数理解析研究所講究録
第 1929 巻 2014 年 19-29

Existence problem of flat projective structures and affine structures

Hironao Kato *
Osaka City University Advanced Mathematical Institute

1 Introduction

In this article we consider the existence problem of flat projective structures on manifolds. Firstly we recall the definition of flat projective structures. Let ∇ and ∇' be torsion-free affine connections on a manifold M of dimension n. ∇ and ∇' are said to be projectively equivalent if there exists a 1-form λ such that $\nabla_X Y - \nabla'_X Y = \lambda(X)Y + \lambda(Y)X$ for vector fields X and Y on M. A projective equivalence class of ∇ is called a projective structure and denoted by $[\nabla]$. The affine connection ∇ is called projectively flat if ∇ is locally projectively equivalent to a flat affine connection. If ∇ is projectively flat, then $[\nabla]$ is called a flat projective structure. We can rephrase projectively flatness by using tensors. For $n \geq 3$ the connection ∇ is projectivley flat if Weyl's projective curvature tensor vanishes, i.e. $W(X,Y)Z = R(X,Y)Z + [P(X,Y) - P(Y,X)]Z - [P(Y,Z)X - P(X,Z)Y] = 0$ (cf. [10]). For $n = 2$, ∇ is projectivley flat if $\nabla_X P(Y,Z) = \nabla_Y P(X,Z)$. Here P is the $(1,1)$-tensor defined by $P(X,Y) = \frac{1}{n^2-1}[nRic(X,Y) + Ric(Y,X)]$.

When the base space is a Lie group, ∇ is called left invariant if it satisfies $L_a^*\nabla = \nabla$ for the left translation by any element a of the group. Concerning a left invariant flat projective structure (abbrev. IFPS) on Lie group, Y.Agaoka [1] made a correspondence between IFPSs and certain Lie algebra homomorphisms called (P)-homomorphisms by using Cartan connections. Let L be a n-dimensional Lie group with Lie algebra $\mathfrak{l}$. Denote by $\{e_1, \ldots, e_{n+1}\}$ the standard basis of $\boldsymbol{R}^{n+1}$ and by $\{X_1, \ldots, X_n\}$ a basis of $\mathfrak{l}$. Then a Lie algebra homomorphism $f : \mathfrak{l} \to \mathfrak{sl}(n+1, \boldsymbol{R})$ is called a (P)-homomorphism if $f(X_i)e_{n+1} = e_i + \alpha e_{n+1}$ for some $\alpha \in \boldsymbol{R}$. By using the Weyl's curvature tensor the correspondence can be directly stated as follows (see [5] for the proof): The set of left invariant projectively flat affine connections ∇ on L is bijectively corresponding to the set of (P)-homomorphisms $f : \mathfrak{l} \to \mathfrak{sl}(n+1, \boldsymbol{R})$. The (P)-homomorphism f corresponding to ∇ is given by

$$f(X) = \begin{pmatrix} \nabla_X - \frac{1}{n+1} tr\nabla_X I_n & X \\ -P(X,\cdot) & -\frac{1}{n+1} tr\nabla_X \end{pmatrix}.$$

*This work was partially supported by JSPS and JSPS Strategic Young Researcher Overseas Visits Program for Accelerating Brain Circulation.

Here we identified the representation space $\boldsymbol{R}^{n+1}$ with $\mathfrak{l} \oplus \boldsymbol{R}$ by the correspondence $e_i + \alpha e_{n+1} \leftrightarrow (X_i, \alpha)$. Denote by $id : R \to R$ the identity representation. Then the tensor product representation $f \otimes id$: $\mathfrak{l} \oplus \boldsymbol{R} \to \mathfrak{gl}(\boldsymbol{R}^{n+1} \otimes \boldsymbol{R})$ satisfies $f \otimes id(\mathfrak{l} \oplus \boldsymbol{R})e_{n+1} = \boldsymbol{R}^{n+1}$. Thus $f \otimes id$ gives an infinitesimal prehomogeneous vector space (abbrev. PV). Conversely from a given infinitesimal prehomogeneous vector space $f \otimes id : \mathfrak{l} \oplus \boldsymbol{R} \to \mathfrak{gl}(n+1, \boldsymbol{R})$ we can obtain a left invariant projectivley flat affine connection on a Lie group having Lie algebra $\mathfrak{l}$. Let ∇ be a left invariant projectively flat affine connection. Then ∇ is affinely flat iff the Ricci tensor vanishes. We also consider the existence problem of Left invariant flat affine connections (abbrev. IFASs) on Lie groups. Note that by the above correspondence we might say a Lie algebra admits an IFPS and IFAS.

Example. $SL(2, \boldsymbol{R})$ acts on the upper half plane $\boldsymbol{R}H^2$ transitively and the subgroup $H = \{\begin{pmatrix} e^x & y \\ 0 & e^{-x} \end{pmatrix} \mid x, y \in \boldsymbol{R}\}$ acts on $\boldsymbol{R}H^2$ freely and transitively. Thus H is identified with $\boldsymbol{R}H^2$ by the mapping $a \mapsto a\sqrt{-1}$. $\boldsymbol{R}H^2$ has the metric $g = \frac{dx^2+dy^2}{y^2}$ and g is left invariant with respect to the action of H. Thus we obtain the Lie group with left invariant metric (H, g). The Lie algebra $\mathfrak{l} := Lie(H)$ is given by $\{\begin{pmatrix} x & y \\ 0 & -x \end{pmatrix} \mid x, y \in \boldsymbol{R}\}$. Put $X_1 := \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $X_2 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then we obtain the 2-dimensional Lie algebra $[X_1, X_2] = X_2$, and the left invariant metric g is described by the matrix $(g(X_i, X_j)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The Levi-Civita connection is left invariant and its Christoffel symbols are given by

$$\nabla_{X_1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \ \nabla_{X_2} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

As a result (H, g) is constant curvature -1 and Einstein $Ric = -g$. The Ric tensor gives 1-forms

$$Ric(X_1, \cdot) = (-1, 0), \ Ric(X_2, \cdot) = (0, -1).$$

From these data we can construct (P)-homomorphism $f : \mathfrak{l} \to \mathfrak{sl}(3, \boldsymbol{R})$.

$$f(X_1) = \begin{pmatrix} \nabla_{X_1} & X_1 \\ -Ric(X_1, \cdot) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ f(X_2) = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Putting $e_3 := {}^t(0, 0, 1)$ yields $f(\mathfrak{l})e_3 \oplus < e_3 >= \boldsymbol{R}^3$. Thus $f \otimes id : \mathfrak{l} \oplus \boldsymbol{R} \to \mathfrak{gl}(\boldsymbol{R}^3)$ gives an infinitesimal PV.

We shall see that the representation f is related to the 2 symmetric product of identity representation $g := S^2 id$: $\mathfrak{sl}(2, \boldsymbol{R}) \to \mathfrak{sl}(3, \boldsymbol{R})$. Put $X_3 := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $v :=$

${}^t(1,0,1)$. We define a matrix P to be $(f(X_1)v, f(X_2)v, v)$. Then we have

$$P^{-1}\{g(X_1), g(X_2), g(X_2 - X_3)\}P = \{f(X_1), f(X_2), \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\}.$$

2 Low dimensional classification

About the sufficient condition for the existence of IFPSs and IFASs the following result is known. Abelian Lie algebras, 3-step nilpotent Lie algebra (J.Scheuneman), positively graded Lie algebras $\mathfrak{k} = \oplus_{i\geq 1}\mathfrak{k}_i$ (S.Yamaguchi) admit an IFAS. Let us consider a semidirect sum $\mathfrak{h} \ltimes \mathfrak{k}$ of a Lie algebra $\mathfrak{h}$ admitting a flat affine connection $\nabla^{\mathfrak{h}}$ with a positively graded Lie algebra $\mathfrak{k} = \oplus_{i\geq 1}\mathfrak{k}_i$. If $\mathfrak{h}$ preserve the grading of $\mathfrak{k}$, then $\mathfrak{h} \ltimes \mathfrak{k}$ admits a flat affine connection. When $\mathfrak{h}$ is abelain $\mathfrak{a}$, this result is due to S.Yamaguchi. The construction of a flat affine connection on $\mathfrak{k}$ and $\mathfrak{h} \otimes \mathfrak{k}$ is given as follows:
On $\mathfrak{k}$ $\quad \nabla_X Y = \frac{j}{i+j}[X,Y]$ for $X \in \mathfrak{k}_i$, $Y \in \mathfrak{k}_j$.

On $\mathfrak{h} \ltimes \mathfrak{k}$

X	Y	$\nabla_X Y$
$\mathfrak{h}$	$\mathfrak{h}$	$\nabla^{\mathfrak{h}}_X Y$
$\mathfrak{k}_i$	$\mathfrak{k}_j$	$\frac{j}{i+j}[X,Y]$
$\mathfrak{h}$	$\oplus\mathfrak{k}_i$	$[X,Y]$
$\oplus\mathfrak{k}_i$	$\mathfrak{h}$	0

Concerning classification, nilpotent Lie algebras of dimension ≤ 6 (H.Fujiwara) and solvable Lie algebras of dimension ≤ 4 (S.Yamaguchi) admit IFASs. On the other hand perfect Lie algebras, i.e. $[\mathfrak{l},\mathfrak{l}] \neq \mathfrak{l}$, do not admit IFASs (J.Helmstetter).
Let $\mathfrak{l}$ be a Lie algebra $\mathfrak{l}$ of dim ≤ 5. Then $\mathfrak{l}$ admits an IFAS iff $\mathfrak{l} \neq \mathfrak{sl}(2,\boldsymbol{R})$, $\mathfrak{o}(3,\boldsymbol{R})$, $\mathfrak{sl}(2,\boldsymbol{R}) \ltimes \boldsymbol{R}^2$ (perfect). However always $\mathfrak{l}$ admits an IFPS (H.Kato [7]). On the other hand $\mathfrak{sl}(2,\boldsymbol{R}) \oplus \mathfrak{sl}(2,\boldsymbol{R})$, $\mathfrak{sl}(2,\boldsymbol{R}) \oplus \mathfrak{o}(3,\boldsymbol{R})$, $\mathfrak{o}(3,\boldsymbol{R}) \oplus \mathfrak{o}(3,\boldsymbol{R})$, $\mathfrak{o}(1,3)$ admit no IFPSs.

Example. We consider the the Lie algebra $\mathfrak{g}_2$. By definition $\mathfrak{g}_2$ is the Lie algebra arising from the Cartan matrix $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$. Precisely $\mathfrak{g}_2$ is the Lie algebra generated by $\{H_i, E_i, F_i\}_{i=1,2}$ by the serre relation $ad(H_i)E_j = a_{ij}E_j$, $ad(H_i)F_j = -a_{ij}F_j$, $[E_i, F_j] = \delta_{ij}H_i$, $ad(E_i)^{1-a_{ij}}E_j = 0$ $(i \neq j)$, $ad(F_i)^{1-a_{ij}}F_j = 0$ $(i \neq j)$. Put $x = E_1$, $y = E_2$. Then $\{x, y\}$ are the vectors corresponding to the set of simple roots. The Lie algebra $\mathfrak{g}_2$ does not admit any IFPS, whilst the standard borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}_2$ admits an IFAS. Indeed the positive root part $\mathfrak{k}$ of $\mathfrak{b}$ is spanned by $\{x, y, e_2, e_3, e_4, e_5\}$, which satisfies the bracket relation $[x, y] = e_2, [x, e_2] = e_3, [x, e_3] = e_4, [y, e_4] = e_5, [e_2, e_3] = e_5$. Hence $\mathfrak{k}$ is graded by positive integers as follows.

$\mathfrak{k}_1$	$\mathfrak{k}_2$	$\mathfrak{k}_3$	$\mathfrak{k}_4$	$\mathfrak{k}_5$
x, y	e_2	e_3	e_4	e_5

Thus $\mathfrak{k}$ admits an IFAS. The Cartan subalgebra spanned by $\mathfrak{h} = \{H_1, H_2\}$ preserves the root space decomposition, thus $\mathfrak{b} = \mathfrak{h} \ltimes \mathfrak{k}$ admits an IFAS. From the serre relation we can explicitly write down the bracket relation between $\{H_i\}$ and $\{x, y, e_i\}$ as follows.

	x	y	e_2	e_3	e_4	e_5
h_1	$2x$	$-3y$	$-e_2$	e_3	$3e_4$	0
h_2	$-x$	$2y$	e_2	0	$-e_4$	e_5

The Lie algebra $\mathfrak{k}$ has a codimension one subalgebra $\mathfrak{k}_5 =< x, e_2, e_3, e_4, e_5 >$. We can modify the bracket relation and obtain another nilpotent Lie algebra $\mathfrak{k}'_5$ defined by $[x, e_2] = e_3,\ [x, e_3] = e_4,\ [x, e_4] = e_5,\ [e_2, e_3] = e_5$. The Lie algebra $\mathfrak{k}'_5$ is also graded by positive integers and hence admits an IFAS. The corresponding (P)-homomorphism $f : \mathfrak{k}'_5 \to \mathfrak{sl}(6, \boldsymbol{R})$ is described as follows:

$$f(x) = \begin{pmatrix} 0 & & & & & 1 \\ 0 & 0 & & & & 0 \\ & \frac{2}{3} & 0 & & & 0 \\ & & \frac{3}{4} & 0 & & 0 \\ & & & \frac{4}{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ f(e_2) = \begin{pmatrix} 0 & & & & & 0 \\ 0 & 0 & & & & 1 \\ -\frac{1}{3} & 0 & 0 & & & 0 \\ & 0 & 0 & 0 & & 0 \\ & & \frac{3}{5} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ f(e_3) = \begin{pmatrix} 0 & & & & & 0 \\ 0 & 0 & & & & 0 \\ 0 & 0 & 0 & & & 1 \\ -\frac{1}{4} & 0 & 0 & 0 & & 0 \\ 0 & -\frac{2}{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$f(e_4) = \begin{pmatrix} 0 & & & & & 0 \\ 0 & 0 & & & & 0 \\ 0 & 0 & 0 & & & 0 \\ 0 & 0 & 0 & 0 & & 1 \\ -\frac{1}{5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},\ f(e_5) = \begin{pmatrix} 0 & & & & & 0 \\ 0 & 0 & & & & 0 \\ 0 & 0 & 0 & & & 0 \\ 0 & 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Consequently semidirect sums of $\mathfrak{h}$ with $\mathfrak{k}$, $\mathfrak{k}_5$, $\mathfrak{k}'_5$ admit an IFAS.

3 Castling transformations

If two manifolds M_1 and M_2 admit a flat affine connection, then naturally the product $M_1 \times M_2$ admits a flat affine connection again. On the other hand we have a different story about flat projective structures. Even if two manifolds admits a flat projective structure, its product manifold does not necessarily admit a flat projective structure again. Indeed the n-dimensional sphere S^n admits a flat projective structure, but $S^n \times S^n$ $(n \geq 2)$ does not admit any one (S.Kobayashi and T.Nagano [9]). Another counter example is $SL(2, \boldsymbol{R})$, which admits a left invariant flat projective structure. In this case $SL(2, \boldsymbol{R}) \times SL(2, \boldsymbol{R})$ also does not admit any one. However $SL(2, \boldsymbol{R}) \times SL(3, \boldsymbol{R})$ admits a left invariant flat projective structure (A.Elduque [3]). We expect the combinatorics of product manifolds admitting a flat projective structure is quite restricted. Concerning this problem castling transformations turned out to be useful tool. Originally castling transformation is a notion for prehomogeneous vector spaces, which can yield a new PV from a given one. Let $f : \mathfrak{g} \to \mathfrak{gl}(\boldsymbol{R}^m)$ be a representation. Assume that $m > n$. The transformation
(1) $f \otimes id$: $\mathfrak{l} \oplus \mathfrak{gl}(n, \boldsymbol{R}) \to \mathfrak{gl}(\boldsymbol{R}^m \otimes \boldsymbol{R}^n)$
(2) $f^* \otimes id$: $\mathfrak{l} \oplus \mathfrak{gl}(m-n, \boldsymbol{R}) \to \mathfrak{gl}(\boldsymbol{R}^{m*} \otimes \boldsymbol{R}^{m-n})$
is called a castling transformation, which preserves the prehomogeneity. We introduce the geometric version of castling transformation (see H.Kato [8] for details): To state this geometric transformation we need flat Grassmannian structures. The definition is due to W.Goldman [4]. Let M be a manifold. Put $G = PGL(l)$, X= $Gr_{m,l}$. Let x be

a pt in X. Denote by G' the isotropy subgroup at x of G. Then we have $G/G' = X$. A flat Grassmannian structure on M is a maximal atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ of M satisfying the following condition:

(1) φ_α: $U_\alpha \to O_\alpha \subset X$ is a diffeomorphism
(2) If $U_\alpha \cap U_\beta \neq \emptyset$, then for each connected component C of $U_\alpha \cap U_\beta$ there exists $\tau(C; \beta, \alpha) \in G$ such that $\varphi_\beta \circ \varphi_\alpha^{-1}$ equals the map $\tau(C; \beta, \alpha)$ on $\varphi_\alpha(C) \subset X$.

If $G = PGL(n+1)$ and $X = P(\boldsymbol{R}^{n+1})$, then a maximal atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ gives a alternative definition of flat projective structures on M. Moreover if $G = PGL(\boldsymbol{C}^{n+1})$ and $X = P(\boldsymbol{C}^{n+1})$, in addition M is a complex manifold and φ_α is a biholomorphic map, then the atlas gives a flat complex projective structure. A flat Grassmannian structure corresponds to an isomorphism class of flat Grassmannian Cartan connections, which is a useful tool to investigate geometric structures.

Now let us consider the model space $G = PGL(l)$, X$= Gr_{m,l}$. Denote by $\mathfrak{g}$ the Lie algebra of G and by $\mathfrak{g}'$ the one of G'. A Grassmannian Cartan connection of type (n, m) is a pair (P, ω) where P is a principal fiber bundle over M with structure group G' and ω is a $\mathfrak{g}'$-valued 1-form satisfying the following condition
(1) for $u \in P$, $\omega_u : T_uP \to \mathfrak{g}$:linear isomorphism
(2) for $g \in G'$, $R_g^*\omega = Ad(g^{-1})\omega$
(3) for $Y \in \mathfrak{g}'$, $\omega(Y^*) = Y$
where Y^* is the fundamental vector field corresponding to Y.

(P, ω) is called flat if $d\omega + \frac{1}{2}[\omega, \omega] = 0$.

Now we recall how a flat Grassmannian structure gives rise to a flat Grassmannian Cartan connection (see H.Kato [6] for the detailed correspondence). A given Grassmannian structure $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ on M has a coordinate map φ_α: $U_\alpha \to O_\alpha \subset X = G/G'$. Denote by $\pi : G \to X$ the projection. Then $\pi^{-1}(O_\alpha)$ is regarded as a principal fiber bundle over U_α with structure group G'.

$$
\begin{tikzcd}
& \pi^{-1}(O_\alpha) \arrow[dl, dashed, "\pi_\alpha"'] \arrow[d] & \subset & G \arrow[d, "\pi"] \\
U_\alpha & O_\alpha \arrow[l, "\varphi_\alpha^{-1}"] & \subset & X
\end{tikzcd}
$$

Denote by ω the Maurer Cartan form of G. Denote by ω_α the restriction $\omega|_{\pi^{-1}(O_\alpha)}$ of ω to the open subset. Thus we obtain a family of Cartan connections $\{(\pi^{-1}(O_\alpha), \omega_\alpha)\}_{\alpha \in A}$. These data can be glued by the following relation: Elements $g \in \pi^{-1}(O_\alpha)$ and $h \in \pi^{-1}(O_\beta)$ are identified if $\pi_\alpha(g) = \pi_\beta(h)$ and $h = \tau(C; \beta, \alpha)g$ for connected component $C \ni \pi_\alpha(g)$ of $U_\alpha \cap U_\beta$. Then by gluing we obtain $P := \bigsqcup_{\alpha \in A} \pi^{-1}(O_\alpha)/_\sim$ and $\omega_P := \omega_\alpha$ on $\pi^{-1}(O_\alpha)$, which give a Grassmannian Cartan connection.

A Grassmannian Cartan connection (Q, ω) induces a certain reduction of the frame bundle $L(M)$ of M as follows. Denote by $< v >$ the subspace of $\boldsymbol{R}^l$ spanned by $\{e_1, e_2, \cdots, e_m\}$. Then we have $Gr_{m,l} = PL(l)/PL(l)_{<v>}$. Consider the isotropy representation $\rho : PL(l)_{<v>} \to GL(M(n,m))$.

$$\rho : \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \mapsto B \otimes {}^t A^{-1}.$$

Then the image is given by $\rho(PL(l)_{<v>}) = GL(n) \otimes GL(m)$. Thus $Q/ker\rho$ gives a $GL(n) \otimes GL(m)$-bundle over M. This quotient bundle $P_t M$ is regarded as a subbundle of $L(M)$.

Finally we state our geometric castling transformations. Let M be a manifold equipped with a Grassmannian Cartan connection (Q, ω) of type (n,m) and Λ_1 a Maurer-Cartan form of $PGL(m)$. Then we have the following:

Proposition 3.1. (1) $(Q \times PGL(m), \omega \times \Lambda^1)$ *is a flat Cartan connection over* N
$\iff$ (2) $(Q \times PGL(n), \omega^* \times \Lambda^1)$ *is a flat Cartan connection over* N'

We call this transformation a castling transformation of projective structures. Note that (1) and (2) can be enlarged to projective Cartan connections. If (1) is flat, then (2) and (Q, ω) are also flat. Thus in that case Cartan connections induces a flat Grassmannian structure on M and flat projective structures on N and N'.

Now we describe the base space appearing in castling transformations. N and N' has the structure of principal fiber bundle indicated in the following diagram.

$$\begin{array}{lcl}
(1)\ (Q \times PGL(m), \omega \times \Lambda_1) & \longleftrightarrow & (2)\ (Q \times PGL(n), \omega^* \times \Lambda_1) \\
\quad \downarrow & & \quad \downarrow \\
\quad N \leftarrow PGL(m) & & \quad N' \leftarrow PGL(n) \\
\quad \downarrow & & \quad \downarrow \\
\quad M & & \quad M
\end{array}$$

Recall that (Q, ω) induces a $GL(n) \otimes GL(m)$-structure $P_t M \subset L(M)$.

Proposition 3.2. *N is isomorphic to the quotient manifold $P_t M / GL(n) \otimes GL(1)$.*

From a given manifold equipped with a flat projective structure by successive castling transformations we can obtain an infinite sequence of projectively flat manifolds, which are connected by manifold equipped with a flat Grassmannian structure. We shall illustrate a sequence of base spaces of successive castling transformations. Let M be a 2 dimensional manifold equipped with a flat projective structure. For instance It is known that any closed surface and also any 2 dimensional Lie group admits a flat projective structure. Then by successive castling transformations we obtain the following sequence:

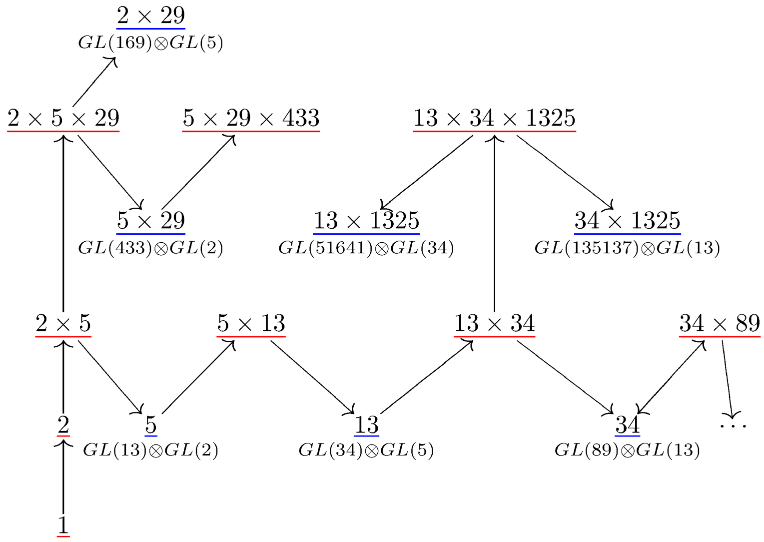
$$M \longrightarrow \overline{L}(M) \longrightarrow \overline{L}(\overline{L}(M)) \longrightarrow \overline{L}(\overline{L}(M))/PGL(2) \longrightarrow \cdots$$

Here is the geometric meaning: $\overline{L}(M)$ is the projective frame bundle of M. Since $PGL(2)$ acts on $\overline{L}(M)$, by the differential $PGL(2)$ also acts on $\overline{L}(\overline{L}(M))$. Then by the quotient we obtain a $PGL(5)$-bundle over M. As a result M, $\overline{L}(M)$ and $\overline{L}(\overline{L}(M))$ admit a flat projective structure. $\overline{L}(\overline{L}(M))/PGL(2)$ admits a flat Grassmannian structure.

When the given base space M is a 2-dimensional Lie group L we can more explicity write down the base spaces as follows:

$$\overline{L}(M) = L \times PGL(2),\ \ \overline{L}(\overline{L}(M)) = L \times PGL(2) \times PGL(5)$$
$$\overline{L}(\overline{L}(M))/PGL(2) = L \times PGL(5).$$

By successive castling transformations we can obtain the following tree of manifolds equipped with a flat projective structure or a flat Grassmannian structure.



The above tree is obtained from successive castling transformations starting from 2-dimensional manifold M equipped with a flat projective structure. The numbers denotes the base spaces. For instance 1 denotes M and 2 denotes a $PGL(2)$-bundle over M, 2×5 denotes a $PGL(2) \times PGL(5)$-bundle over M. A manifold having only the underline is equipped with a flat projective structure, on the other hand a manifold under which has a tensor product group is equipped with a flat Grassmannian structure. The combinatorics of base spaces are described in the following way.

Theorem 3.3. *The set of manifolds equipped with a flat projective structure on the tree corresponds to the set of solution of the equation*

$$(*)\quad 2 + k_1^2 + \cdots + k_j^2 - j - 3k_1 \cdots k_j + 1 = 0.$$

Note that we can obtain the same kind of tree and quadratic equation by starting from any dimensional manifold equipped with a flat projective structure or a Grassmannian structure (cf. H.Kato [8]).

As an application we can achieve a development in the classification problem of projectively flat semisimple Lie groups. The preceding result given by Y.Agaoka [1], H.Urakawa [14], A.Elduque [3] is stated as follows: Let L be a simple Lie group. Then L admits a left invariant flat projective structure iff $Lie(L) = \mathfrak{sl}(n, \boldsymbol{R})$ or $\mathfrak{sl}(n, \boldsymbol{H})$. In the same paper Elduque [3] obtained the semisimple Lie group admitting a left invariant flat projective structure $SL(2, \boldsymbol{R}) \times SL(3, \boldsymbol{R})$.

By using castling transformations we can obtain an infinite sequence of semisimple Lie groups admitting a left invariant flat projective structure. In the classification of reduced irreducible complex prehomogeneous vector spaces M.Sato and T.Kimura [11] obtained the following PVs:

• $\rho = S^3id$: $GL(2, \boldsymbol{C}) \to GL(\boldsymbol{C}^4)$

$$d\rho : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 3a & b & 0 & 0 \\ 3c & 2a+d & 2b & 0 \\ 0 & 2c & a+2d & 3b \\ 0 & 0 & c & 3d \end{pmatrix}$$

The point $v = {}^t(1,0,0,1) \in \boldsymbol{C}^4$ satisfies $d\rho(\mathfrak{gl}(2,\boldsymbol{C}))v = \boldsymbol{C}^4$.

• $\rho = S^2id \otimes id : SL(3) \times GL(2) \to GL(\boldsymbol{C}^6 \otimes \boldsymbol{C}^2)$

$S^2id \otimes id(A,B)(X_1,X_2) = (A(aX_1 + bX_2)^tA, A(cX_1 + dX_2)^tA) \quad X_1, X_2 \in Sym(3,\boldsymbol{R})$.

A generic point is given by $(X_1, X_2) = \{\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 2 & \\ & & 3 \end{pmatrix}\}$.

• $\wedge^2 id \otimes id$: $SL(5) \times GL(4) \to GL(\boldsymbol{C}^{10} \otimes \boldsymbol{C}^4)$

Combining successive castling transformations with Sato-Kimura's classification of reduced irreducible PVs yields the following (cf. H.Kato [6]):

Theorem 3.4. *A complex Lie group L admits an irreducible invariant flat complex projective structure iff its Lie algebra is of the form $\mathfrak{sl}(a) \oplus \mathfrak{sl}(m_1) \oplus \cdots \oplus \mathfrak{sl}(m_k)$, where $a = 2,\ 3,\ or\ 5$ $(k \geq 1,\ m_i \geq 1)$ and satisfies the equality $(**)$ $\quad a^2 + m_1^2 + \cdots + m_k^2 - k - 2am_1m_2 \cdots m_k = 0$.*

4 projectively flat parabolic subgroups

Y.Takemoto and S.Yamaguchi [12] proved that solvable part $\mathfrak{a} \oplus \mathfrak{n}$ of the Iwasawa decomposition $\mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ of semisimple real Lie algebra admits a left invariant flat affine connection. However on parabolic subalgebras the existence problem has not been settled yet. From the viewpoint of submanifolds we investigate this problem concerning the parabolic subalgebras of special linear Lie algebras.

Y.Agaoka [1], H.Urakawa [14], A.Elduque [3] proved that a simple Lie group L admits a left invariant flat projective structure iff $Lie(L) = \mathfrak{sl}(n, \boldsymbol{R})$ or $\mathfrak{sl}(n, \boldsymbol{H})$. The left invariant projectively flat affine connection is described as follows:

$$\nabla_X Y = XY - \frac{\mathrm{tr}XY}{n} I_n \text{ for } X, Y \in \mathfrak{sl}(n, \boldsymbol{R}) \tag{4.1}$$

$$\nabla_X Y = XY - \frac{\mathrm{Retr}XY}{n} I_n \text{ for } X, Y \in \mathfrak{sl}(n, \boldsymbol{H}) \tag{4.2}$$

Now we define parabolic sublalgebras, following H.Tamaru [13]. Let $\mathfrak{a}$ be the diagonal of $\mathfrak{g} = \mathfrak{sl}(n, \boldsymbol{R})$. Then the reduced root system $\triangle$ of $\mathfrak{g}$ with respect to $\mathfrak{a}$ is given by $\triangle = \{\lambda_i - \lambda_j (i \neq j)\}$. Here λ_i is defined by $\lambda_i(\mathrm{diag}(a_1, \ldots, a_n)) = a_i$. Put

$\alpha_i = \lambda_i - \lambda_{i+1}$ and then $\Lambda := \{\alpha_1, \ldots, \alpha_{n-1}\}$ gives a set of simple roots. Let Λ' be a proper subset of Λ. Then $\mathfrak{q}_{\Lambda'} = \mathfrak{a} + \sum_{\alpha \in \Lambda \cup <\Lambda'>}$ is called a parabolic subalgebra. Analogously we can define parabolic subalgebras in $\mathfrak{sl}(n, \boldsymbol{H})$. Then we can show that parabolic subalgebras are autoparallel in $\mathfrak{sl}(n, \boldsymbol{R})$ and $\mathfrak{sl}(n, \boldsymbol{H})$ with respect to the connection ∇ (4.1) and (4.2). Thus we obtain the induced left invariant projectivley flat affine connection ∇ on $\mathfrak{q}_{\Lambda'}$.

Question: the connection ∇ on $\mathfrak{q}_{\Lambda'}$ is projectively equivalent to a flat affine connection?

To deal with this question we introduce the invariants associated to representations. Let $\mathfrak{l}$ be a Lie algebra of dimension n and $f : \mathfrak{l} \to gl(n+1, \boldsymbol{R})$ a Lie algebra representation. Put $v := {}^t(x_1, \ldots, x_{n+1})$. Denote by $\{X_1, \cdots, X_n\}$ a basis of $\mathfrak{l}$. We define a function $\phi : \boldsymbol{R}^{n+1} \to \boldsymbol{R}$ by $\phi(v) := det(f(X_1)v, \cdots, f(X_n)v, v)$. Then Y.Agaoka [2] showed that ϕ is a relative invariant polynomial, i.e. $d\phi_v(f(X)v) = \alpha(X)\phi(v)$ for some representation $\alpha : \mathfrak{l} \to \mathfrak{gl}(1)$. Note that if f is a (P)-homomorpshim, then the associated invariant is not zero. Practically the invariant ϕ can be used as follows: Let ∇ be a left invariant projectively flat affine connection on L^n and $\phi : \boldsymbol{R}^{n+1} \to \boldsymbol{R}$ the invariant associated to ∇. Then ∇ is projectively equivalent to a flat affine connection iff $\phi(v)$ has a linear factor involving x_{n+1} (cf. [2]). Here are two examples of invariants.

1. $id \oplus id : \mathfrak{gl}(2) \to \mathfrak{gl}(4)$ gives a PV. Put $v := (a, b, c, d)$. Then the invariant associated to $id \oplus id$ is calculated as follows:

$$\begin{aligned} \phi(v) &= \det(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.v, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.v, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.v, v) \\ &= \det \begin{pmatrix} a & b & 0 & a \\ -b & 0 & a & b \\ c & d & 0 & c \\ -d & 0 & c & d \end{pmatrix} \\ &= 2(ad - bc)^2. \end{aligned}$$

2. Let $\mathfrak{s}_{\Lambda'}$ be a solvable subalgebra of $\mathfrak{sl}(3, \boldsymbol{R})$, which is defined to be $\mathfrak{s}_{\Lambda'} =< H^1, E_{12}, E_{13} >$. Then $\mathfrak{s}_{\Lambda'}$ is autoparallel in $\mathfrak{sl}(3, \boldsymbol{R})$ with respect to the connection (4.1). Let ∇ be the induced projective flat affine connection on $\mathfrak{s}_{\Lambda'}$. Denote by $f : \mathfrak{s}_{\Lambda'} \to \mathfrak{sl}(4, \boldsymbol{R})$ the induced representation from ∇, which is given by

$$f(H_1) = \begin{pmatrix} \frac{1}{3} & & & 1 \\ & \frac{2}{3} & & 0 \\ & & \frac{2}{3} & 0 \\ \frac{2}{9} & 0 & 0 & 0 \end{pmatrix}, \ f(X_2) = \begin{pmatrix} & & & 0 \\ -\frac{1}{3} & 0 & 0 & 1 \\ & & & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ f(X_3) = \begin{pmatrix} & & & 0 \\ & & & 0 \\ -\frac{1}{3} & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Put $v = {}^t(a, b, c, d)$. The invariant is defined by $\phi(v) := \det(f(H^1)v, f(E_{12})v, f(E_{13})v, v)$. Then we have

$$\phi(v) = (-\frac{1}{3}a + d)(-\frac{2}{9}a^2 + \frac{1}{3}ad + d^2).$$

Thus ∇ is projectively equivalent to a flat affine connection ∇', which is described as follows:

$$\nabla'_{H^1} = id_{\mathfrak{s}_{\Lambda'}},\ \nabla'_{E_{12}} = 0,\ \nabla'_{E_{13}} = 0.$$

By using invariants we can answer our question. Let us express Λ' as $\{\alpha_{i_1}, \alpha_2, \dots, \alpha_{i_k}\}$ satisfying $i_1 < i_2 < \cdots < i_k$.

Theorem 4.1. *(H.Kato [5]) The induced affine connection ∇ on $\mathfrak{q}_{\Lambda'}$ is not projectively equivalent to any flat affine connection iff we have $i_1 = 1$, $i_k = n-1$ and $|i_r - i_{r+1}| \le 2$ for $1 \le r \le k-1$.*

Examples. The parabolic subalgebra $\mathfrak{q}_{\Lambda'}$ is checked if ∇ is not projectively equivalent to any flat affine connection.

(1) $\Lambda' = \{\alpha_1, \alpha_2, \alpha_4, \alpha_5\}$

● — ● — ○ — ● — ● ✓

(2) $\Lambda' = \{\alpha_1, \alpha_2, \alpha_4, \alpha_7\}$

● — ● — ○ — ● — ○ — ○ — ●

(3)

● — ○ — ● ✓

$$\left(\begin{array}{c|c|c|c} * & * & * & * \\ \hline * & * & * & * \\ \hline & & * & * \\ \hline & & * & * \end{array}\right)$$

○ — ● — ●

$$\left(\begin{array}{c|c|c|c} * & * & * & * \\ \hline & * & * & * \\ \hline & * & * & * \\ \hline & * & * & * \end{array}\right)$$

○ — ● — ○

$$\left(\begin{array}{c|c|c|c} * & * & * & * \\ \hline & * & * & * \\ \hline & * & * & * \\ \hline & & & * \end{array}\right)$$

○ — ○ — ●

$$\left(\begin{array}{c|c|c|c} * & * & * & * \\ \hline & * & * & * \\ \hline & & * & * \\ \hline & & * & * \end{array}\right)$$

On the other hand concerning the affine connection (4.2) on $\mathfrak{sl}(n, \boldsymbol{H})$ we have the following:

Theorem 4.2. *([5]) The induced projectively flat affine connection ∇ on $\mathfrak{q}_{\Lambda'}$ is not projectively equivalent to any flat affine connection.*

References

[1] Yoshio Agaoka. Invariant flat projective structures on homogeneous spaces. *Hokkaido Math. J.*, 11(2):125–172, 1982.

[2] Yoshio Agaoka and Hironao Kato. Invariants and left invariant flat projective structures on lie groups. *In preparation.*

[3] Alberto Elduque. Invariant projectively flat affine connections on Lie groups. *Hokkaido Math. J.*, 30(1):231–239, 2001.

[4] William M. Goldman. Geometric structures on manifolds and varieties of representations. In *Geometry of group representations (Boulder, CO, 1987)*, volume 74 of *Contemp. Math.*, pages 169–198. Amer. Math. Soc., Providence, RI, 1988.

[5] Hironao Kato. Projectively flat parabolic subgroups of special linear groups. *Preprint.*

[6] Hironao Kato. Left invariant flat projective structures on Lie groups and prehomogeneous vector spaces. *Hiroshima Math. J.*, 42(1):1–35, 2012.

[7] Hironao Kato. Low dimensional Lie groups admitting left invariant flat projective or affine structures. *Differential Geom. Appl.*, 30(2):153–163, 2012.

[8] Hironao Kato. Castling transformations of projective structures. *J. Lie Theory*, 23(4):1129–1160, 2013.

[9] Shoshichi Kobayashi and Tadashi Nagano. On projective connections. *J. Math. Mech.*, 13:215–235, 1964.

[10] Katsumi Nomizu and Takeshi Sasaki. *Affine differential geometry*, volume 111 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 1994. Geometry of affine immersions.

[11] Mikio Sato and Tatsuo Kimura. A classification of irreducible prehomogeneous vector spaces and their relative invariants. *Nagoya Math. J.*, 65:1–155, 1977.

[12] Yoshio Takemoto and Satoru Yamaguchi. Affine structures of maximal solvable subalgebras of noncompact semisimple Lie algebras. *Mem. Fac. Sci. Kyushu Univ. Ser. A*, 35(1):39–44, 1981.

[13] Hiroshi Tamaru. Parabolic subgroups of semisimple Lie groups and Einstein solvmanifolds. *Math. Ann.*, 351(1):51–66, 2011.

[14] Hajime Urakawa. On invariant projectively flat affine connections. *Hokkaido Math. J.*, 28(2):333–356, 1999.