数理解析研究所講究録
第 1929 巻 2014 年 54-63

EXAMPLES OF GENERALIZED LAGRANGIAN MEAN CURVATURE FLOWS IN TORIC ALMOST CALABI–YAU MANIFOLDS

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ABSTRACT. In the former half of this paper, Section 1 and Section 2, we summarize some basic notions and facts of calibrated geometries and special Lagrangian geometries and Lagrangian mean curvature flows. In the later half, Section 3, we review the results of [10] which give some examples of generalized Lagrangian mean curvature flows in toric almost Calabi–Yau manifolds.

1. Calibrated geometries

In this section, we briefly sketch some basic notions of calibrated geometries introduced by Harvey and Lawson [3]. Let (M, g) be a real n-dimensional Riemannian manifold.

Definition 1.1. A closed k-form φ on M is called a *calibration* if it satisfies

$$\varphi|_\xi \leq dV_\xi, \tag{1}$$

for all points p in M and all k-dimensional oriented subvector spaces $\xi \subset T_pM$.

Let $e_1, \dots, e_k$ be an oriented orthogonal basis of ξ with respect to the metric g. Then the inequality (1) means that $\varphi(e_1, \dots, e_k) \leq 1$. Note that the notion of calibration depends on ambient Riemannian metrics. Let φ be a calibration on (M, g) and k-from.

Definition 1.2. A real k-dimensional oriented submanifold $L \subset M$ is called a *calibrated submanifold* or *φ-submanifold* if all its tangent spaces attain the equality of (1), that is, we have

$$\varphi|_L = dV_{g|_L}, \tag{2}$$

where $dV_{g|_L}$ is the volume form on L with the induced Riemannian metric $g|_L$.

The notion of calibrated submanifolds can be easily extended for an immersion $F : L \to M$ from a real k-dimensional oriented manifold L. Actually we call an immersion $F : L \to M$ also a calibrated submanifold or φ-submanifold if $F^*\varphi = dV_{F^*g}$. In [3], Harvey and Lawson studied geometries of calibrated submanifolds and showed that calibrated submanifolds are homologically volume minimizing in (M, g). Hence it is clear that calibrated submanifolds are automatically minimal submanifolds. This is the most important property of calibrated submanifolds. The precise statement and its very fundamental proof is the following.

Theorem 1.3 (Harvey and Lawson [3]). *Let (M, g) be a Riemannian manifold with a calibration φ and $L \subset M$ be a compact φ-submanifold. Then for any submanifold L' in $[L]$, the homolog class of L, we have $\mathrm{Vol}_g(L) \leq \mathrm{Vol}_g(L')$, where $\mathrm{Vol}_g(\cdot)$ is the volume of the submanifold measured by the metric g.*

Proof. We have $\mathrm{Vol}_g(L) = \int_L \varphi = \int_{L'} \varphi \leq \mathrm{Vol}_g(L')$. The first equality follows from (2). The middle equality follows from that the pairing of a closed form and a homology class, it is given by the integration, does not depend on the choice of representations. The final inequality follows from (1). □

As explained in Section 4.2 in Joyce [5], interesting calibrations can be constructed naturally if the ambient Riemannian manifold (M, g) has a special holonomy. One of such examples is a Kähler manifold. Actually, let (M, ω, g, J) be a complex m-dimensional Kähler manifold with a symplectic form ω, Riemannian metric g and complex structure J. Then one can see that $\varphi := \omega^p/p!$ for some $1 \leq p \leq m$ becomes a calibration on (M, g) by Wirtinger's inequality. It is well-known that the φ-submanifolds are just the canonically oriented complex submanifolds of complex dimension p in M. Hence it follows that the complex submanifolds are homologically volume minimizing in the Kähler manifold. Another such example is an (almost) Calabi–Yau manifold, a main subject in this paper, which is explained in the next section and calibrated submanifolds in it are called special Lagrangian submanifolds.

2. Almost Calabi–Yau manifolds and Lagrangian submanifolds

First of all, we introduce the notion of almost Calabi–Yau manifolds following Definition 8.4.3 of Joyce [5]. Almost Calabi–Yau manifolds are ambient spaces for special Lagrangian submanifolds, weighted hamiltonian stationary Lagrangian submanifolds and generalized Lagrangian mean curvature flows defined also in this section. Let (M, ω, g, J) be a complex m-dimensional Kähler manifold.

Definition 2.1. If there exists a non-vanishing holomorphic $(m, 0)$-form Ω over M, we call this quintuplet $(M, \omega, g, J, \Omega)$ an *almost Calabi–Yau manifold* and Ω a holomorphic volume form over M.

It is clear that the canonical line bundle of an almost Calabi–Yau manifold is trivial and its 1st Chern class, denoted by $c_1(M)$, is zero, since its holomorphic volume form gives a global trivialization of it. On an almost Calabi–Yau manifold $(M, \omega, g, J, \Omega)$, we define a real valued function $\psi : M \to \mathbb{R}$ by

$$e^{2m\psi}\frac{\omega^m}{m!} = (-1)^{\frac{m(m-1)}{2}}\left(\frac{i}{2}\right)^m \Omega \wedge \bar{\Omega}. \tag{3}$$

Then one can easily see that the Ricci form $\rho(\omega)$ of this almost Calabi–Yau manifold is given by

$$\rho(\omega) = 2mi\partial\bar{\partial}\psi.$$

Thus it follows that ω is a Ricci flat Kähler metric if and only if ψ is a constant. Especially if $\psi = 0$, we call $(M, \omega, g, J, \Omega)$ a Calabi–Yau manifold.

Note that if an almost Calabi–Yau manifold M is compact then we can give a new Kähler structure on it so that M becomes a Calabi–Yau manifold by Calabi Ansatz since $c_1(M) = 0$. However this given Calabi–Yau metric is not explicit in general. On the other hand, there are many examples of almost Calabi–Yau metrics which have explicit forms. Hence, we prefer to work on almost Calabi–Yau manifolds rather than Calabi–Yau manifolds to observe some concrete examples of Lagrangian submanifolds in these.

The most typical example of almost Calabi–Yau manifolds is a complex space $\mathbb{C}^m$ with the standard structure described precisely below, and actually it is a Calabi–Yau manifold.

Example 2.2. Let $(z_1, \ldots, z_m)$ be the standard complex coordinates on $\mathbb{C}^m$ with the standard complex structure J. If we define a Kähler form ω and a holomorphic volume form Ω by

$$\omega := \frac{i}{2}\sum_{j=1}^{m} dz_j \wedge d\bar{z}_j \quad \text{and} \quad \Omega := dz_1 \wedge \cdots \wedge dz_m,$$

then $(\mathbb{C}^m, \omega, g, J, \Omega)$ is an (almost) Calabi–Yau manifold, where g is the standard Euclidean metric on $\mathbb{R}^{2m} \cong \mathbb{C}^m$.

In the equality (3), the term $\omega^m/m!$ is equal to dV_g, the volume form of g, and it is clear that $e^{2m\psi}\omega^m/m!$ is equal to $dV_{\tilde{g}}$ where $\tilde{g} := e^{2\psi}g$ is the conformal rescaling of g with ψ. Hence the equality (3) is reformulateted as

$$dV_{\tilde{g}} = (-1)^{\frac{m(m-1)}{2}} \left(\frac{i}{2}\right)^m \Omega \wedge \bar{\Omega}.$$

By this equality, we have $|\Omega|_{\tilde{g}} = \sqrt{2}^m$, note that the reft hand side is the norm of Ω with respect to the metric $\tilde{g}$, not g. Then we can see that

$$\varphi_\beta := \mathrm{Re}(e^{-i\beta}\Omega)$$

is a calibration on Riemannian manifold $(M, \tilde{g})$, for all $\beta \in \mathbb{R}$. This method is also mentioned in Section V.3 in the paper of Harvey and Lawson [3].

Definition 2.3. We call a φ_β-submanifold in Riemannian manifold $(M, \tilde{g})$ a *special Lagrangian submanifold* with phase $e^{i\beta}$.

The name of special Lagrangian submanifold comes from the fact that a special Lagrangian submanifold is a Lagrangian submanifold in the symplectic manifold (M, ω), though it is non-trivial by the definition. The outline of the proof is the following. Take a point p in M and a real m-dimensional subvector space ξ in T_pM. Let $e_1, \ldots, e_m$ be an orthogonal basis of ξ with respect to the metric $\tilde{g}(= e^{2\psi}g)$. Then we define dV_ξ by $e_1^* \wedge \cdots \wedge e_m^*$, where e_j^* is the dual. Note that here we do not assume that ξ is oriented, hence dV_ξ is defined up to sign. We define a complex number $\alpha_\xi \in \mathbb{C}/\{\pm 1\}$ (up to multiplication by ± 1) by

$$\Omega|_\xi = \alpha_\xi dV_\xi.$$

Then one can show that $|\alpha_\xi| \leq 1$, and $|\alpha_\xi| = 1$ if and only if ξ is a Lagrangian subvector space in (T_pM, ω_p). Hence if L is a $\mathrm{Re}(e^{-i\beta}\Omega)$-submanifold, then it is necessary that L is a Lagrangian submanifold.

For a Lagrangian subvector space ξ, we take the argument of α_ξ (it is defined modulo $\pi\mathbb{Z}$) and denote it by $\theta_\xi := \arg \alpha_\xi \in \mathbb{R}/\pi\mathbb{Z}$.

Definition 2.4. For a Lagrangian submanifold L, we define a function $\theta_L : L \to \mathbb{R}/\pi\mathbb{Z}$ by

$$\theta_L(p) := \theta_{T_pL},$$

and call it the *Lagrangian angle* of L.

Note that if L is oriented then the Lagrangian angle $\theta_L : L \to \mathbb{R}/\pi\mathbb{Z}$ has a lift $\theta_L : L \to \mathbb{R}/2\pi\mathbb{Z}$. One can show that a special Lagrangian submanifold with phase $e^{i\beta}$ is a Lagrangian submanifold whose Lagrangian angle θ_L is a constant β. We call the cohomology class $[d\theta_L] \in H^1(L, \mathbb{R})$ the *Maslov class* of L. It is equivalent to that the Maslov class is zero and the Lagrangian angle $\theta_L : L \to \mathbb{R}/\pi\mathbb{Z}$ has a lift $\theta_L : L \to \mathbb{R}$, and if these two hold then L is called *graded.* Hence it is clear that a special Lagrangian submanifold is graded, since its Lagrangian angle is constant.

As explained in Section 1, a special Lagrangian submanifold is volume-minimizing in its homology class (note that the volume is measured by the metric $\tilde{g}$) since it is a calibrated submanifold. Especially it is a minimal submanifold in $(M, \tilde{g})$. Furthermore the converse is true under the Lagrangian assumption, that is, if L is a minimal Lagrangian submanifold in $(M, \tilde{g})$ then it is a special Lagrangian submanifold. We explain this below. Let L be a real m-dimensional submanifold in M. We denote the mean curvature vector field of L defined by the ambient metic g by H_L and the one defined by $\tilde{g}(= e^{2\psi}g)$ by $\tilde{H}_L$. Then there is a relation as

$$\tilde{H}_L = e^{-2\psi}(H_L - m\nabla\psi^{\perp}),$$

where $\nabla\psi^{\perp}$ is the normal part (with respect to TL) of the gradient of ψ defined by the equation (3). Here we introduce the notion of generalized mean curvature vector field following Behrndt [1].

Definition 2.5. We call a vector field defined by

$$K_L := e^{2\psi}\tilde{H}_L = H_L - m\nabla\psi^{\perp}$$

the *generalized mean curvature vector field* on L.

Note that if M is a Calabi–Yau manifold, that is, $\psi = 0$ then K_L coincides with H_L. If L is a Lagrangian submanifold then the (generalized) mean curvature vector field and the Lagrangian angle has a relation stated precisely below, it is proved in Proposition 2.17 in Harvey and Lawson [3, III.2.D.] for the case that M is $\mathbb{C}^m$ and for Calabi–Yau case see the paper of Thomas and Yau [8], and Behrndt proved in almost Calabi–Yau cases.

Proposition 2.6 (Behrndt [1]). *On a Lagrangian submanifold L, we have*

$$K_L = J\nabla\theta_L. \tag{4}$$

Hence we see that if a Lagrangian submanifold is minimal in $(M, \tilde{g})$, that is, $\tilde{H}_L = 0$, it is equivalent to that $K_L = 0$, then the Lagrangian angle θ_L is constant by the identity (4). Here we summarize some equivalent conditions for special Lagrangian submanifolds.

Proposition 2.7. *For an real m-dimensional submanifold L in M, the following four conditions are equivalent.*

(1) *L is a special Lagrangian submanifold with phase $e^{i\beta}$ for some $\beta \in \mathbb{R}$.*
(2) *L is a Lagrangian submanifold whose Lagrangian angle θ is a constant β.*
(3) *L is a minimal Lagrangian submanifold in $(M, \tilde{g})$.*
(4) *L is a Lagrangian submanifold and* $\mathrm{Im}(e^{-i\beta}\Omega)|_L = 0$.

Note that in the conditions (2)–(4) it follows that L is orientable, and we have to admit an orientation on L so that $\mathrm{Re}(e^{-i\beta}\Omega)|_L$ becomes the volume form on L with the induced metric $\tilde{g}|_L$.

Locally, a Lagrangian submanifold is expressed as a graph of 1-form df of some function $f : (\mathbb{R}^m \supset)\Omega \to \mathbb{R}$ under a Darboux chart. Then the condition $\mathrm{Im}(e^{-i\beta}\Omega)|_L = 0$ in (4) in Proposition 2.7 is expressed as a non-linear second order elliptic equation of f, Monge–Ampère type, as explained in [5]. Hence, to construct examples of special Lagrangian submanifolds in a given almost Calabi–Yau manifold is a difficult problem in general. As a method to avoid this difficulty, in this paper, we focus on the condition (2) in Proposition 2.7, the minimality of special Lagrangian submanifolds. To get minimal submanifold, we can consider mean curvature flows. This is a parabolic equation like a heat equation. The precise definition is the following. Let (N, h) be an n-dimensional Riemannian manifold, L be an k-dimensional manifold and $F_0 : L \to N$ be an immersion.

Definition 2.8. We say that a smooth 1-parameter family of immersions $F : L \times [0, T) \to N$, which is continuous up to $t = 0$, is evolving by mean curvature flow with the initial condition $F_0 : L \to N$ if it satisfies

$$\left.\frac{\partial F}{\partial t}\right|_{t=s} = H(F_s) \text{ for } s \in (0, T) \quad \text{and} \quad F(\cdot, 0) = F_0. \tag{5}$$

Here we denote the mean curvature vector field of the immersion $F_s := F(\cdot, s)$ by $H(F_s)$. It is known that there is the short-time existence and uniqueness result for the mean curvature flow in the case that M is compact. It is proved by Hamilton's theorem [2] used for the short-time existence and uniqueness result for the Ricci flow. Mean curvature flows appear naturally as the backward L^2 gradient flow of the volume functional. We explain this below. Let $\mathrm{Imm}(L, N)$ be the set of all immersion maps from L to N. Then we define the volume functional

$$\mathrm{Vol}_h : \mathrm{Imm}(L, N) \to \mathbb{R} \quad \text{by} \quad \mathrm{Vol}_h(F) := \int_L 1 dV_{F^*h},$$

it is just the volume of $F(L)$ measured by the metric h. Then it is well-known that the first variation of Vol_h is given as follows.

Proposition 2.9. *Let $F : L \to (N, h)$ be an immersion and $\{F_s : L \to N\}_{s \in (-\epsilon, \epsilon)}$ be a smooth 1-parameter family of immersions (that is a curve in $\mathrm{Imm}(L, N)$) with*

$$F_0 = F \quad \text{and} \quad V := \left.\frac{\partial F}{\partial s}\right|_{s=0}.$$

Then we have

$$\left.\frac{d}{ds}\right|_{s=0} \mathrm{Vol}_h(F_s) = -\int_L h(V, H(F)) dV_{F^*h}.$$

By this proposition, it is clear that minimal submanifolds are critical points of the volume functional, and the mean curvature flow is the backward L^2 gradient flow of the volume functional and the volume is monotone decreasing along a mean curvature flow. This is one of characterizations of mean curvature flows.

Let us come back to the case that $(N, h) = (M^{2m}, \tilde{g})$. In this case, as an analog of mean curvature flows, Behrndt introduces generalized mean curvature flows in [1]. Let L be a real m-dimensional manifold.

Definition 2.10. We say that a smooth 1-parameter family of immersions $F : L \times [0, T) \to M$, which is continuous up to $t = 0$, is evolving by *generalized mean curvature flow* with the initial condition $F_0 : L \to M$ if it satisfies

$$\left.\frac{\partial F}{\partial t}\right|_{t=s} = K(F_s) \text{ for } s \in (0, T) \quad \text{and} \quad F(\cdot, 0) = F_0. \tag{6}$$

Note that in the original definition of [1] Behrndt considers the normal part of $\partial F / \partial t$. The advantage of considering a generalized mean curvature flow in an almost Calabi–Yau manifold is that the Lagrangian condition is preserved along the flow. It was first proved by Smoczyk in [7] for Calabi–Yau cases by the parabolic maximum principle for the norm of $F_t^* \omega$ on L, and Behrndt generalized this result for almost Calabi–Yau cases.

Proposition 2.11. *Let $F : L \times [0, T) \to M$ be a solution of generalized mean curvature flows. If the initial condition $F_0 : L \to M$ is a Lagrangian immersion, then F_t is also a Lagrangian immersion for every $t \in [0, T)$.*

If $F : L \times [0, T) \to M$ is a solution of generalized mean curvature flows and each F_t is a Lagrangian immersion for every $t \in [0, T)$, then we call it the generalized Lagrangian mean curvature flow. Hence we hope that if there exists a long time solution $F : L \times [0, \infty) \to M$ of generalized mean curvature flows for a given Lagrangian immersion $F_0 : L \to M$ and F_t converges to some immersion as $t \to \infty$ then we can get a special Lagrangian immersion $F_\infty : L \to M$, since it is also a Lagrangian submanifold by Proposition 2.11

and $K(F_\infty) = 0$ (see also Proposition2.7). Actually in some cases this hope is confirmed to be true however in generic cases generalized mean curvature flows develop singularities in finite time. In this paper, we construct examples of generalized Lagrangian mean curvature flows in toric almost Calabi–Yau manifolds which have finite time singularities and can be continued over singular times in some sense.

Before we step into the next section, we define weighted hamiltonian stationary Lagrangian submanifolds. It can be considered as a weak notion of special Lagrangian submanifold. Remember that a special Lagrangian submanifold L (or immersion $F : L \to M$) is a minimal Lagrangian submanifold (or immersion) by Proposition2.7. Hence it is a critical point of the *weighted volume functional*;

$$\mathrm{Vol}_{\tilde{g}} : \mathrm{Imm}(L, N) \to \mathbb{R} \quad \text{by} \quad \mathrm{Vol}_{\tilde{g}}(F) := \int_L 1 dV_{F^*\tilde{g}},$$

along all infinitesimal deformations as submanifolds. In some sense, a weighted hamiltonian stationary Lagrangian submanifolds is also a critical point of the weighted volume functional, however its variations are restricted to only *Hamiltonian deformations.* The precise meaning is the following. First, Let $F : L \to M$ be a Lagrangian immersion to an almost Calabi–Yau manifold $(M, \omega, g, J, \Omega)$. Next take an infinitesimal Hamiltonian deformation $\{F_s : L \to M\}_{s\in(-\epsilon,\epsilon)}$ of F, that is, we assume that it satisfies $F_0 = F$ and there exists a function $f \in C^\infty(L)$ such that

$$\left.\frac{\partial F}{\partial t}\right|_{t=0} = J\nabla f. \tag{7}$$

Under these assumptions, taking the first variation of the weighted volume functional by using Proposition 2.9, we have

$$\left.\frac{d}{ds}\right|_{s=0} \mathrm{Vol}_{\tilde{g}}(F_s) = -\int_L \tilde{g}(J\nabla f, \tilde{H}(F)) dV_{F^*\tilde{g}} = -\int_L g(J\nabla f, K(F)) dV_{F^*\tilde{g}}.$$

In the second equality, we used the relations $\tilde{g} = e^{2\psi} g$ and $\tilde{H} = e^{-2\psi} K$. Furthermore we can use the relation $K(F) = J\nabla\theta_F$ by (4). Hence we have

$$\begin{aligned}
\left.\frac{d}{ds}\right|_{s=0} \mathrm{Vol}_{\tilde{g}}(F_s) &= -\int_L g(J\nabla f, J\nabla\theta_F) dV_{F^*\tilde{g}} \\
&= -\int_L \langle df, d\theta_F\rangle_g dV_{F^*\tilde{g}} \\
&= -\int_L f\Delta_\psi\theta_F dV_{F^*\tilde{g}},
\end{aligned}$$

where Δ_ψ is the weighted Laplacian on L defined by $\Delta_\psi u := \Delta u - mg(\nabla\psi, \nabla u)$, here Δ is the Laplacian on Riemannian manifold (L, F^*g). Thus we can see that the first variations of the weighted volume functional along all Hamiltonian deformations are zero if and only if the Lagrangian angle $\theta_F : L \to \mathbb{R}/\pi\mathbb{Z}$ is weighted harmonic, that is, $\Delta_\psi\theta_F = 0$.

Definition 2.12. We call a Lagrangian submanifold with weighted harmonic Lagrangian angle a weighted hamiltonian stationary Lagrangian submanifold.

It is clear that a special Lagrangian submanifold is a weighted hamiltonian stationary Lagrangian submanifold, since its Lagrangian angle is constant. Note that since θ_F is a $\mathbb{R}/\pi\mathbb{Z}$-valued function the condition $\Delta_\psi\theta_F = 0$ does not imply θ_F is constant whenever L is compact. For example, $S^1 := \{ e^{i\theta} \mid \theta \in \mathbb{R} \} \subset \mathbb{C}$ is a (weighted) hamiltonian stationary Lagrangian submanifold since its Lagrangian angle is given by $\theta(e^{i\theta}) := \theta$, this is locally linear and the second derivative is clearly zero, however this is not a special Lagrangian submanifold since this is not minimal.

3. Examples of generalized Lagrangian mean curvature flows in toric almost Calabi–Yau manifolds

In this section, we review examples of generalized Lagrangian mean curvature flows in toric almost Calabi–Yau manifolds constructed in [10]. First of all, we introduce some basic notions of toric Kähler geometries. Let $(M^{2m}, \omega, g, J) \curvearrowleft T^m$ be a complex m-dimensional toric Kähler manifold with a Hamiltonial T^m action. Then we have a moment map $\mu : M \to \mathbb{R}^m$ and its moment polytope $\Delta := \mu(M)$. Here we assume that Δ is given by

$$\Delta = \{\, y \in \mathbb{R}^m \mid \langle y, \lambda_i \rangle \geqq \kappa_i,\ i = 1, \ldots, d \,\}$$

for some primitive integral inward normal vectors λ_i and constants κ_i. Note that on a toric Kähler manifold (M^{2m}, ω, g, J), there exists an anti-holomorphic and anti-symplectic involution $\sigma : M \to M$ $(\sigma^2 = id)$. We call the set of fixed points of σ a real form, and denote it by

$$M^\sigma := \{\, p \in M \mid \sigma(p) = p \,\}.$$

We restrict the moment map $\mu : M \to \Delta$ to M^σ, and denote it by

$$\mu^\sigma := \mu|_{M^\sigma} : M^\sigma \to \Delta.$$

Note that this is a 2^m-fold ramified covering map over Δ. In $\mathbb{C}^m$, the most typical example of toric Kähler manifolds, the involution σ is just the complex conjugation given by $\sigma(z_1, \ldots, z_m) := (\bar{z}_1, \ldots, \bar{z}_m)$, and the real form $(\mathbb{C}^m)^\sigma$ is just the real plane $\mathbb{R}^m \subset \mathbb{C}^m$.

We can construct a Lagrangian submanifold in M by an affine plane in Δ. We explain this construction below. Fix $0 \leq k \leq m$ arbitrary. Let $A(V, c) := V + c$ be a k-dimensional affine plane in $\mathbb{R}^m$, where $V \subset \mathbb{R}^m$ is a k-dimensional subspace and $c \in \mathbb{R}^m$ is a vector. We assume that $A(V, c)$ intersects the interior of Δ. Then we put

$$\begin{aligned}
&M^\sigma(V, c) := (\mu^\sigma)^{-1}(\Delta \cap A(V, c)), \\
&T(V^\perp) := V^\perp / (V^\perp \cap \mathbb{Z}^m) \cong T^{m-k} \subset T^m \text{ and} \\
&L(V, c) := T(V^\perp) \cdot M^\sigma(V, c).
\end{aligned}$$

Here $V^\perp$ is the orthogonal complement of V. Note that we assume that $V^\perp/(V^\perp \cap \mathbb{Z}^m)$ is isomorphic to a subtorus T^{m-k} in T^m and we also assume that $M^\sigma(V, c)$ becomes a smooth real k-dimensional submanifold in M. Then $L(V, c)$, the $T(V^\perp)$-orbit of $M^\sigma(V, c)$, becomes a Lagrangian submanifold in M automatically. Especially, if we take $A(V, c)$ as a 0-dimensional affine plane, that is a point c in Δ, then $L(V, c)$ becomes just a torus fiber of $\mu^{-1}(c) \cong T^m$, and if we take $A(V, c)$ as a m-dimensional affine plane, that is just $\mathbb{R}^m$, then $L(V, c)$ becomes just the real form M^σ. These two are typical Lagrangian submanifolds in M. Hence, roughly speaking, $L(V, c)$ is a hybrid (or interpolation) of a torus fiber T^m and the real form M^σ, and $m - k$ is the dimension of torus factors in $L(V, c)$

Example 3.1. The complex space $\mathbb{C}^m$ is the standard toric Kähler manifold with a moment map $\mu(z_1, \ldots, z_m) := \frac{1}{2}(|z_1|^2, \ldots, |z_m|^2)$. For example, let $\xi \in \mathbb{Z}^m$ be a primitive integral vector and define the $(m-1)$-dimensional vector space V by

$$V := \{\, y \in \mathbb{R}^m \mid \langle y, \xi \rangle = y_1\xi_1 + \ldots y_m\xi_m = 0 \,\}.$$

Fix a vector $c \in \mathbb{R}^m$. Then we have an $(m-1)$-dimensional affine plane $A(V, c) := V + c$. Put $\kappa := 2\langle c, \xi\rangle$. Then $L(V, c)$ becomes a T^1-invariant Lagrangian submanifold defined by

$$L(V,c) = \{ (x_1 e^{2\pi i \xi_1 \theta}, \ldots, x_m e^{2\pi i \xi_m \theta}) \in \mathbb{C}^m \mid \theta \in \mathbb{R},\ \xi_1 x_1^2 + \cdots + \xi_m x_m^2 = \kappa \},$$

where $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$. Lagrangian submanifolds defined as the form above are constructed and studied by Joyce [4, Example 9.4].

To make sense of notions of special Lagrangian submanifolds, weighted Hamiltonian stationary Lagrangian submanifolds and generalized Lagrangian mean curvature flows, we have to admit an almost Calabi–Yau structure on the toric Kähler manifold (M^{2m}, ω, g, J). It is known that the canonical line bundle of M is trivial if and only if there exists a vector γ in $\mathbb{Z}^m$ such that $\langle \gamma, \lambda_i\rangle = 1$ for all $i = 1, \ldots, d$. From now on, we assume that the existence of such γ. Actually, using γ, a nonvanishing holomorphic volume form Ω_γ on M is given by

$$\Omega_\gamma = e^{\gamma_1 w_1 + \cdots + \gamma_m w_m} dw_1 \wedge \cdots \wedge dw_m,$$

where $(w_i)_{i=1}^m$ are logarithmic holomorphic coordinates on an open dense $T^m_{\mathbb{C}}$-orbit over M, that is, $w_i := \log z_i$ for the standard holomorphic coordinates z_i of $(\mathbb{C}^*)^m \cong T^m_{\mathbb{C}}$. Note that Ω_γ is only defined on an open dense $T^m_{\mathbb{C}}$-orbit over M, however it can be extended globally as a non-vanishing holomorphic $(m, 0)$-form on M. We call this quintuplet $(M, \omega, g, J, \Omega_\gamma) \curvearrowleft T^m$ a *toric almost Calabi–Yau manifold*.

Example 3.2. We check the $\mathbb{C}^m$ case. The moment polytope Δ of $\mathbb{C}^m$ is given by

$$\Delta := \{ y \in \mathbb{R}^m \mid \langle y, e_i\rangle \geq 0\, (i = 1, \ldots, m) \},$$

where e_i is the standard basis of $\mathbb{R}^m$. If we take $\gamma = (1, \ldots, 1)$, then it satisfies that $\langle \gamma, e_i\rangle = 1$ for all $i = 1, \ldots, d$. Let $(z_1, \ldots, z_m)$ be the standard holomorphic coordinates on $\mathbb{C}^m$. On $(\mathbb{C}^*)^m$, we can define the logarithmic holomorphic coordinates $(w_1, \ldots, w_m)$ by $w_i := \log z_i$. Then Ω_γ becomes

$$e^{w_1 + \cdots + w_m} dw_1 \wedge \cdots \wedge dw_m,$$

and this coincides with the standard holomorphic volume form $dz_1 \wedge \cdots \wedge dz_m$.

For a given k-dimensional affine plane $A(V, c) := V + c$, let $L(V, c)$ be the Lagrangian submanifold constructed above and we denote the normal part of γ with respect to V by $\gamma^{\perp_V}$. Then main results in the paper [10] are stated as follows.

Theorem 3.3. *In the toric almost Calabi–Yau manifold $(M, \omega, g, J, \Omega_\gamma)$, the Lagrangian submanifold $L(V, c)$ is a weighted Hamiltonian stationary Lagrangian submanifold and its Lagrangian angle θ of $L(V, c)$ is given by $\theta([b] \cdot p) = 2\pi\langle \gamma, b\rangle + \frac{\pi}{2}(m - k)$ for $b \in V^\perp$ and $p \in M^\sigma(V, c)$. Hence it is clear that $L(V, c)$ is a special Lagrangian submanifold if and only if $\gamma^{\perp_V} = 0$.*

Theorem 3.4. *If we put $c(t) := c - t\gamma^{\perp_V}$, then a one parameter family of Lagrangian submanifolds $\{ L(V, c(t)) \}_{t\in[0,T)}$ is a solution of generalized Lagrangian mean curvature flows with singularities and topological changes with the initial condition $L(V, c)$. Here T is the first time such that $L(V, c(t))$ becomes the empty set.*

Here the precise definition of a generalized Lagrangian mean curvature flow with singularities and topological changes is the following.

Definition 3.5. Let $(M, \omega, g, J, \Omega)$ be a real $2m$-dimensional almost Calabi–Yau manifold and $\{L_t\}_{t\in I}$ be a one parameter family of subsets in M. Then we call $\{L_t\}_{t\in I}$ a solution of a generalized Lagrangian mean curvature flow with singularities and topological changes if there exists a real m-dimensional manifold L and a solution of generalized Lagrangian mean curvature flows $F : L \times I \to M$ such that $F_t : L \to M$ is an embedding into L_t and m-dimensional Hausdorff measure of $L_t \setminus F_t(L)$ is zero, that is,

$$F_t(L) \subset L_t \quad \text{and} \quad \mathcal{H}^m(L_t \setminus F_t(L)) = 0.$$

It means that $\{L_t\}_{t\in I}$ is almost parametrized by a smooth solution of generalized Lagrangian mean curvature flows.

Example 3.6. In Example 3.1, we consider a T^1-invariant Lagrangian submanifold

$$L(V, c) = \{\, (x_1 e^{2\pi i \xi_1 \theta}, \dots, x_m e^{2\pi i \xi_m \theta}) \in \mathbb{C}^m \mid \theta \in \mathbb{R},\ \xi_1 x_1^2 + \cdots + \xi_m x_m^2 = \kappa \}.$$

in $\mathbb{C}^m$ constructed by $\xi \in \mathbb{Z}^m$. By applying Theorem 3.3, $L(V, c)$ is a special Lagrangian submanifold if and only if

$$\xi_1 + \cdots + \xi_m = 0.$$

Actually, this is proved by Joyce in [4]. Next, we apply Theorem 3.4. Put $c = 0$ in Example 3.1. Remember that $\gamma = (1, \dots, 1)$ in $\mathbb{C}^m$ case. Then we have $c(t) = c - t\gamma^{\perp_V} = -t((\xi_1 + \cdots + \xi_m)/|\xi|^2)\xi$ and

$$L(V, c(t)) = \left\{ (x_1 e^{2\pi i \xi_1 s}, \dots, x_m e^{2\pi i \xi_m s}) \in \mathbb{C}^m \,\middle|\, 0 \le s \le 1, \right.$$
$$\left. \sum_{j=1}^m \xi_j x_j^2 = -2t \sum_{j=1}^m \xi_j,\ x = (x_1, \dots, x_m) \in \mathbb{R}^m \right\}.$$

By Theorem 3.4, we can see that $\{\, L(V, c(t)) \,\}_{t\in[0,T)}$ is a solution of (generalized) Lagrangian mean curvature flows with singularities and topological changes with the initial condition $L(V, c)$. Actually, $L(V, c(t))$ coincides with V_t in Theorem 1.1 in [6], and Lee and Wang proved that V_t is Hamiltonian stationary and $\{V_t\}_{t\in\mathbb{R}}$ form an eternal solution for Brakke flows. Hence our two theorems above can be considered as some kind of generalization of results of Joyce [4] and Lee and Wang [6] in $\mathbb{C}^m$ to the one in toric almost Calabi–Yau manifolds.

Finally, we give an example of generalized Lagrangian mean curvature flows with singularities and topological changes.

Example 3.7. Let $M := K_{\mathbb{P}^2}$ be the total space of the canonical line bundle of $\mathbb{P}^2$. Then a moment polytope is given by $\Delta = \{\, y \in \mathbb{R}^3 \mid \langle y, \lambda_i \rangle \ge \kappa_i,\ i = 1, \dots, 4 \,\}$ where

$$\lambda_1 = (0, 0, 1),\ \lambda_2 = (1, 0, 1),\ \lambda_3 = (0, 1, 1),\ \lambda_4 = (-1, -1, 1)$$

and $\kappa_1 = \kappa_2 = \kappa_3 = 0$, $\kappa_4 = -1$. Then M is a toric almost Calabi–Yau manifold since we can take $\gamma = (0, 0, 1)$ so that $\langle \gamma, \lambda_i \rangle = 1$ for all $i = 1, 2, 3, 4$. For example, take $\xi = (3, 1, 5)$ and consider a 2-dimensional subvector space $V := \{\, y \in \mathbb{R}^m \mid \langle y, \xi \rangle = 0 \,\}$. Take $c = (0, 0, 1)$. Then we have

$$A(V, c(t)) = \{\, y \in \mathbb{R}^m \mid \langle y, \xi \rangle = 5 - 5t \,\}$$

and denote the intersection of $A(V, c(t))$ and Δ by $\Delta(V, c(t))$. We write each facet of Δ by $F_i := \{\, y \in \Delta \mid \langle y, \lambda_i \rangle = \kappa_i \,\}$ for $i = 1, 2, 3, 4$.

By simple calculation, one can easily see that when $0 \le t < \frac{2}{5}$ then $A(V, c(t))$ intersects with F_2, F_3 and F_4 hence $\Delta(V, c(t))$ is a triangle, when $t = \frac{2}{5}$ then $A(V, c(t))$ across $(1, 0, 0)$

a vertex of Δ and a topological change happens, when $\frac{2}{5} < t < \frac{4}{5}$ then $A(V, c(t))$ intersects with F_1, F_2, F_3 and F_4 hence $\Delta(V, c(t))$ is a square, when $t = \frac{4}{5}$ then $A(V, c(t))$ across $(0, 1, 0)$ a vertex of Δ and a topological change happens, when $\frac{4}{5} < t < 1$ then $A(V, c(t))$ intersects with F_1, F_2 and F_3 so $\Delta(V, c(t))$ is a triangle, and when $t = 1$ then $\Delta(V, c(t))$ is one point $\{(0, 0, 0)\}$ this means that $L(V, c(t))$ vanishes. Hence a solution $\{L(V, c(t))\}_{t \in I}$ of generalized Lagrangian mean curvature flows with singularities and topological changes exists for $t \in I = [0, 1)$. It forms singularities and topological changes when $t = \frac{2}{5}$ and $t = \frac{4}{5}$, and vanishes when $t = 1$.

One can see the topology of $L(V, c(t))$, the S^1-orbit of $M^\sigma(V, c(t))$, by the same argument as explained in the proof of Proposition A.3 in [9]. In fact the topology of $M^\sigma(V, c(t))$ is S^2 when $0 \leq t < \frac{2}{5}$, is T^2 when $\frac{2}{5} < t < \frac{4}{5}$, and is S^2 when $\frac{4}{5} < t < 1$.

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