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GROUP ACTIONS ON SYMMETRIC SPACES RELATED TO LEFT-INVARIANT GEOMETRIC STRUCTURES

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ABSTRACT. In this paper, we summarize how the theory and results of group actions on symmetric spaces can be applied to the study of left-invariant geometric structures on Lie groups. We also present a list of problems on group actions, which naturally arise from this framework.

1. INTRODUCTION

Isometric actions on Riemannian symmetric spaces of noncompact type, such as cohomogeneity one actions and (hyper)polar actions, have been studied actively in these decades (see [1–7, 19] and references therein). The theory and results of these actions have recently been applied to the study of left-invariant geometric structures on Lie groups ([10–12, 17, 18, 38]). The aim of this survey paper is to present our framework and recent results. We also propose several problems on group actions, which naturally arise from our framework. The answers of these problems would be interesting, not only from the viewpoint of group actions and submanifold geometry, but also for possible applications to the further studies on left-invariant geometric structures.

Left-invariant geometric structures on Lie groups, such as (pseudo-)Riemannian metrics, symplectic structures, and (generalized) complex structures, have provided a lot of interesting examples, and have been studied very actively (for left-invariant metrics, we refer to [8, 9, 13–16, 20, 22–27, 29–36, 39–44]). One of the central problems is the following existence and nonexistence problem.

Problem 1.1. For a given Lie group, determine whether it admits "nice" left-invariant geometric structures or not.

Note that, for a given Lie group and a given left-invariant structure, one can directly study the properties of them. For example, the following can be studied in the Lie algebra level — curvatures of a left-invariant metric, the integrability condition of a left-invariant almost complex structure, and so on. However, this does not mean that the above mentioned problem is easy. One of the difficulties comes from the fact that a Lie group admits so many left-invariant geometric structures. For left-invariant metrics, one knows the following.

Fact 1.2. *Let G be a Lie group of dimension n. Then there are identifications*

$$\begin{aligned}
\widetilde{\mathfrak{M}} &:= \{\textit{left-invariant Riemannian metrics on } G\} \\
&\cong \{\textit{inner products on } \mathfrak{g} := \mathrm{Lie}(G)\} \\
&\cong \mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n), \\
\widetilde{\mathfrak{M}}_{(p,q)} &:= \{\textit{left-invariant metrics on } G \textit{ with signature } (p,q)\} \\
&\cong \mathrm{GL}(n,\mathbb{R})/\mathrm{O}(p,q).
\end{aligned}$$

Therefore, for studying the existence and nonexistence of "nice" left-invariant metrics, such as Einstein or Ricci soliton, one has to study all points on the above spaces. For left-invariant metrics, of course the Einstein equation is a linear equation, but it contains $(1/2)n(n+1)$ variables, which is in general very hard to be solved. In order to avoid this difficulty, we have proposed an approach from the following viewpoint.

Fact 1.3. *The above spaces $\widetilde{\mathfrak{M}}$ and $\widetilde{\mathfrak{M}}_{(p,q)}$ are symmetric spaces (the former is noncompact Riemannian, but the latter is pseudo-Riemannian). Furthermore, there are natural actions of the automorphism groups of the Lie algebras on these spaces.*

This connects naturally the studies of left-invariant geometric structures and of group actions on symmetric spaces.

Remark 1.4. We only mention left-invariant metrics in this paper, but believe that similar frameworks also work for other left-invariant geometric structures. In many cases, the set of some geometric structures forms a symmetric space.

Isometric actions on Riemannian symmetric spaces of compact type are certainly interesting topics, and have been studied by many authors. We would like to say that, isometric actions on Riemannian symmetric spaces of noncompact type, and group actions on pseudo-Riemannian symmetric spaces, are also interesting topics. They are of course interesting from the viewpoint of group actions and submanifold geometry, and also interesting because of possible applications to the studies on left-invariant geometric structures.

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2. Isometric actions on Riemannian symmetric spaces of noncompact type (1)

Let us consider isometric actions of H on Riemannian symmetric spaces M of noncompact type. In this section, we consider the orbit spaces $H\backslash M$. We mention some known results, and their applications to the study of left-invariant Riemannian metrics on Lie groups.

2.1. **Results on cohomogeneity one actions.** First of all, we recall some fundamental notions on group actions, and review a result on cohomogeneity one actions.

Definition 2.1. Consider an isometric action of a Lie group H on a Riemannian manifold M. Then, orbits of maximal dimension are said to be *regular*, and other orbits *singular.* The codimension of a regular orbit is called the *cohomogeneity* of the action.

For cohomogeneity one actions on symmetric spaces of noncompact type, possible topological types of the orbit spaces have been studied.

Theorem 2.2 (Berndt-Brück ([1])). *Let M be a Riemannian symmetric space of noncompact type, and consider a cohomogeneity one action of H on M with H being connected. Then, the orbit space $H\backslash M$ is homeomorphic to either $\mathbb{R}$ or $[0,+\infty)$.*

As an example, we here draw a picture of this situation for cohomogeneity one actions on the real hyperbolic plane

$$\mathbb{R}\mathrm{H}^2 = \mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2).$$

Example 2.3. Consider the Iwasawa decomposition $\mathrm{SL}(2,\mathbb{R}) = KAN$, where

$$K = \mathrm{SO}(2), \quad A = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \mid a > 0 \right\}, \quad N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{R} \right\}.$$

Then, the actions of K, A, and N on $\mathbb{R}\mathrm{H}^2$ are of cohomogeneity one (and in fact they exhaust all, up to orbit equivalence). The orbits and the orbit spaces of these actions are as in Figure 1. More precisely,

(K) The action of K has a fixed point, the center o. Other orbits are circles centered at o. In this case, the orbit space is homeomorphic to $[0,+\infty)$.

(A) The orbit of A through the center o is a geodesic, and other orbits are its equidistant lines. The orbit space is homeomorphic to $\mathbb{R}$.

(N) The orbits of N are horocycles. In this case, all orbits are congruent to each other, and the orbit space is homeomorphic to $\mathbb{R}$.

2.2. **Results on left-invariant metrics.** We now consider $\widetilde{\mathfrak{M}}$, the set of all left-invariant Riemannian metrics on a given Lie group G. Assume that G is simply-connected, for simplicity.

Remark 2.4. Denote by $\mathfrak{g}$ the Lie algebra of G. Let $n := \dim \mathfrak{g}$, and fix a basis of $\mathfrak{g}$. Then one has identifications

$$\mathfrak{g} \cong \mathbb{R}^n, \quad \mathrm{GL}(\mathfrak{g}) \cong \mathrm{GL}(n,\mathbb{R}), \quad \widetilde{\mathfrak{M}} \cong \mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n).$$

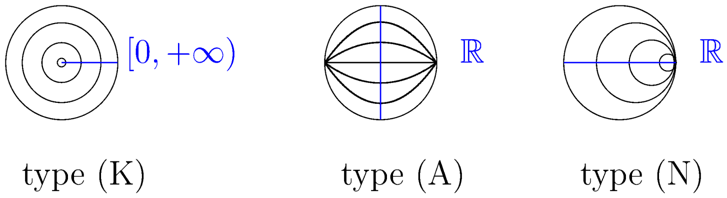

FIGURE 1. The orbits and the orbit spaces of actions on $\mathbb{R}\mathrm{H}^2$

Then, it is well-known that $\widetilde{\mathfrak{M}}$ equipped with the canonical $\mathrm{GL}(n, \mathbb{R})$-invariant Riemannian metric is a Riemannian symmetric space, which is noncompact. In this paper, this Riemannian metric is always assumed to be equipped.

We consider the group actions on $\widetilde{\mathfrak{M}}$ given by

$$\mathbb{R}^\times := \{c \cdot \mathrm{id} : \mathfrak{g} \to \mathfrak{g} \mid c \in \mathbb{R}_{\neq 0}\},$$
$$\mathrm{Aut}(\mathfrak{g}) := \{\varphi \in \mathrm{GL}(\mathfrak{g}) \mid \varphi([\cdot, \cdot]) = [\varphi(\cdot), \varphi(\cdot)]\}.$$

Definition 2.5. The orbit space of the action of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ on $\widetilde{\mathfrak{M}}$,

$$\mathfrak{PM} := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}},$$

is called the *moduli space* of left-invariant Riemannian metrics on G.

Remark 2.6. The action of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ gives rise to isometry up to scaling of left-invariant Riemannian metrics. Therefore, all Riemannian geometric properties of left-invariant metrics are preserved by this action. Thus, in order to examine the existence and nonexistence of a "nice" metric, one has only to study the moduli space $\mathfrak{PM}$.

We here give one easy example of a description of the moduli space $\mathfrak{PM}$. Throughout this paper, the canonical basis of $\mathfrak{g} = \mathbb{R}^n$ is denoted by $\{e_1, \ldots, e_n\}$.

Proposition 2.7 (Hashinaga-Tamaru-Terada ([12])). *Consider the Lie algebra* $\mathfrak{g} := (\mathbb{R}^3, [,])$ *with* $[e_1, e_2] = e_2$ *(and others are zero). Denote by* $\langle, \rangle_0$ *the canonical inner product on* $\mathfrak{g} = \mathbb{R}^3$*. Then one has*

(1) *the action of* $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ *on* $\widetilde{\mathfrak{M}}$ *is of cohomogeneity one,*
(2) *the moduli space* $\mathfrak{PM}$ *can be expressed as*

$$\mathfrak{PM} = \left\{ \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).(\begin{pmatrix} 1 & & \\ & 1 & \\ & \lambda & 1 \end{pmatrix} .\langle, \rangle_0) \mid \lambda \in \mathbb{R} \right\}.$$

Remark 2.8. In order to give an expression of $\mathfrak{PM}$, one needs direct matrix calculations. On the other hand, one knows possible topological types of the orbit spaces in Theorem 2.2. This general theory of cohomogeneity one actions does not give an expression of $\mathfrak{PM}$, but gives us a certification.

This expression of the moduli space $\mathfrak{PM}$ provides the following Milnor-type theorem. The reason of this naming is that the basis mentioned below is a kind of a generalization of "Milnor frames", obtained by Milnor ([29]).

Proposition 2.9 ([12]). *Let $\mathfrak{g} := (\mathbb{R}^3, [,])$ be the Lie algebra in Proposition 2.7, and $\langle,\rangle$ be an arbitrary inner product on $\mathfrak{g}$. Then, there exist $\lambda \in \mathbb{R}$, $k > 0$, and an orthonormal basis $\{x_1, x_2, x_3\}$ with respect to $k\langle,\rangle$ such that the nontrivial bracket relations are given by*

$$[x_1, x_2] = x_2 - \lambda x_3.$$

When we apply this Milnor-type theorem, one can assume $k = 1$ without loss of generality, since $k > 0$ is just a scaling factor. In terms of the basis $\{x_1, x_2, x_3\}$, one can directly calculate the curvatures, which proves the following.

Corollary 2.10 ([12]). *The Lie algebra in Proposition 2.7 does not admit left-invariant Einstein metrics, and furthermore, it does not admit left-invariant metrics of negative Ricci curvature.*

The above Lie algebra is just an easy example. We have studied some other Lie algebras as well, including the following ones.

Remark 2.11. Milnor-type theorems have been obtained for

(1) all three-dimensional solvable Lie algebras ([11]),
(2) all four-dimensional nilpotent Lie algebras ([10]),
(3) some higher-dimensional Lie algebras ([10, 12, 38]).

It is remarkable that the action of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ on $\widetilde{\mathfrak{M}}$ can be of cohomogeneity one, even if the dimension of $\mathfrak{g}$ is high. In fact, for any $n \geq 3$, there exists a Lie algebra $\mathfrak{g}$ of dimension n whose corresponding action is of cohomogeneity one.

2.3. **Problems.** In the last of this section, we propose some problems concerning with the orbit spaces of isometric actions on symmetric spaces of noncompact type.

Problem 2.12. For Riemannian symmetric spaces of noncompact type, classify possible topological type of orbit spaces of some particular classes of actions (for examples, cohomogeneity two, (hyper)polar, and so on).

The answers of the above problem would be useful to obtain Milnor-type theorems for more complicated Lie algebras. On the Lie algebra side, the following problems would be natural.

Problem 2.13. Classify Lie algebras $\mathfrak{g}$ so that the actions of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ on $\widetilde{\mathfrak{M}}$ have some particular properties (for examples, cohomogeneity one or two, (hyper)polar, and so on).

Note that Taketomi ([37]) has constructed Lie algebras $\mathfrak{g}$ so that the actions of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ are hyperpolar, having a singular orbit and higher cohomogeneity. In this way, the study of left-invariant metrics would also contribute to the study of isometric actions on symmetric spaces.

3. Isometric actions on Riemannian symmetric spaces of noncompact type (2)

In this section, we continue to consider isometric actions on Riemannian symmetric spaces of noncompact type. We here focus on the geometry of the orbits, in particular, some "distinguished" orbits.

3.1. **Results on cohomogeneity one actions.** First of all, we recall a result on cohomogeneity one actions. The following is not explicitly written in [4], but can be seen from the classification result (we also refer to [7]).

Theorem 3.1 (Berndt-Tamaru ([4])). *Let M be an irreducible Riemannian symmetric space of noncompact type, and consider a cohomogeneity one action of H on M with H being connected. Then, this action satisfies one of the following:*

- (K) *There exists a unique singular orbit.*
- (A) *All orbits are regular, and there exists a unique minimal orbit.*
- (N) *All orbits are regular, and furthermore, all orbits are isometrically congruent to each other.*

One already knows examples of these actions in Figure 1. Recall that the names (K), (A), and (N) come from the Iwasawa decomposition of $\mathrm{SL}(2, \mathbb{R})$.

Remark 3.2. For a cohomogeneity one action of type (K) or (A), there exists a unique "distinguished" orbit, that is, the unique singular orbit for type (K), or the unique minimal orbit for type (A). For the case of type (N), it looks that there does not exist a distinguished orbit.

3.2. **Results on left-invariant metrics.** We now see how the above pictures of cohomogeneity one actions are related to the study of left-invariant Riemannian metrics. We are interested in the following class of left-invariant metrics.

Definition 3.3. Let $(\mathfrak{g}, \langle , \rangle)$ be a metric Lie algebra, and denote by $\mathrm{Ric} : \mathfrak{g} \to \mathfrak{g}$ the Ricci operator. Then, $\langle , \rangle$ is said to be *algebraic Ricci soliton* if there exist $c \in \mathbb{R}$ and a derivation $D \in \mathrm{Der}(\mathfrak{g})$ such that

$$\mathrm{Ric} = c \cdot \mathrm{id} + D.$$

It is easy to see that "Einstein" implies "algebraic Ricci soliton". Recall that a derivation $D : \mathfrak{g} \to \mathfrak{g}$ is a linear map satisfying

$$D([\cdot, \cdot]) = [D(\cdot), \cdot] + [\cdot, D(\cdot)].$$

Remark 3.4. If $(\mathfrak{g}, \langle , \rangle)$ is algebraic Ricci soliton, then it gives rise to a Ricci soliton metric ([23, 25]). More precisely, if $(\mathfrak{g}, \langle , \rangle)$ is algebraic Ricci soliton, then the corresponding simply-connected Lie group with the induced left-invariant metric $(G, \langle , \rangle)$ is Ricci soliton, in the sense that there exist $c \in \mathbb{R}$ and $X \in \mathfrak{X}(G)$ such that

$$\mathrm{ric} = c\langle , \rangle + \mathfrak{L}_X \langle , \rangle .$$

Here, ric denotes the Ricci $(0,2)$-tensor, and $\mathfrak{L}_X$ the Lie derivative along X.

We study algebraic Ricci soliton metrics, from the view point of the actions of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$. In the following case, there is a very nice correspondence.

Theorem 3.5 (Hashinaga-Tamaru ([11])). *Let $\mathfrak{g}$ be a three-dimensional solvable Lie algebra. Then, an inner product $\langle , \rangle$ on $\mathfrak{g}$ is algebraic Ricci soliton if and only if the orbit $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle$ is a minimal submanifold.*

Recall that the ambient space $\widetilde{\mathfrak{M}} = \mathrm{GL}(3, \mathbb{R})/\mathrm{O}(3)$ is equipped with the natural $\mathrm{GL}(3, \mathbb{R})$-invariant metric, and is a noncompact Riemannian symmetric space.

Remark 3.6. A key point of Theorem 3.5 is that, for a three-dimensional solvable Lie algebra $\mathfrak{g}$, the action of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ is of cohomogeneity at most one (that is, transitive or cohomogeneity one). For the cohomogeneity one cases, algebraic Ricci soliton metrics are exactly corresponding to the "distinguished" orbits described in Remark 3.2.

We checked several Lie algebras $\mathfrak{g}$ whether it has the similar property or not. The answer is affirmative for some cases, but not for some cases. For examples,

Theorem 3.7 (Hashinaga ([10])). *For a four-dimensional nilpotent Lie algebra $\mathfrak{g}$, an inner product $\langle , \rangle$ is algebraic Ricci soliton if and only if $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle$ is a minimal submanifold. On the other hand, there exists a four-dimensional solvable Lie algebra $\mathfrak{g}$ such that the both implications do not hold.*

Despite of the above result, we still expect that (algebraic) Ricci soliton metrics $\langle , \rangle$ are corresponding to "distinguished" $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle$. One naive certification is that, if $\langle , \rangle$ is bi-invariant, then $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}).\langle , \rangle$ is totally geodesic.

3.3. **Problems.** In the last of this section, we propose some problems on isometric actions, which naturally arise from our framework.

Problem 3.8. For Riemannian symmetric spaces of noncompact type, study the geometry of orbits of some particular actions. For examples, cohomogeneity one actions (with H not necessarily connected), cohomogeneity two actions, (hyper)polar actions, and so on.

The next problem is about a characterization of particular left-invariant metrics in terms of the corresponding submanifolds.

Problem 3.9. Examine whether properties of $\mathfrak{g}$ can be understood by the corresponding group actions. For examples, the following properties can be characterized by submanifolds? — (algebraic) Ricci soliton, flat, positive or negative curvatures, and so on.

4. Nonisometric actions on R-spaces

We here consider group actions, not necessarily isometric, on R-spaces. In this section, we mention one simple example, and some possible problems. We refer to [28, 21] and references therein for related and deeper results.

4.1. **Results on group actions.** We here see one example of a (nonisometric) group action on an R-space.

Definition 4.1. Let L be a semisimple Lie group with trivial center, and Q be a parabolic subgroup of L. Then the coset space $M := L/Q$ is called an *R-space.*

An R-space is also called a *real flag manifold.* An example on which we concentrate in this section is the real projective space.

Example 4.2. The real projective space $\mathbb{RP}^n$ is an R-space. In fact, one has an expression

$$\mathbb{RP}^{n-1} = \mathrm{SL}(n,\mathbb{R}) / \left\{ \left(\begin{array}{c|ccc} * & * & \cdots & * \\ \hline 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{array} \right) \mid \det = 1 \right\}.$$

Remark 4.3. It is known that an R-space can be realized as an orbit of the isotropy representation of a Riemannian symmetric space. In fact, $\mathbb{RP}^{n-1}$ can be realized as an orbit of the isotropy representation of $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$.

We consider the natural action of $\mathrm{SO}(p,q)$ on $\mathbb{RP}^{p+q-1}$. The reason of this choice will be mentioned in the next section.

Proposition 4.4 (Kubo-Onda-Taketomi-Tamaru ([18])). *Consider* $\mathrm{SO}(p,q)$ *with* $p,q \geq 1$. *Let* $\langle , \rangle_0$ *be the canonical inner product on* $\mathbb{R}^{p+q}$ *with signature* (p,q), *which is invariant under the action of* $\mathrm{SO}(p,q)$. *Then, the action of* $\mathrm{SO}(p,q)$ *on* $\mathbb{RP}^{p+q-1}$ *has exactly three orbits, namely,*

$$\begin{aligned} \mathcal{O}^+ &:= \{[v] \in \mathbb{RP}^{p+q-1} \mid \langle v, v\rangle_0 > 0\}, \\ \mathcal{O}^0 &:= \{[v] \in \mathbb{RP}^{p+q-1} \mid \langle v, v\rangle_0 = 0\}, \\ \mathcal{O}^- &:= \{[v] \in \mathbb{RP}^{p+q-1} \mid \langle v, v\rangle_0 < 0\}. \end{aligned}$$

Note that $\mathcal{O}^+$ and $\mathcal{O}^-$ are open orbits. Since $\mathrm{SO}(p,q)$ preserves $\langle , \rangle_0$, it is easy to see that these orbits are different to each other.

Remark 4.5. Note that $\mathrm{SO}(p,q)$ is a symmetric subgroup of $\mathrm{SL}(n,\mathbb{R})$, that is, $(\mathrm{SL}(n,\mathbb{R}),\mathrm{SO}(p,q))$ is a symmetric pair. The above mentioned action would be interesting from this view point, since it can be considered as an analogy of a "Hermann action".

4.2. **Problems.** In the last of this section, we propose some problems on group actions on R-spaces. The following problems must be related to representation theory.

Problem 4.6. Let $M = L/Q$ be an R-space, and H be a Lie subgroup of L. Consider the action of H on $M = L/Q$.

(1) Among such actions, find interesting examples.
(2) Study what happens if (L, H) is a symmetric pair.
(3) Examine when it has an open orbit.

Next problem is about submanifold geometry. At the moment, the author knows no interesting examples, but it would be natural.

Problem 4.7. Consider the same situation as Problem 4.6. Then, orbits $H.p$ can be inhomogeneous with respect to $\mathrm{Isom}(M)$. However, study whether the orbits still have some "nice" properties (as Riemannian submanifold) or not.

5. Isometric actions on pseudo-Riemannian symmetric spaces

In this section, we consider isometric actions on pseudo-Riemannian symmetric spaces. This is related to the topic of the previous section, and also has applications to the study of left-invariant pseudo-Riemannian metrics on Lie groups.

5.1. **Results on group actions.** We here mention the following only one example of an isometric action.

Proposition 5.1 (Kubo-Onda-Taketomi-Tamaru ([18])). *The action of the following Q on $\mathrm{GL}(p+q,\mathbb{R})/\mathrm{O}(p,q)$ has exactly three orbits*:

$$Q := \left\{ \left(\begin{array}{c|ccc} * & * & \cdots & * \\ \hline 0 & & & \\ \vdots & & * & \\ 0 & & & \end{array} \right) \in \mathrm{GL}(p+q,\mathbb{R}) \right\}.$$

One can certificate this from Proposition 4.4. In fact, the orbit space of this action coincides with the orbit space of the action of $\mathrm{O}(p,q)$ on

$$\mathrm{GL}(p+q,\mathbb{R})/Q = \mathbb{R}\mathrm{P}^{p+q-1}.$$

5.2. **Results on left-invariant metrics.** We now recall $\widetilde{\mathfrak{M}}_{p,q}$, the set of all left-invariant metrics on a Lie group G with signature (p, q). Denote by $\mathfrak{g}$ the Lie algebra of G, and consider the group $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ as in the Riemannian case.

Definition 5.2. The orbit space of the action of $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ on $\widetilde{\mathfrak{M}}_{p,q}$,

$$\mathfrak{P}\mathfrak{M}_{p,q} := \mathbb{R}^\times \mathrm{Aut}(\mathfrak{g}) \backslash \widetilde{\mathfrak{M}}_{p,q},$$

is called the *moduli space* of left-invariant metrics on G with signature (p, q).

We pic up the following two Lie algebras. Recall that $\{e_1, \ldots, e_n\}$ denotes the canonical basis of $\mathbb{R}^n$, and we only write nonzero bracket relations.

Definition 5.3. We define the following two Lie algebras:

(1) $\mathfrak{h}^3 = (\mathbb{R}^3, [,])$ with $[e_1, e_2] = e_3$ is called the *Heisenberg Lie algebra.*
(2) $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n} = (\mathbb{R}^n, [,])$ with the following bracket relations is called the *solvable Lie algebra of* $\mathbb{R}\mathrm{H}^n$:

$$[e_1, e_2] = e_2, \ldots, [e_1, e_n] = e_n.$$

Note that $\mathbb{R}\mathrm{H}^n$ is the real hyperbolic space. The simply-connected Lie group $G_{\mathbb{R}\mathrm{H}^n}$ with Lie algebra $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$ acts simply-transitively on $\mathbb{R}\mathrm{H}^n$.

Theorem 5.4 ([18]). *Let $\mathfrak{g} := \mathfrak{h}^3$ or $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$. Then, for all $p, q \in \mathbb{N}$ with $\dim \mathfrak{g} = p + q$, there exists exactly three left-invariant metrics on $\mathfrak{g}$ with signature (p, q), up to isometry and scaling.*

The proof is based on the fact that, for the Lie algebras $\mathfrak{g}$ mentioned above, $\mathbb{R}^\times \mathrm{Aut}(\mathfrak{g})$ is a parabolic subgroup. In fact, it coincides with Q in Proposition 5.1.

Remark 5.5. For the case of $\mathfrak{h}^3$, the above result has been known by Rahmani ([36]), but the method is different. The properties of these three metrics have also been studied by Onda ([35]).

Remark 5.6. For the case of $\mathfrak{g}_{\mathbb{R}\mathrm{H}^n}$, we have proved that these three left-invariant metrics have constant sectional curvatures ([18]). Note that the Lorentzian case of these results have been known by Nomizu ([34]), but the method is different. Our argument simplifies the proof of his results, and also extends it to the case of generic signatures.

5.3. **Problems.** As mentioned above, the study of isometric actions on $\mathrm{GL}(p + q, \mathbb{R})/\mathrm{O}(p, q)$ has applications to left-invariant pseudo-Riemannian metrics. This motivates to study the following problems.

Problem 5.7. Study isometric actions of H on pseudo-Riemannian symmetric spaces M. First of all, study the case when H is a parabolic subgroup (this case corresponds to the "symmetric actions" on R-spaces, mentioned in Problem 4.6).

Problem 5.8. Construct "nice" isometric actions on a pseudo-Riemannian symmetric space $M = U/K$. For example, let K' be a maximal compact subgroup of U, and consider the Riemannian symmetric space $M' = U/K'$ of noncompact type. If the action of H on M' is nice in some sense, then so is the action of H on M?

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