}_{i=0}^d$ satisfies (21), (22). Let us call a trio $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\sigma_i\}_{i=0}^d)$ *feasible* if the conditions (i), (ii), (iii) above hold. Then in a word, the following holds.

THEOREM 1. *The isomorphism classes of TD-systems are in one-to-one correspondence with the feasible trios.*

This correspondence is shown in [5, Theorem 1.6, Theorem 1.7] for q-geometric TD-pairs, in [7, Theorem 3.3] for q-Racah TD-pairs, in [8, Corollary 1.12] for generic TD-pairs, i.e., for the case of $\beta = q + q^{-1}$, where q is not a root of unity, and in [2, Theorem 3.1] for all TD-pairs. It is in a way a classification of TD-pairs [1], but it is rather a parametrization theorem, not a structure theorem. It does not answer the following questions:

(Q1) Given a feasible trio $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\sigma_i\}_{i=0}^d)$, how to find the shape $\{\rho_i\}_{i=0}^d$ of a TD-system that corresponds to it, in particular how to show the character formula (12).

(Q2) Given a feasible trio $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\sigma_i\}_{i=0}^d)$, how to construct a TD-system that corresponds to it.

The fact is that the problems (Q1), (Q2) were solved first, Theorem 1 following as a corollary, for q-geometric TD-pairs [5] and then for generic TD-pairs [8]. To solve (Q1) for generic TD-pairs, the paper [8] defines a polynomial by the trio $(\{\theta_i\}_{i=0}^d, \{\theta_i^*\}_{i=0}^d, \{\sigma_i\}_{i=0}^d)$, which is called a Drinfel'd polynomial, and finds the shape $\{\rho_i\}_{i=0}^d$ by analyzing the zeros of the polynomial. As for (Q2), all the generic TD-pairs are explicitly constructed in [8] as certain sort of tensor product of L-pairs that corresponds to the structure of the zeros of the Drinfel'd polynomial.

[1]It turns into a classification of TD-pairs with the aid of [9, Theorem 9.3].

There are two different lines of approaches to the classification problem of TD-pairs. The one that appeared first is the series of papers [4], [5], [6], [8]. This line aims to solve the problems (Q1), (Q2) for each of the three types of TD-pairs separately, assuming the ground field is $\mathbb{C}$, where necessary, to avoid complications caused in the course of arguments. It was so far successful only for generic TD-pairs. The other one is the series of [9], [10], [7], [11], [2] (plus some more related papers) that came later. This line aims to prove Theorem 1 in a uniform way, regardless of the types of TD-pairs and the ground field (so long as it is algebraically closed). It has already accomplished that aim. The problems (Q1), (Q2) are now completely solved over $\mathbb{C}$ for each of the three types of TD-pairs along the lines of [4], [5], [6], [8]. In particular, the character formula is promoted from a conjecture to a theorem. We shall give a summary of the classification of TD-pairs in this sense in the following sections.

2 Standardization

Let A, A^* be a TD-pair. We fix a TD-system $(\{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ for A, A^*. For the parameter β in the tridiagonal relations (13), (14), we set

$$\beta = q^2 + q^{-2}. \tag{23}$$

Then by solving (15), (16) we find the eigenvalues of A, A^* as follows: if $q^2 \neq \pm 1$, i.e., A, A^* are of type I, then

$$\begin{aligned} \theta_i &= c_0 + c_1 q^{2i} + c_2 q^{-2i}, \quad 0 \leq i \leq d, \\ \theta_i^* &= c_0^* + c_1^* q^{2i} + c_2^* q^{-2i}, \quad 0 \leq i \leq d; \end{aligned}$$

if $q^2 = 1$, i.e., A, A^* are of type II, then

$$\begin{aligned} \theta_i &= c_0 + c_1 i + c_2 i^2, \quad 0 \leq i \leq d, \\ \theta_i^* &= c_0^* + c_1^* i + c_2^* i^2, \quad 0 \leq i \leq d; \end{aligned}$$

if $q^2 = -1$, i.e., A, A^* are of type III, then

$$\begin{aligned} \theta_i &= c_0 + (-1)^i (c_1 + c_2 i), \quad 0 \leq i \leq d, \\ \theta_i^* &= c_0^* + (-1)^i (c_1^* + c_2^* i), \quad 0 \leq i \leq d. \end{aligned}$$

In the above expressions of the eigenvalues, c_0, c_1, c_2, c_0^*, c_1^*, c_2^* are some constant from $\mathbb{C}$. If the TD-pair A, A^* is of type III, we need to assume $c_2 \neq 0$, $c_2^* \neq 0$, since the eigenvalues are distinct. For TD-pairs of type I, II, three cases occur: (1) $c_2 \neq 0$, $c_2^* \neq 0$, (2) $c_2 \neq 0$, $c_2^* = 0$ or $c_2 = 0$, $c_2^* \neq 0$, (3) $c_2 = c_2^* = 0$. Accordingly, the TD-pair A, A^* is called the *first*, the *second*, the *third kind.* If it is the second kind, we may assume $c_2 \neq 0$, $c_2^* = 0$ by interchanging A and A^* if necessary.

Observe that if A, A^* are a TD-pair, then affine transformations $\lambda A + \mu$, $\lambda^* A^* + \mu^*$ of them are also a TD-pair ($\lambda, \mu, \lambda^*, \mu^* \in \mathbb{C}$, $\lambda \neq 0$, $\lambda^* \neq 0$). Affine transformations do not change the type of a TD-pair. So we may assume the eigenvalues of A, A^* are *standardized* as follows, by applying some affine transformations or by choosing another TD-system $(\{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$ for A, A^*, if necessary:

type I : With non-zero scalars b, b^* and $(\varepsilon, \varepsilon^*) \in \{(1,1), (1,0), (0,0)\}$,

$$\theta_i = b\, q^{2i-d} + \varepsilon b^{-1} q^{-2i+d}, \quad 0 \le i \le d, \tag{24}$$
$$\theta_i^* = \varepsilon^* b^* q^{2i-d} + b^{*-1} q^{-2i+d}, \quad 0 \le i \le d. \tag{25}$$

type II : With scalars b, b^* and $(\varepsilon, \varepsilon^*) \in \{(1,1), (1,0), (0,0)\}$,

$$\theta_i = \frac{2i-d+b-1}{2}\left(\varepsilon\frac{2i-d+b-1}{2} + 1\right), \quad 0 \le i \le d, \tag{26}$$
$$\theta_i^* = \frac{2i-d+b^*-1}{2}\left(\varepsilon^*\frac{2i-d+b^*-1}{2} + 1\right), \quad 0 \le i \le d. \tag{27}$$

type III : With scalars b, b^*,

$$\theta_i = (-1)^i(2i-d+b), \quad 0 \le i \le d, \tag{28}$$
$$\theta_i^* = (-1)^i(2i-d+b^*), \quad 0 \le i \le d. \tag{29}$$

If A, A^* are of type I or type II, then they are the first, the second, the third kind according to $(\varepsilon, \varepsilon^*) = (1,1), (1,0), (0,0)$. A TD-pair A, A^* is called *standardized*, if their eigenvalues are standardized. It is enough to classify standardized TD-pairs.

3 The TD-algebra $\mathcal{A}$, the augmented TD-algebra $\mathcal{T}$

Let q be a non-zero scalar and $(\varepsilon, \varepsilon^*)$ one of $(1,1)$, $(1,0)$, $(0,0)$. We call q the *base*.

Let $\mathcal{A} = \mathcal{A}_q^{(\varepsilon,\varepsilon^*)}$ denote the associative $\mathbb{C}$-algebra with 1 generated by z, z^* subject to the following defining relations.

case of $q^2 \neq \pm 1$:

$$(\mathrm{TD})_{\mathrm{I}} \begin{cases} z^3 z^* - [3]_q z^2 z^* z + [3]_q z z^* z^2 - z^* z^3 &= -\varepsilon(q^2 - q^{-2})^2 [z, z^*], \\ z^{*3} z - [3]_q z^{*2} z z^* + [3]_q z^* z z^{*2} - z z^{*3} &= -\varepsilon^*(q^2 - q^{-2})^2 [z^*, z], \end{cases} \tag{30}$$

where $[3]_q = (q^3 - q^{-3})/(q - q^{-1}) = q^2 + 1 + q^{-2}$, $[X, Y] = XY - YX$.

case of $q^2 = 1$:

$$(\mathrm{TD})_{\mathrm{II}} \begin{cases} z^3 z^* - 3 z^2 z^* z + 3 z z^* z^2 - z^* z^3 &= 2\varepsilon[z^2, z^*] + (1-\varepsilon)[z, z^*], \\ z^{*3} z - 3 z^{*2} z z^* + 3 z^* z z^{*2} - z z^{*3} &= 2\varepsilon^*[z^{*2}, z] + (1-\varepsilon^*)[z^*, z]. \end{cases} \tag{31}$$

case of $q^2 = -1$:

$$(\mathrm{TD})_{\mathrm{III}} \begin{cases} z^3 z^* + z^2 z^* z - z z^* z^2 - z^* z^3 &= 4[z, z^*], \\ z^{*3} z + z^{*2} z z^* - z^* z z^{*2} - z z^{*3} &= 4[z^*, z]. \end{cases} \tag{32}$$

This algebra $\mathcal{A}$ is called the *TD-algebra of type I, type II, type III* according to the cases of $q^2 \neq \pm 1$, $q^2 = 1$, $q^2 = -1$. If $\mathcal{A}$ is of type I or type II, the algebra $\mathcal{A}$ is called the *first*, the *second*, the *third kind* accorging to $(\varepsilon, \varepsilon^*) = (1,1)$, $(1,0)$, $(0,0)$. The third kind TD-algebra of type I is isomorphic to the positive part of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$ and the third kind TD-algebra of type II is isomorphic to the universal enveloping algebra of the Onsager algebra.

Let $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ denote the associative $\mathbb{C}$-algebra with 1 generated by $x, y, k^{\pm 1}$ subject to the following defining relations.

case of $q^2 \neq \pm 1$:

$$(\mathrm{augTD})_{\mathrm{I}} \begin{cases} kk^{-1} = k^{-1}k = 1, \\ kxk^{-1} = q^2 x, \quad kyk^{-1} = q^{-2}y, \\ x^3y - [3]_q x^2yx + [3]_q xyx^2 - yx^3 = \delta x(\varepsilon^* k^2 - \varepsilon k^{-2})x, \\ y^3x - [3]_q y^2xy + [3]_q yxy^2 - xy^3 = -\delta y(\varepsilon^* k^2 - \varepsilon k^{-2})y, \end{cases} \tag{33}$$

where $\delta = -(q - q^{-1})(q^2 - q^{-2})(q^3 - q^{-3})$.

case of $q^2 = 1$:

$$(\mathrm{augTD})_{\mathrm{II}} \begin{cases} kx - xk = 2x, \quad ky - yk = -2y, \\ x^3y - 3x^2yx + 3xyx^2 - yx^3 = \begin{cases} -12xkx & \text{if } (\varepsilon,\varepsilon^*) = (1,1), \\ -6x^2 & \text{if } (\varepsilon,\varepsilon^*) = (1,0), \\ 0 & \text{if } (\varepsilon,\varepsilon^*) = (0,0), \end{cases} \\ y^3x - 3y^2xy + 3yxy^2 - xy^3 = \begin{cases} 12yky & \text{if } (\varepsilon,\varepsilon^*) = (1,1), \\ 6y^2 & \text{if } (\varepsilon,\varepsilon^*) = (1,0), \\ 0 & \text{if } (\varepsilon,\varepsilon^*) = (0,0). \end{cases} \end{cases} \tag{34}$$

This algebra $\mathcal{T}$ is called the *augmented TD-algebra of type I, type II* according to the cases of $q^2 \neq \pm 1$, $q^2 = 1$. The augmented TD-algebra $\mathcal{T}$ of type I or type II is called the *first*, the *second*, the *third kind* accorging to $(\varepsilon, \varepsilon^*) = (1,1)$, $(1,0)$, $(0,0)$.

Let $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ with $q^2 = -1$, $(\varepsilon, \varepsilon^*) = (1,1)$ denote the associative $\mathbb{C}$-algebra with 1 generated by x, y, k, τ subject to the defining relations of

case of $q^2 = -1$:

$$(\mathrm{augTD})_{\mathrm{III}} \begin{cases} \tau^2 = 1, \\ \tau k - k\tau = 0, \quad \tau x + x\tau = 0, \quad \tau y + y\tau = 0, \\ kx - xk = 2x, \quad ky - yk = -2y, \\ x^3y + x^2yx - xyx^2 - yx^3 = 16xkx, \\ y^3x + y^2xy - yxy^2 - xy^3 = -16yky. \end{cases} \tag{35}$$

The algebra $\mathcal{T}$ is called the *augmented TD-algebra of type III.*

In the rest of this section, we shall explain how TD-pairs are related to finite-dimensional irreducible representations of the algebras $\mathcal{A}$ and $\mathcal{T}$. To observe $\mathcal{A} = \mathcal{A}_q^{(\varepsilon,\varepsilon^*)}$ is embedded in $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$, we first introduce the elements z_t, z_t^* of $\mathcal{T}$ defined as follows. Actually the algebra homomorphism ι_t from $\mathcal{A}$ to $\mathcal{T}$ in the next proposition is injective, but we do not need the injectivity of ι_t for the classification of TD-pairs.

type I : For $t \in \mathbb{C}$ $(t \neq 0)$,

$$z_t = x + tk + \varepsilon t^{-1}k^{-1}, \tag{36}$$
$$z_t^* = y + \varepsilon^* t^{-1}k + tk^{-1}. \tag{37}$$

type II : For $t \in \mathbb{C}$,

$$z_t = x + \frac{k+t-1}{2}(\varepsilon\frac{k+t-1}{2} + 1), \tag{38}$$
$$z_t^* = y + \frac{k-t-1}{2}(\varepsilon^*\frac{k-t-1}{2} + 1). \tag{39}$$

type III : For $t \in \mathbb{C}$,

$$z_t = x + \tau(k+t), \tag{40}$$
$$z_t^* = y + \tau(k-t). \tag{41}$$

PROPOSITION 1. *For each t, there exists an algebra homomorphisn $\iota_t : \mathcal{A}_q^{(\varepsilon,\varepsilon^*)} \longrightarrow \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ that sends z, z^* to z_t, z_t^*, respectively.*

Let V be a finite-dimensional irreducible $\mathcal{T}$-module. Then k is diagonalizable on V. Let us denote the number of the eigenvalues of k by $d+1$. We call the integer d the *diameter* and the pair (q,d) the *ground parameters* of the $\mathcal{T}$-module V, where q is the base we chose at the beginning. The ground parameters (q,d) are defined to be *exceptional* if it satisfies $q^{2(d+1)} = 1$ and $q^2 \neq \pm 1$. So an irreducible $\mathcal{T}$-module V has exceptional ground parameters only when $\mathcal{T}$ is of type I.

Let U_i, $0 \leq i \leq d$, denote the eigenspaces of k on the irreducible $\mathcal{T}$-module V. We call U_i, $0 \leq i \leq d$, the *weight spaces* and $V = \bigoplus_{i=0}^d U_i$ the *weight space decomposition*. The weight spaces are ordered by the action of x, y, namely it holds that

$$xU_i \subseteq U_{i+1}, \quad yU_i \subseteq U_{i-1}, \quad 0 \leq i \leq d, \tag{42}$$

where $U_{-1} = 0$, $U_{d+1} = 0$ if the ground parameters are not exceptional, and $U_{-1} = U_d$, $U_{d+1} = U_0$ if the ground parameters are exceptional. The eigenvalues of k are

$$k|_{U_i} = \begin{cases} sq^{2i-d} & \text{if } \mathcal{T} \text{ is of type I,} \\ s + 2i - d & \text{if } \mathcal{T} \text{ is of type II or type III,} \end{cases} \quad 0 \leq i \leq d \tag{43}$$

for some scalar s. Notice that $q^{2i} \neq 1$ holds for $1 \leq i \leq d$ in the case where $\mathcal{T}$ is of type I.

If $xU_d = 0$ and $yU_0 = 0$, then the $\mathcal{T}$-module V is called *non-circulant*. Notice that an irreducible $\mathcal{T}$-module V is always non-circulant if it does not have exceptional ground parameters. If an irreducible $\mathcal{T}$-module is non-circulant, then the eigenspace U_0 is uniquely determined and so is the scalar s in (43). In this case, the eigenspace U_0 is called the *highest weight space* and the scalar s the *type* of the $\mathcal{T}$-module, while the eigenspace U_d is called the *lowest weight space.*

Let $\mathcal{T}$ be the augmented TD-algebra of type III. Then the weight space U_i, $0 \leq i \leq d$, is invariant under τ and the eigenvalue of τ on U_i is either $(-1)^i$, $0 \leq i \leq d$, or $(-1)^{i+1}$,

$0 \leq i \leq d$. Since the augmented TD-algebra of type III has the automorphism that sends τ to $-\tau$ and keeps x, y, k invariant, we may assume

$$\tau|_{U_i} = (-1)^i, \quad 0 \leq i \leq d. \tag{44}$$

If the ground parameters (q, d) of the $\mathcal{T}$-module V are not exceptional, the $\mathcal{T}$-module V is called *regular*. Notice that the $\mathcal{T}$-module V is always regular if $\mathcal{T}$ is of type II or type III. Assume $\mathcal{T}$ is of type I and the $\mathcal{T}$-module V is regular. Then not only is the $\mathcal{T}$-module V non-circulant and hence the type s is determined, it satisfies

$$x^d yx - xyx^d = (q - q^{-1})^3 \, [d-1] \, [d] \, (\varepsilon^* s^2 - \varepsilon s^{-2}) x^d \quad \text{on } U_0, \tag{45}$$

$$y^d xy - yxy^d = -(q - q^{-1})^3 \, [d-1] \, [d] \, (\varepsilon^* s^2 - \varepsilon s^{-2}) y^d \quad \text{on } U_d, \tag{46}$$

where $[j] = (q^j - q^{-j})/(q - q^{-1})$ and U_0 (resp. U_d) is the highest (resp. lowest) weight space. We extend the range of the teminology 'regular' to the case where the ground parameters (q, d) are exceptional. When the ground parameters (q, d) are exceptional, the $\mathcal{T}$-module V is defined to be *regular* if (i) it is non-circulant and (ii) it satisfies (45), (46).

THEOREM 2. *Let $\mathcal{T}$ be the augmented TD-algebra. Let V be a finite-dimensional irreducible $\mathcal{T}$-module. If the $\mathcal{T}$-module V is regular, then the highest weight space has dimension one: dim $U_0 = 1$.*

Let V be a finite-dimensional vector space over $\mathbb{C}$. Let $A, A^* \in \mathrm{End}\,(V)$ be a standardized TD-pair, where $\mathrm{End}\,(V)$ is the endmorphism algebra consisting of all the linear transformations of V. We fix a TD-system $(\{V_i\}_{i=0}^d, \{V_i^*\}_{i=0}^d)$. Let U_i, $0 \leq i \leq d$, be the weight spaces from (3). The eigenvalues θ_i, θ_i^* of A, A^* are expressed as in (24), (25), (26), (27), (28), (29) according to the type of A, A^*, using the scalars b, b^*. Accordingly we choose scalars s, t as follows:

type I : $b = st, \ b^* = st^{-1}; \ \ s^2 = bb^*, \ t^2 = bb^{*-1}$.

type II, III : $b = s + t, \ b^* = s - t; \ \ 2s = b + b^*, \ 2t = b - b^*$.

Let K denote the diagonalizable linear transformation of V whose eigenspaces coincide with the weight spaces U_i, $0 \leq i \leq d$, as follows:

type I : $K|_{U_i} = sq^{2i-d}, \ 0 \leq i \leq d$.

type II, III : $K|_{U_i} = s + 2i - d, \ 0 \leq i \leq d$.

PROPOSITION 2. *Let A, A^* be a standardized TD-pair. Let $\mathcal{A} = \mathcal{A}_q^{(\varepsilon,\varepsilon^*)}$ and $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ be the TD-algebra and the augmented TD-algebra whose type and kind are in accordance with those of A, A^*. From A, A^*, there arise finite-dimensional irreducible representations of $\mathcal{A}$ and $\mathcal{T}$ as follows.*

(i) There exists a finite-dimensional irreducible representation $\rho_0 : \mathcal{A} \longrightarrow \mathrm{End}(V)$ of $\mathcal{A}$ that sends z, z^* to A, A^*, respectively.

(ii) There exists a finite-dimensional irreducible representation $\rho_1 : \mathcal{T} \longrightarrow \mathrm{End}(V)$ of $\mathcal{T}$ that sends x, y, k to R, L, K, respectively, where R, L are the raising, lowering maps from (5), (6). Moreover, the irreducible $\mathcal{T}$-module V via ρ_1 is regular.

(iii) It holds that $\rho_0 = \rho_1 \circ \iota_t$, where ι_t is from Proposition 1.

A converse of Proposition 2 holds in the following sense. Let $\rho_1 : \mathcal{T} \longrightarrow \mathrm{End}(V)$ be a finite-dimensional irreducible representation of the augmented TD-algebra $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ such that the $\mathcal{T}$-module V via ρ_1 is regular. Let $\mathcal{A} = \mathcal{A}_q^{(\varepsilon,\varepsilon^*)}$ be the TD-algebra. We consider the representation

$$\rho_0 = \rho_1 \circ \iota_t : \mathcal{A} \longrightarrow \mathrm{End}(V) \tag{47}$$

of $\mathcal{A}$, where $\iota_t : \mathcal{A} \longrightarrow \mathcal{T}$ is the algebra homomorphism from Proposition 1. We set

$$A = \rho_0(z), \quad A^* = \rho_0(z^*), \tag{48}$$

where z, z^* are the generators of $\mathcal{A}$. Then A, A^* are diagonalizable linear transformations of V and have the following eigenvalues $\{\theta_i\}_{i=0}^d$, $\{\theta_i^*\}_{i=0}^d$ respectively, according to the type of the TD-algebra $\mathcal{A} = \mathcal{A}_q^{(\varepsilon,\varepsilon^*)}$, where d is the diameter of the $\mathcal{T}$-module V:

type I : With s the type of the $\mathcal{T}$-module V and t the parameter of ι_t,

$$\theta_i = st\, q^{2i-d} + \varepsilon s^{-1}t^{-1}q^{-2i+d}, \quad 0 \le i \le d, \tag{49}$$

$$\theta_i^* = \varepsilon^* st^{-1}q^{2i-d} + s^{-1}tq^{-2i+d}, \quad 0 \le i \le d. \tag{50}$$

type II : With s the type of the $\mathcal{T}$-module V and t the parameter of ι_t,

$$\theta_i = \frac{2i-d+s+t-1}{2}\left(\varepsilon\frac{2i-d+s+t-1}{2} + 1\right), \quad 0 \le i \le d, \tag{51}$$

$$\theta_i^* = \frac{2i-d+s-t-1}{2}\left(\varepsilon^*\frac{2i-d+s-t-1}{2} + 1\right), \quad 0 \le i \le d. \tag{52}$$

type III : With s the type of the $\mathcal{T}$-module V and t the parameter of ι_t,

$$\theta_i = (-1)^i(2i-d+s+t), \quad 0 \le i \le d, \tag{53}$$

$$\theta_i^* = (-1)^i(2i-d+s-t), \quad 0 \le i \le d. \tag{54}$$

PROPOSITION 3. *Let A, A^* be the diagonalizable linear transformations from (48). Assume $\theta_i \neq \theta_j$ and $\theta_i^* \neq \theta_j^*$ for distinct $i, j \in \{0, \cdots, d\}$, where θ_i, θ_i^* are from (49), (50), (51), (52), (53), (54) according to the type of the TD-algebra $\mathcal{A} = \mathcal{A}_q^{(\varepsilon,\varepsilon^*)}$. Then the pair A, A^* satisfies the conditions (i), (ii) for the definition of a TD-pair. In particular, if ρ_0 is irreducible, then A, A^* become a TD-pair.*

Thus the classification problem of standardized TD-pairs is reduced to the following.

PROBLEM 1. *Solve the problems (1), (2), (3) below.*

(1) Construct, up to isomorphism, all the finite-dimensional irreducible representations $\rho_1 : \mathcal{T} \longrightarrow End(V)$ of $\mathcal{T}$ such that the irreducible $\mathcal{T}$-module V via ρ_1 is regular.

(2) For a finite-dimensional irreducible representation ρ_1 of $\mathcal{T}$ constructed in (1) above, determine when the representation $\rho_0 = \rho_1 \circ \iota_t : \mathcal{A} \longrightarrow End(V)$ of $\mathcal{A}$ is irreducible, where ι_t is from Proposition 1.

(3) Determine the isomorphism class of the $\mathcal{A}$-module V via ρ_0 constructed in (2) above.

4 Construction of a $\mathcal{T}$-module V as a tensor product of evaluation modules

Let $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ be the augmented TD-algebra of type I. First we construct an evaluation module for $\mathcal{T}$. Choose a positive integer ℓ and non-zero scalars a, $s \in \mathbb{C}$ arbitrarily. Let $V(\ell, a)$ be a vector space over $\mathbb{C}$ of dimension $\ell + 1$ with a basis $v_0, v_1, \cdots, v_\ell$, and define linear transformations $k(s)$, $x(s)$, $y(s)$ of $V(\ell, a)$ by

$$k(s)v_i = s\,q^{2i-\ell}v_i, \tag{55}$$

$$x(s)v_i = -(q-q^{-1})^2\,[i+1](a\,s + \varepsilon s^{-1}q^{-2i+\ell-1})v_{i+1}, \tag{56}$$

$$y(s)v_i = [\ell - i + 1](\varepsilon^* a^{-1} s\,q^{2i-\ell-1} + s^{-1})v_{i-1}, \tag{57}$$

for $0 \le i \le \ell$, where $[j] = (q^j - q^{-j})/(q - q^{-1})$, $v_{-1} \doteq 0$, $v_{\ell+1} = 0$. If $\mathcal{T}$ is the second kind, ie., $(\varepsilon, \varepsilon^*) = (1, 0)$, then the scalar a is allowed to be zero, in which case the line (57) should be read with the understanding of $\varepsilon^* a^{-1} = 0$. The vector space V is called an *evaluation module* and a the *evaluation parameter.* In fact, V becomes a $\mathcal{T}$-module on which k, x, y act as $k(s)$, $x(s)$, $y(s)$ respectively. The basis $v_0, v_1, \cdots, v_\ell$ is called a *standard basis* of the evaluation module V. The scalar s is called the *type* of the evaluation module V.

Next we construct a $\mathcal{T}$-module as a tensor product of evaluation modules. Let $V(\ell_i, a_i)$, $1 \le i \le n$, be an evaluation module with $v_0^{(i)}, v_1^{(i)}, \cdots, v_{\ell_i}^{(i)}$ a standard basis. Consider the tensor product of the evaluation modules:

$$V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n). \tag{58}$$

The set $\{u_{j_1,\cdots,j_n} \mid 0 \le j_1 \le \ell_1, \cdots, 0 \le j_n \le \ell_n\}$ is a basis of V, where

$$u_{j_1,\cdots,j_n} = v_{j_1}^{(1)} \otimes \cdots \otimes v_{j_n}^{(n)}. \tag{59}$$

Let U_i be the subspace of V spanned by $u_{j_1,\cdots,j_n}$ over all $j_1, \cdots, j_n$ with $j_1 + \cdots + j_n = i$:

$$U_i = \langle u_{j_1,\cdots,j_n} \mid j_1 + \cdots + j_n = i \rangle. \tag{60}$$

Then V is decomposed into the direct sum of U_i, $0 \le i \le d$, where $d = \ell_1 + \cdots + \ell_n$. Note that U_0 (resp. U_d) is a subspace of dimension 1 spanned by $u_{0,\cdots,0}$ (resp. $u_{\ell_1,\cdots,\ell_n}$).

THEOREM 3. *Let $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ be the augmented TD-algebra of type I. For an arbitrary non-zero scalar $s \in \mathbb{C}$, there exists a $\mathcal{T}$-module structure on V such that k, x, y act on the basis $\{u_{j_1,\cdots,j_n} \mid 0 \le j_1 \le \ell_1, \cdots, 0 \le j_n \le \ell_n\}$ from (59) as follows:*

$$k\,u_{j_1,\cdots,j_n} = s\,q^{2i-d}u_{j_1,\cdots,j_n}, \quad i = j_1 + \cdots + j_n, \tag{61}$$

$$x\,u_{j_1,\cdots,j_n} = \sum_{i=1}^{n}(1 \otimes \cdots \otimes 1 \otimes x(s_i) \otimes 1 \otimes \cdots \otimes 1)u_{j_1,\cdots,j_n}, \tag{62}$$

$$y\,u_{j_1,\cdots,j_n} = \sum_{i=1}^{n}(1 \otimes \cdots \otimes 1 \otimes y(s_i) \otimes 1 \otimes \cdots \otimes 1)u_{j_1,\cdots,j_n}, \tag{63}$$

where $s_i = s\,q^{\sum_{\nu=1}^{i-1}(2j_\nu - \ell_\nu)}$, $d = \ell_1 + \cdots + \ell_n$.

For the $\mathcal{T}$-module V constructed above, *we always assume* $q^{2i} \neq 1$, $1 \leq i \leq d$, so that $V = \bigoplus_{i=0}^{d} U_i$ is the eigenspace decomposition of k on V. Note that $k|_{U_i} = s\,q^{2i-d}$, $xU_i \subseteq U_{i+1}$, $yU_i \subseteq U_{i-1}$ for $0 \leq i \leq d$, where $U_{-1} = 0$, $U_{d+1} = 0$. Moreover (45), (46) hold for the $\mathcal{T}$-module V. So if the $\mathcal{T}$-module V is irreducible, it is regular. The non-zero scalar s and the integer d are called the *type* and the *diameter* of the $\mathcal{T}$-module V, respectively.

Let $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ be the augmented TD-algebra of type II, i.e, $q^2 = 1$. Choose a positive integer ℓ and scalars a, $s \in \mathbb{C}$ arbitrarily. The scalars a, s are allowed to be zero. Let $V(\ell, a)$ be a vector space over $\mathbb{C}$ of dimension $\ell + 1$ with a basis $v_0, v_1, \cdots, v_\ell$, and define linear transformations $k(s)$, $x(s)$, $y(s)$ of $V(\ell, a)$ by

$$k(s)v_i = (2i - \ell + s)v_i, \tag{64}$$

$$x(s)v_i = (i+1)(a + \varepsilon(s + \frac{2i - \ell + 1}{2}))v_{i+1}, \tag{65}$$

$$y(s)v_i = (\ell - i + 1)(1 + \varepsilon^*(a - s - \frac{2i - \ell + 1}{2}))v_{i-1} \tag{66}$$

for $0 \leq i \leq \ell$, where $v_{-1} = 0$, $v_{\ell+1} = 0$. The vector space V is called an *evaluation module* and a the *evaluation parameter.* In fact, V becomes a $\mathcal{T}$-module on which k, x, y act as $k(s)$, $x(s)$, $y(s)$ respectively. The basis $v_0, v_1, \cdots, v_\ell$ is called a *standard basis* of the evaluation module V. The scalar s is called the *type* of the evaluation module V.

Let $V(\ell_i, a_i)$, $1 \leq i \leq n$, be an evaluation module with $v_0^{(i)}, v_1^{(i)}, \cdots, v_{\ell_i}^{(i)}$ a standard basis. Consider the tensor product of the evaluation modules $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ as in (58) and the basis $\{u_{j_1,\cdots,j_n} \mid 0 \leq j_1 \leq \ell_1, \cdots, 0 \leq j_n \leq \ell_n\}$ of V as in (59).

THEOREM 4. *Let $\mathcal{T}$ be the augmented TD-algebra of type II. For an arbitrary scalar $s \in \mathbb{C}$, there exists a $\mathcal{T}$-module structure on V such that k, x, y act on the basis $\{u_{j_1,\cdots,j_n} \mid 0 \leq j_1 \leq \ell_1, \cdots, 0 \leq j_n \leq \ell_n\}$ from (59) as follows:*

$$k\,u_{j_1,\cdots,j_n} = (2i - d + s)\,u_{j_1,\cdots,j_n}, \quad i = j_1 + \cdots + j_n, \tag{67}$$

$$x\,u_{j_1,\cdots,j_n} = \sum_{i=1}^{n} (1 \otimes \cdots \otimes 1 \otimes x(s_i) \otimes 1 \otimes \cdots \otimes 1)u_{j_1,\cdots,j_n}, \tag{68}$$

$$y\,u_{j_1,\cdots,j_n} = \sum_{i=1}^{n} (1 \otimes \cdots \otimes 1 \otimes y(s_i) \otimes 1 \otimes \cdots \otimes 1)u_{j_1,\cdots,j_n}, \tag{69}$$

where $s_i = s + \sum_{\nu=1}^{i-1}(2j_\nu - \ell_\nu)$, $d = \ell_1 + \cdots + \ell_n$.

Note that the direct sum decomposition $V = \bigoplus_{i=0}^{d} U_i$ given by (60) is the eigenspace decomposition of k and it holds that $k|_{U_i} = 2i - d + s$, $xU_i \subseteq U_{i+1}$, $yU_i \subseteq U_{i-1}$ for $0 \leq i \leq d$, where $U_{-1} = 0$, $U_{d+1} = 0$. Note also that U_0 (resp. U_d) is a subspace of dimension 1 spanned by $u_{0,\cdots,0}$ (resp. $u_{\ell_1,\cdots,\ell_n}$). The scalar s and the integer d are called the *type* and the *diameter* of the $\mathcal{T}$-module V, respectively.

Let $\mathcal{T}$ be the augmented TD-algebra of type III. Choose a positive integer ℓ and scalars a, $s \in \mathbb{C}$ arbitrarily. The scalars a, s are allowed to be zero. Let $V(\ell, a)$ be a vector space over $\mathbb{C}$ of dimension $\ell + 1$ with a basis $v_0, v_1, \cdots, v_\ell$, and define linear transformations

$\tau(s)$, $k(s)$, $x(s)$, $y(s)$ of $V(\ell, a)$ by

$$\tau(s)v_i = (-1)^i v_i, \tag{70}$$
$$k(s)v_i = (2i - \ell + s)v_i, \tag{71}$$
$$x(s)v_i = \begin{cases} 2(-1)^i\,(i+1)\,v_{i+1} & \text{if } i \text{ is odd,} \\ 2(-1)^i\,(a+s+\frac{2i-\ell+1}{2})\,v_{i+1} & \text{if } i \text{ is even,} \end{cases} \tag{72}$$
$$y(s)v_i = \begin{cases} 2(-1)^{i-1}\,(\ell-i+1)\,v_{i-1} & \text{if } \ell - i \text{ is odd,} \\ 2(-1)^{i-1}\,(a-s-\frac{2i-\ell-1}{2})\,v_{i-1} & \text{if } \ell - i \text{ is even,} \end{cases} \tag{73}$$

for $0 \leq i \leq \ell$, where $v_{-1} = 0$, $v_{\ell+1} = 0$. The vector space V is called an *evaluation module* and a the *evaluation parameter.* In fact, V becomes a $\mathcal{T}$-module on which τ, k, x, y act as $\tau(s)$, $k(s)$, $x(s)$, $y(s)$ respectively. The basis $v_0, v_1, \cdots, v_\ell$ is called a *standard basis* of the evaluation module V. The scalar s is called the *type* of the evaluation module V.

Let $V(\ell_i, a_i)$, $1 \leq i \leq n$, be an evaluation module with $v_0^{(i)}, v_1^{(i)}, \cdots, v_{\ell_i}^{(i)}$ a standard basis. Consider the tensor product of the evaluation modules $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ as in (58) and the basis $\{u_{j_1,\cdots,j_n} \mid 0 \leq j_1 \leq \ell_1, \cdots, 0 \leq j_n \leq \ell_n\}$ of V as in (59).

THEOREM 5. *Let $\mathcal{T}$ be the augmented TD-algebra of type III. For an arbitrary scalar $s \in \mathbb{C}$, there exists a $\mathcal{T}$-module structure on V such that τ, k, x, y act on the basis $\{u_{j_1,\cdots,j_n} \mid 0 \leq j_1 \leq \ell_1, \cdots, 0 \leq j_n \leq \ell_n\}$ from (59) as follows:*

$$\tau\, u_{j_1,\cdots,j_n} = (-1)^i\, u_{j_1,\cdots,j_n}, \quad i = j_1 + \cdots + j_n, \tag{74}$$
$$k\, u_{j_1,\cdots,j_n} = (2i - d + s)\, u_{j_1,\cdots,j_n}, \quad i = j_1 + \cdots + j_n, \tag{75}$$
$$x\, u_{j_1,\cdots,j_n} = \sum_{i=1}^{n} (\tau(s) \otimes \cdots \otimes \tau(s) \otimes x(s_i) \otimes 1 \otimes \cdots \otimes 1) u_{j_1,\cdots,j_n}, \tag{76}$$
$$y\, u_{j_1,\cdots,j_n} = (-1)^d \sum_{i=1}^{n} (-1)^{\sum_{\nu=1}^{i} \ell_\nu} (\tau(s) \otimes \cdots \otimes \tau(s) \otimes y(s_i) \otimes 1 \otimes \cdots \otimes 1) u_{j_1,\cdots,j_n}, \tag{77}$$

where $s_i = s + \sum_{\nu=1}^{i-1}(2j_\nu - \ell_\nu)$, $d = \ell_1 + \cdots + \ell_n$.

Note that the direct sum decomposition $V = \bigoplus_{i=0}^{d} U_i$ given by (60) is the eigenspace decomposition of k and it holds that $\tau|_{U_i} = (-1)^i$, $k|_{U_i} = 2i - d + s$, $xU_i \subseteq U_{i+1}$, $yU_i \subseteq U_{i-1}$ for $0 \leq i \leq d$, where $U_{-1} = 0$, $U_{d+1} = 0$. Note also that U_0 (resp. U_d) is a subspace of dimension 1 spanned by $u_{0,\cdots,0}$ (resp. $u_{\ell_1,\cdots,\ell_n}$). The scalar s and the integer d are called the *type* and the *diameter* of the $\mathcal{T}$-module V, respectively.

REMARK 1. *Theorems 3, 4, 5 suggest that the augmented TD-algebra can be embedded into some bigger algebra equipped with a coproduct.*

(1) The augmented TD-algebra of type I can be embedded into the $U_q(\mathfrak{sl}_2)$-loop algebra $U_q(L(\mathfrak{sl}_2))$. Theorem 3 is obtained through this embedding. For further details, see [8].

(2) The third kind augmented TD-algebra of type II can be embedded into the $\mathfrak{sl}_2$-loop algebra $L(\mathfrak{sl}_2)$, but it is unlikely that $L(\mathfrak{sl}_2)$ is big enough to accommodate the first or the second kind augmented TD-algebra of type II.

(3) *We can define the $U_q(\mathfrak{sl}_2)$-loop algebra $U_q(L(\mathfrak{sl}_2))$ at $q^2 = -1$ in such a way that the augmented TD-algebra of type III is embedded into it and Theorem 5 is obtained through this embedding.*

5 The Drinfel'd polynomial $P_V(\lambda)$ of the $\mathcal{T}$-module V

Let $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ be the augmented TD-algebra. Let $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ be the $\mathcal{T}$-module constructed in Theorem 3, Theorem 4, or Theorem 5 as a tensor product of evaluation modules. Note that the direct sum decomposition $V = \bigoplus_{i=0}^{d} U_i$ given by (60) is the eigenspace decomposition of k, and it holds that (i) $xU_i \subseteq U_{i+1}$, $yU_i \subseteq U_{i-1}$ for $0 \le i \le d$, where $U_{-1} = 0$, $U_{d+1} = 0$, and (ii) $\dim U_0 = 1$. So $y^i x^i$ acts on U_0 as a scalar σ_i for $i \in \mathbb{Z}_{\ge 0}$:

$$y^i x^i|_{U_0} = \sigma_i. \tag{78}$$

Note that $\sigma_0 = 1$ and $\sigma_i = 0$ for $i \ge d+1$.

First assume that $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ is of type I or type II. Let d and s be the diameter and the type of the $\mathcal{T}$-module V, respectively. We define a monic polynomial $P_V(\lambda)$ in λ of degree d, which is called the *Drinfel'd polynomial* of the $\mathcal{T}$-module V. For $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ of type I, i.e., $q^2 \neq \pm 1$, it is defined by

$$P_V(\lambda) = \sum_{i=0}^{d} (-1)^i \frac{\sigma_i}{(q-q^{-1})^{2i}([i]!)^2} \prod_{j=i+1}^{d} (\lambda - \varepsilon s^{-2} q^{2(d-j)} - \varepsilon^* s^2 q^{-2(d-j)}), \tag{79}$$

where $[i] = (q^i - q^{-i})/(q - q^{-1})$, $[i]! = [i][i-1]\cdots[1]$. For $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ of type II, i.e., $q^2 = 1$, it is defined by

$$P_V(\lambda) = \sum_{i=0}^{d} (-1)^i \frac{\sigma_i}{(i!)^2} \prod_{j=i+1}^{d} (\lambda - (j+s-d)^2) \quad \text{if } (\varepsilon, \varepsilon^*) = (1,1), \tag{80}$$

$$P_V(\lambda) = \sum_{i=0}^{d} (-1)^i \frac{\sigma_i}{(i!)^2} \prod_{j=i+1}^{d} (\lambda - (j+s-d)) \quad \textit{if } (\varepsilon, \varepsilon^*) = (1,0), \tag{81}$$

$$P_V(\lambda) = \sum_{i=0}^{d} (-1)^i \frac{\sigma_i}{(i!)^2} \lambda^{d-i} \quad \textit{if } (\varepsilon, \varepsilon^*) = (0,0). \tag{82}$$

Recall we can embed the third kind augmented TD-algebra $\mathcal{T}$ of type I into the $U_q(\mathfrak{sl}_2)$-loop algebra $U_q(L(\mathfrak{sl}_2))$. If the $\mathcal{T}$-module V comes from an irreducible $U_q(L(\mathfrak{sl}_2))$-module V via the embedding, then $\lambda^d P_V(\lambda^{-1})$ turns out to be the original Drinfel'd polynomial up to a scalar multiple, where d is the degree of $P_V(\lambda)$ [5].

Next assume that $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ is of type III, i.e., $q^2 = -1$. Let d and s be the diameter and the type of the $\mathcal{T}$-module V, respectively. We define two *Drinfel'd polynomials* $P_V^{(0)}(\lambda)$, $P_V^{(1)}(\lambda)$ for the $\mathcal{T}$-module V. The first one is the monic polynomial of degree $[\frac{d+1}{2}]$ defined by

$$P_V^{(0)}(\lambda) = \sum_{i=0}^{[\frac{d+1}{2}]} (-1)^i \frac{\sigma_{2i} + 16 i^2 \sigma_{2i-1}}{4^{3i}(i!)^2} \prod_{j=i+1}^{[\frac{d+1}{2}]} (\lambda - (2j-1+s-d)^2). \tag{83}$$

The second one is the monic polynomial of degree $[\frac{d}{2}]$ defined by

$$P_V^{(1)}(\lambda) = \sum_{i=0}^{[\frac{d}{2}]} (-1)^i \frac{\sigma_{2i}}{4^{3i}(i!)^2} \prod_{j=i+1}^{[\frac{d}{2}]} (\lambda - (2j + s - d)^2). \tag{84}$$

Proposition 4. *Let $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ be the augmented TD-algebra and $V(\ell, a)$ the evaluation module for $\mathcal{T}$ constructed in Section 4. The zeros of the Drinfel'd polynomial of the $\mathcal{T}$-module $V(\ell, a)$ are given as follows.*

(1) For $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ of type I,

$$P_{V(\ell,a)}(\lambda) = \prod_{i=0}^{\ell-1} (\lambda + a\, q^{2i-\ell+1} + \varepsilon^* a^{-1} q^{-2i+\ell-1}) \quad \text{if } a \neq 0, \tag{85}$$

$$P_{V(\ell,a)}(\lambda) = \lambda^\ell \quad \text{if } (\varepsilon, \varepsilon^*) = (1, 0) \text{ and } a = 0. \tag{86}$$

(2) For $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ of type II,

$$P_{V(\ell,a)}(\lambda) = \prod_{i=0}^{\ell-1} (\lambda - (a + \frac{2i - \ell + 1}{2})^2) \quad \text{if } (\varepsilon, \varepsilon^*) = (1, 1), \tag{87}$$

$$P_{V(\ell,a)}(\lambda) = \prod_{i=0}^{\ell-1} (\lambda + a + \frac{2i - \ell + 1}{2}) \quad \text{if } (\varepsilon, \varepsilon^*) = (1, 0), \tag{88}$$

$$P_{V(\ell,a)}(\lambda) = (\lambda + a)^\ell \quad \text{if } (\varepsilon, \varepsilon^*) = (0, 0). \tag{89}$$

(3) For $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ of type III,

$$P_{V(\ell,a)}^{(0)}(\lambda) = \prod_{\substack{0 \leq i \leq \ell - 1 \\ i \equiv \ell - 1 \bmod 2}} (\lambda - (a + \frac{2i - \ell + 1}{2})^2), \tag{90}$$

$$P_{V(\ell,a)}^{(1)}(\lambda) = \prod_{\substack{0 \leq i \leq \ell - 1 \\ i \equiv \ell \bmod 2}} (\lambda - (a + \frac{2i - \ell + 1}{2})^2). \tag{91}$$

Theorem 6. *Let $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ be the augmented TD-algebra. Let $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ be the $\mathcal{T}$-module constructed in Theorem 3, Theorem 4, or Theorem 5 as a tensor product of evaluation modules. The zeros of the Drinfel'd polynomial of the $\mathcal{T}$-module V are given as follows with the aid of Proposition 4. In particular, the type s of the $\mathcal{T}$-module V does not affect the Drinfel'd polynomial, although it appears in the definition of it.*

(1) For $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ of type I or type II,

$$P_V(\lambda) = \prod_{i=1}^{n} P_{V(\ell_i,a_i)}(\lambda). \tag{92}$$

(2) For $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ of type III,

$$P_V^{(0)}(\lambda) = \prod_{i=1}^{n} P_{V(\ell_i,a_i)}^{(\eta_i)}(\lambda), \tag{93}$$

$$P_V^{(1)}(\lambda) = \prod_{i=1}^{n} P_{V(\ell_i,a_i)}^{(1-\eta_i)}(\lambda), \tag{94}$$

where $\eta_i \equiv \sum_{\nu=i+1}^{n} \ell_\nu$ mod 2 ($\eta_i \in \{0,1\}$) with the understanding of $\eta_n = 0$.

6 The irreducibility and the isomorphism class of the T-module V

Let $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ be the augmented TD-algebra. Let $V(\ell,a)$ be the evaluation module for $\mathcal{T}$ constructed in Section 4.

If $\mathcal{T}$ is of type I, we define a *q-string* $S(\ell,a)$ associated with $V(\ell,a)$ to be

$$S(\ell,a) = \{aq^{2i-\ell+1} \mid 0 \leq i \leq \ell-1\}. \tag{95}$$

If $(\varepsilon,\varepsilon^*) = (1,0)$, notice that a is allowed to be zero, in which case we understand that $S(\ell,0)$ is a multi-set of ℓ zeros. The positive integer ℓ is called the *length* of the q-string $S(\ell,a)$. Two q-strings $S(\ell,a)$ and $S(\ell',a')$ are said to be *in general position* if either (i) one is contained in the other as sets or (ii) the union as sets of the two q-strings does not make up a q-string. So two q-strings are not in general position if and only if the union as sets of the two q-strings becomes a q-string which has a longer length than that of each of the two. In the case of $(\varepsilon,\varepsilon^*) = (1,0)$, we understand that $S(\ell,0)$ is in general position with any $S(\ell',a')$ ($a' \neq 0$) and not in general position with any $S(\ell',0)$.

If $\mathcal{T}$ is of type II, we define a *string* $S(\ell,a)$ associated with $V(\ell,a)$ to be

$$S(\ell,a) = \{a + \frac{2i-\ell+1}{2} \mid 0 \leq i \leq \ell-1\}. \tag{96}$$

The positive integer ℓ is called the *length* of the string $S(\ell,a)$. Two strings $S(\ell,a)$ and $S(\ell',a')$ are said to be *in general position* if either (i) one is contained in the other as sets or (ii) the union as sets of the two strings does not make up a string.

If $\mathcal{T}$ is of type III, we define a *signed string* $S^{(\eta)}(\ell,a)$ associated with $V(\ell,a)$ to be

$$S^{(\eta)}(\ell,a) = \{(a + \frac{2i-\ell+1}{2}, (-1)^{\eta+\ell-1-i}) \mid 0 \leq i \leq \ell-1\}, \tag{97}$$

where $\eta \in \{0,1\}$. The positive integer ℓ is called the *length* of the signed string $S^{(\eta)}(\ell,a)$. Two signed strings $S^{(\eta)}(\ell,a)$ and $S^{(\eta')}(\ell',a')$ are said to be *in general position* if either (i) one is contained in the other as sets or (ii) the union as sets of the two signed strings does not make up a signed string. The *transpose* of a signed string $S^{(\eta)}(\ell,a)$, which is denoted by ${}^tS^{(\eta)}(\ell,a)$, is defined to be

$${}^tS^{(\eta)}(\ell,a) = \{(-a - \frac{2i-\ell+1}{2}, (-1)^{\eta+\ell-1-i}) \mid 0 \leq i \leq \ell-1\}. \tag{98}$$

Note that ${}^tS^{(\eta)}(\ell,a) = S^{(\eta)}(\ell,-a)$ if ℓ is odd, and ${}^tS^{(\eta)}(\ell,a) = S^{(1-\eta)}(\ell,-a)$ if ℓ is even.

THEOREM 7. *Let $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ be the augmented TD-algebra. Let $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ be the $\mathcal{T}$-module constructed in Theorem 3, Theorem 4, or Theorem 5 as a tensor product of evaluation modules. Let d and s be the diameter and the type of the $\mathcal{T}$-module V. Let $P_V(\lambda)$ (resp. $P_V^{(\eta)}(\lambda)$ for $\eta = 0, 1$) be the Drinfel'd polynomial of the $\mathcal{T}$-module V in the case of type I or type II (resp. type III).*

(1) *Assume that $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ is of type I. The $\mathcal{T}$-module V is irreducible if and only if the following (a) and (b) hold:*

   (a) *$P_V(\lambda)$ does not vanish at $\lambda = \varepsilon s^{-2} + \varepsilon^* s^2$, i.e., $\sigma_d \neq 0$.*

   (b) *For all distinct $i, j \in \{1, \cdots, n\}$, the q-strings $S(\ell_i, a_i^{\varepsilon_i})$, $S(\ell_j, a_j^{\varepsilon_j})$ are in general position for any $\varepsilon_i, \varepsilon_j \in \{\pm 1\}$ in the case of $(\varepsilon, \varepsilon^*) = (1, 1)$; the q-strings $S(\ell_i, a_i)$, $S(\ell_j, a_j)$ are in general position in the case of $(\varepsilon, \varepsilon^*) = (1, 0)$ or $(0, 0)$.*

(2) *Assume that $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ is of type II. The $\mathcal{T}$-module V is irreducible if and only if the following (a) and (b) hold:*

   (a) *$P_V(\lambda)$ does not vanish at $\lambda = \varepsilon\varepsilon^* s^2 + \varepsilon(1 - \varepsilon^*)s$, i.e., $\sigma_d \neq 0$.*

   (b) *For all distinct $i, j \in \{1, \cdots, n\}$, the strings $S(\ell_i, \varepsilon_i a_i)$, $S(\ell_j, \varepsilon_j a_j)$ are in general position for any $\varepsilon_i, \varepsilon_j \in \{\pm 1\}$ in the case of $(\varepsilon, \varepsilon^*) = (1, 1)$; the strings $S(\ell_i, a_i)$, $S(\ell_j, a_j)$ are in general position in the case of $(\varepsilon, \varepsilon^*) = (1, 0)$; $a_i \neq a_j$ in the case of $(\varepsilon, \varepsilon^*) = (0, 0)$.*

(3) *Assume that $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ is of type III. The $\mathcal{T}$-module V is irreducible if and only if the following (a) and (b) hold:*

   (a) *$P_V^{(\eta)}(\lambda)$ does not vanish at $\lambda = s^2$ for $\eta \equiv d + 1 \bmod 2$, i.e., $\sigma_d \neq 0$.*

   (b) *For all distinct $i, j \in \{1, \cdots, n\}$, the signed strings $S^{(\eta_i)}(\ell_i, a_i)$, $S^{(\eta_j)}(\ell_j, a_j)$ are in general position and the signed strings $S^{(\eta_i)}(\ell_i, a_i)$, ${}^tS^{(\eta_j)}(\ell_j, a_j)$ are in general position, where $\eta_i \equiv \sum_{\nu=i+1}^{n} \ell_\nu \bmod 2$, $\eta_j \equiv \sum_{\nu=j+1}^{n} \ell_\nu \bmod 2$.*

Note that the irreducible $\mathcal{T}$-modules V that appear in Theorem 7 are regular.

THEOREM 8. *Let $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ be the augmented TD-algebra. Let $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$, $V' = V(\ell'_1, a'_1) \otimes \cdots \otimes V(\ell'_{n'}, a'_{n'})$ be $\mathcal{T}$-modules constructed in Theorem 3, Theorem 4, or Theorem 5 as a tensor product of evaluation modules. Let s, s' be the types of the $\mathcal{T}$-modules V, V', respectively. Assume that V and V' are irreducible as $\mathcal{T}$-modules. Then the $\mathcal{T}$-modules V and V' are isomorphic if and only if (a) $s = s'$ and (b) $P_V(\lambda) = P_{V'}(\lambda)$ in the case of type I or type II; $P_V^{(\eta)}(\lambda) = P_{V'}^{(\eta)}(\lambda)$ for $\eta = 0, 1$ in the case of type III. The condition (b) is equivalent to saying that there exist a permutation π of $1, \cdots, n$ such that the following holds for $1 \leq i \leq n$:*

(1) *In the case of type I,*

$$S(\ell'_i, a'_i) = S(\ell_{\pi(i)}, a_{\pi(i)}^{\varepsilon_{\pi(i)}}) \text{ for some } \varepsilon_{\pi(i)} \in \{\pm 1\} \text{ if } (\varepsilon, \varepsilon^*) = (1, 1)\ ;$$
$$S(\ell'_i, a'_i) = S(\ell_{\pi(i)}, a_{\pi(i)}) \text{ if } (\varepsilon, \varepsilon^*) = (1, 0) \text{ or } (0, 0).$$

(2) *In the case of type II,*

$$S(\ell'_i, a'_i) = S(\ell_{\pi(i)}, \varepsilon_{\pi(i)} a_{\pi(i)}) \text{ for some } \varepsilon_{\pi(i)} \in \{\pm 1\} \text{ if } (\varepsilon, \varepsilon^*) = (1,1);$$
$$S(\ell'_i, a'_i) = S(\ell_{\pi(i)}, a_{\pi(i)}) \text{ if } (\varepsilon, \varepsilon^*) = (1,0) \text{ or } (0,0).$$

(3) *In the case of type III,*

$$S^{(\eta'_i)}(\ell'_i, a'_i) = S^{(\eta_{\pi(i)})}(\ell_{\pi(i)}, a_{\pi(i)}) \text{ or } {}^tS^{(\eta_{\pi(i)})}(\ell_{\pi(i)}, a_{\pi(i)}),$$
$$\text{where } \eta'_i \equiv \textstyle\sum_{\nu=i+1}^n \ell'_\nu \bmod 2,\ \eta_j \equiv \sum_{\nu=j+1}^n \ell_\nu \bmod 2.$$

THEOREM 9. *Let $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ be the augmented TD-algebra of type I, type II or type III. For any finite-dimensional irreducible $\mathcal{T}$-module V that is regular, there exist evaluation modules $V(\ell_i, a_i)$, $1 \leq i \leq n$, for some n such that the $\mathcal{T}$-module V is isomorphic to the $\mathcal{T}$-module $V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$ that is constructed in Theorem 3, Theorem 4, or Theorem 5 as a tensor product of evaluation modules.*

7 The classification of standardized TD-pairs

Let $\mathcal{T} = \mathcal{T}_q^{(\varepsilon,\varepsilon^*)}$ be the augmented TD-algebra. Let V be a finite-dimensional irreducible T-module that is regular. By Theorem 9, we may assume V is one of the $\mathcal{T}$-modules constructed in Theorem 3, Theorem 4, or Theorem 5 as a tensor product of evaluation modules: $V = V(\ell_1, a_1) \otimes \cdots \otimes V(\ell_n, a_n)$. Let $P_V(\lambda)$ (resp. $P_V^{(\eta)}(\lambda)$ for $\eta = 0, 1$) be the Drinfel'd polynomial of the $\mathcal{T}$-module V in the case where $\mathcal{T}$ is of type I or type II (resp. type III). We denote the diameter and the type of the T-module V by d and s, respectively. Recall that $q^{2i} \neq 1$, $1 \leq i \leq d$, is assumed if $\mathcal{T}$ is of type I.

Let $\rho_1 : \mathcal{T} \longrightarrow \mathrm{End}(V)$ be the finite-dimensional irreducible representation of $\mathcal{T}$ afforded by the $\mathcal{T}$-module V. Let $\mathcal{A} = \mathcal{A}_q^{(\varepsilon,\varepsilon^*)}$ be the TD-algebra. As in (47), we consider the representation $\rho_0 = \rho_1 \circ \iota_t : \mathcal{A} \longrightarrow \mathrm{End}(V)$ of $\mathcal{A}$, where $\iota_t : \mathcal{A} \longrightarrow \mathcal{T}$ is the algebra homomorphism from Proposition 1.

THEOREM 10. *Let $A = \rho_0(z)$, $A^* = \rho_0(z^*)$ be the diagonalizable linear transformations from (48). Assume $\theta_i \neq \theta_j$ and $\theta^*_i \neq \theta^*_j$ for distinct $i, j \in \{0, \cdots, d\}$, where θ_i, θ^*_i are from (49), (50), (51), (52), (53), (54) according to the type of the TD-algebra $\mathcal{A} = \mathcal{A}_q^{(\varepsilon,\varepsilon^*)}$.*

(1) *The representation ρ_0 of $\mathcal{A}$ is irreducible if and only if $P_V(\lambda)$ does not vanish at $\lambda = t^2 + \varepsilon\varepsilon^* t^{-2}$ in the case of type I; $P_V(\lambda)$ does not vanish at $\lambda = \varepsilon\varepsilon^* t^2 - \varepsilon(1-\varepsilon^*)t - (1-\varepsilon)(1-\varepsilon^*)$ in the case of type II; $P_V^{(0)}(\lambda)$ does not vanish at $\lambda = t^2$ in the case of type III.*

(2) *If ρ_0 is irreducible, then the pair A, A^* becomes a standardized TD-pair. This construction by means of (48) exhausts all the standardized TD-pairs. Moreover in this construction, the isomorphism class of a standardized TD-pair is in one-to-one correspondence with the trio $(\{\theta_i\}_{i=0}^d, \{\theta^*_i\}_{i=0}^d, P_V(\lambda))$ in the case of type I or type II, the trio $(\{\theta_i\}_{i=0}^d, \{\theta^*_i\}_{i=0}^d, (P_V^{(0)}(\lambda), P_V^{(1)}(\lambda)))$ in the case of type III. More precisely, if a construction produces a standardized TD-pair and a trio, then any other construction that results in the same standardized TD-pair up to isomorphism (resp. the same trio) has to produce the same trio (resp. the same standardized TD-pair up to isomorphism).*

REMARK 2. By Theorem10, we define the *Drinfel'd polynomial* of a standardized TD-pair A, A^* to be the polynomial $P_V(\lambda)$ in the case of type I and type II; to be the pair $(P_V^{(0)}(\lambda), P_V^{(1)}(\lambda))$ of polynomials in the case of type III. This definition agrees with that of [6] for type I and type II, but not for type III; the definition of [6] needs to be amended in the case of type III.

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