

# On the Baire property of a function space

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## 1 Introduction

In this article, we define a hypo-graph of each continuous function from a compact metrizable space to a non-degenerate dendrite, endow the space of hypo-graphs with certain topology and discuss the topological properties of that space. In geometric functional analysis, being a Baire space is one of the most important topological properties for a function space, and hence, it is natural to ask when a function space is a Baire space. The main purpose of this article is to provide necessary and sufficient conditions for the space of hypo-graphs to be a Baire space. In the last section, we will consider the topological type of that space. This article is a résumé of the paper [2].

Throughout the article, we assume that all maps are continuous, but functions are not necessarily continuous. Moreover, let  $X$  be a compact metrizable space and  $Y$  be a non-degenerate dendrite with a distinguished end point  $\mathbf{0}$ . We recall that a *dendrite* is a Peano continuum containing no simple closed curves. The following fact is well-known [10, Chapter V, (1.2)]:

*Fact 1.* Any two distinct points of a dendrite are joined by one and only one arc.

From now on, for any two points  $x, y \in Y$ , the symbol  $[x, y]$  means the one and only one arc between  $x$  and  $y$  if  $x \neq y$ , or the singleton  $\{x\} = \{y\}$  if  $x = y$ .

For each function  $f : X \rightarrow Y$ , we define the *hypo-graph*  $\downarrow f$  of  $f$  as follows:

$$\downarrow f = \bigcup_{x \in X} \{x\} \times [\mathbf{0}, f(x)] \subset X \times Y.$$

Note that if  $f$  is continuous, then the hypo-graph  $\downarrow f$  is closed in  $X \times Y$ . By  $\text{Cld}(X \times Y)$  we denote the hyperspace of non-empty closed subsets of  $X \times Y$  endowed with the Vietoris topology. Then we can regard the set

$$\downarrow \mathcal{C}(X, Y) = \{\downarrow f \mid f : X \rightarrow Y \text{ is continuous}\}$$

of hypo-graphs of continuous functions from  $X$  to  $Y$  as a subset of  $\text{Cld}(X \times Y)$ . We equip  $\downarrow \mathcal{C}(X, Y)$  with the subspace topology of  $\text{Cld}(X \times Y)$ .

A closed set  $A$  in a space  $W$  is called a  $Z$ -set in  $W$  if for any open cover  $\mathcal{U}$  of  $W$ , there is a map  $f : W \rightarrow W$  such that for each point  $x \in W$ , the both  $x$  and  $f(x)$  are contained in some  $U \in \mathcal{U}$  and  $f(W) \cap A = \emptyset$ . This concept plays a central role in infinite-dimensional topology. A  $Z_\sigma$ -set is a countable union of  $Z$ -sets. As is easily observed, every  $Z$ -set is nowhere dense, and hence any space that is a  $Z_\sigma$ -set in itself is not a Baire space. We shall give necessary and sufficient conditions for  $\downarrow C(X, Y)$  to be a Baire space as follows (Z. Yang [8] showed the case that  $Y$  is the closed unit interval  $\mathbf{I} = [0, 1]$  and  $\mathbf{0} = 0$ ):

**Main Theorem.** *The following are equivalent:*

- (1)  $\downarrow C(X, Y)$  is a Baire space;
- (2)  $\downarrow C(X, Y)$  is not a  $Z_\sigma$ -set in itself;
- (3) The set of isolated points of  $X$  is dense.

## 2 Preliminaries

In this section, we introduce some notation and lemmas used later. The natural numbers is denoted by  $\mathbb{N}$ . For a metric space  $W = (W, d)$  and  $\epsilon > 0$ , let  $B_d(x, \epsilon) = \{y \in W \mid d(x, y) < \epsilon\}$ . A metric  $d$  is *convex* if any two points  $x$  and  $y$  in  $W$  have a mid point  $z$ . When  $d$  is convex and complete, there exists a path between  $x$  and  $y$  isometric to the interval  $[0, d(x, y)]$ . Every Peano continuum admits a convex metric, see [1] and [5, 6]. In the remaining of this article, we use an admissible metric  $d_X$  on  $X$  and an admissible convex metric  $d_Y$  on  $Y$ . Arcs in a dendrite have the following nice property with respect to its admissible convex metric [3]:

**Lemma 2.1.** *There exists a map  $\gamma : Y^2 \times \mathbf{I} \rightarrow Y$  such that for any distinct points  $x, y \in Y$ , the map  $\gamma(x, y, *) : \mathbf{I} \ni t \mapsto \gamma(x, y, t) \in Y$  is an arc from  $x$  to  $y$  and the following holds:*

- For each  $x_i, y_i \in Y$ ,  $i = 1, 2$ ,  $d_Y(\gamma(x_1, y_1, t), \gamma(x_2, y_2, t)) \leq \max\{d_Y(x_1, x_2), d_Y(y_1, y_2)\}$  for all  $t \in \mathbf{I}$ .

Since  $X$  and  $Y$  are compact, the topology of  $\text{Cld}(X \times Y)$  is induced by the *Hausdorff metric*  $\rho_H$  of an admissible metric  $\rho$  on  $X \times Y$  defined as follows:

$$\rho_H(A, B) = \inf \left\{ r > 0 \mid A \subset \bigcup_{(x,y) \in B} B_\rho((x,y), r), B \subset \bigcup_{(x,y) \in A} B_\rho((x,y), r) \right\}.$$

Fix any  $A \in \text{Cld}(X \times Y)$ . For each point  $x \in X$ , let  $A(x) = \{y \in Y \mid (x, y) \in A\}$ . Moreover, for each subset  $B \subset X$ , let  $A|_B = \{(x, y) \in A \mid x \in B\}$ . The following lemma, that has been proved in [7], is a key lemma of this article.

**Lemma 2.2 (Digging Lemma).** *Let  $Z$  be a metrizable space and  $\phi : Z \rightarrow \downarrow C(X, Y)$  be a map. Suppose that  $X$  contains a non-isolated point  $a$ . Then for each map  $\epsilon : Z \rightarrow (0, 1)$ , there exist maps  $\psi : Z \rightarrow \downarrow C(X, Y)$  and  $\delta : Z \rightarrow (0, 1)$  such that for each  $x \in Z$ ,*

- (a)  $\rho_H(\psi(x), \phi(x)) < \epsilon(x)$ ,
- (b)  $\psi(x)(B_{d_X}(a, \delta(x))) = \{\mathbf{0}\}$ .

### 3 Proof of Main Theorem

This section is devoted to proving the main theorem. For the sake of convenience, by  $X_0$  we denote the set of isolated points of  $X$ . Let  $\overline{\downarrow C(X, Y)}$  be the closure of  $\downarrow C(X, Y)$  in  $\text{Cld}(X \times Y)$ . Then  $\overline{\downarrow C(X, Y)}$  is a compactification of  $\downarrow C(X, Y)$ .

**Lemma 3.1.** *The space  $\overline{\downarrow C(X, Y)} = \{A \in \text{Cld}(X \times Y) \mid (*)\}$ , where*

(\*) for each  $x \in X$ , (i)  $A(x) \neq \emptyset$ , (ii)  $[0, y] \subset A(x)$  if  $y \in A(x)$ , and (iii)  $A(x)$  is an arc or the singleton  $\{0\}$  if  $x \in X_0$ .

*Sketch of proof.* For simplicity, let  $\mathcal{A} = \{A \in \text{Cld}(X \times Y) \mid (*)\}$ . Obviously,  $\downarrow C(X, Y) \subset \mathcal{A}$ . To show that  $\mathcal{A}$  is a closed set in  $\text{Cld}(X \times Y)$ , take any sequence  $\{A_n\}_{n \in \mathbb{N}}$  in  $\mathcal{A}$  that converges to  $A \in \text{Cld}(X \times Y)$ . Noting that

$$A = \left\{ (x, y) \in X \times Y \mid \begin{array}{l} \text{for each } n \in \mathbb{N}, \text{ there is } (x_n, y_n) \in A_n \\ \text{such that } \lim_{n \rightarrow \infty} (x_n, y_n) = (x, y) \end{array} \right\},$$

we can easily prove that  $A \in \mathcal{A}$ . Consequently,  $\mathcal{A}$  is closed in  $\text{Cld}(X \times Y)$ .

We shall prove that  $\downarrow C(X, Y)$  is dense in  $\mathcal{A}$ . Take any  $A \in \mathcal{A}$  and  $\epsilon > 0$ . We need only to construct a map  $f : X \rightarrow Y$  such that  $\rho_H(\downarrow f, A) < \epsilon$ . Since  $A$  is compact, we can choose points  $(x_i, y_i) \in X \times Y$ ,  $i = 1, \dots, m$ , such that  $x_i \neq x_j$  if  $i \neq j$ , and

$$\rho_H(X \times \{0\} \cup \bigcup_{i=1}^m \{x_i\} \times [0, y_i], A) < \epsilon/2.$$

Let  $\lambda = \min\{\epsilon, d_X(x_i, x_j) \mid 1 \leq i < j \leq m\}/3 > 0$ . Using the map  $\gamma : Y^2 \times \mathbf{I} \rightarrow Y$  as in Lemma 2.1, we can define a map  $f : X \rightarrow Y$  as follows:

$$f(x) = \begin{cases} \gamma(0, y_i, (\lambda - d_X(x, x_i))/\lambda) & \text{if } x \in B_{d_X}(x_i, \lambda), i = 1, \dots, m, \\ 0 & \text{if } x \in X \setminus \bigcup_{i=1}^m B_{d_X}(x_i, \lambda), \end{cases}$$

which is the desired map.  $\square$

We prove the implication (3)  $\rightarrow$  (1) in the main theorem.

**Proposition 3.2.** *If  $X_0$  is dense in  $X$ , then  $\downarrow C(X, Y)$  is a Baire space.*

*Sketch of proof.* Let  $\mathcal{F}$  be the collection of finite subsets of  $X_0$ . For each  $F \in \mathcal{F}$  and  $n \in \mathbb{N}$ , the set

$$\mathcal{U}_{F,n} = \{A \in \overline{\downarrow C(X, Y)} \mid A(x) \subset B_{d_Y}(0, 1/n) \text{ for all } x \in X \setminus F\}$$

is open in  $\overline{\downarrow C(X, Y)}$ . Then the union  $\mathcal{U}_n = \bigcup_{F \in \mathcal{F}} \mathcal{U}_{F,n}$  is dense in  $\overline{\downarrow C(X, Y)}$ . Indeed, for each  $\downarrow f \in \overline{\downarrow C(X, Y)}$  and  $\epsilon > 0$ , we can choose  $F \in \mathcal{F}$  so that  $\rho_H(\downarrow f|_F, \downarrow f) < \epsilon$  because  $X_0$  is dense in  $X$ . Define a map  $g : X \rightarrow Y$  as follows:

$$g(x) = \begin{cases} f(x) & \text{if } x \in F, \\ 0 & \text{if } x \in X \setminus F. \end{cases}$$

Then we have  $\downarrow g \in \mathcal{U}_{F,n} \subset \mathcal{U}_n$  and  $\rho_H(\downarrow g, \downarrow f) < \epsilon$ . Since  $\overline{\downarrow C(X, Y)}$  is compact, the  $G_\delta$ -set  $\mathcal{G} = \bigcap_{n \in \mathbb{N}} \mathcal{U}_n$  is a Baire space and dense in  $\overline{\downarrow C(X, Y)}$ .

Next, we show that  $\mathcal{G} \subset \downarrow C(X, Y)$ . Take any  $A \in \mathcal{G}$ . Observe that for any  $x \in X \setminus X_0$ ,  $A(x) = \{\mathbf{0}\}$ . According to Lemma 3.1, for each  $x \in X_0$ ,  $A(x)$  is an arc or the singleton  $\{\mathbf{0}\}$ , and therefore  $A$  is a hypo-graph of some function  $f : X \rightarrow Y$ . Then  $f$  is continuous. Hence  $A = \downarrow f \in \downarrow C(X, Y)$ , so  $\mathcal{G} \subset \downarrow C(X, Y)$ . Consequently,  $\downarrow C(X, Y)$  is a Baire space.  $\square$

The following lemma is a counterpart to Lemma 5 of [8], but we can not prove it by the same way. The reason is because for hypo-graphs  $\downarrow f, \downarrow g \in \downarrow C(X, Y)$  and a point  $x \in X$ ,  $(\downarrow f \cup \downarrow g)(x) = \downarrow f(x) \cup \downarrow g(x)$  is not necessarily an arc or the singleton  $\{\mathbf{0}\}$  in  $Y$ , so  $\downarrow f \cup \downarrow g \notin \downarrow C(X, Y)$ . Using the Digging Lemma 2.2, we prove the following:

**Lemma 3.3.** *Suppose that  $\mathcal{A} = \mathcal{B} \cup \mathcal{Z} \subset \downarrow C(X, Y)$  is a closed set such that  $\mathcal{Z}$  is a  $Z$ -set in  $\downarrow C(X, Y)$ , and there exists a point  $x \in X$  such that for every  $\downarrow f \in \mathcal{B}$ ,  $\downarrow f(x) = \{\mathbf{0}\}$ . Then  $\mathcal{A}$  is a  $Z$ -set in  $\downarrow C(X, Y)$ .*

*Sketch of proof.* It is sufficient to show that for any map  $\epsilon : \downarrow C(X, Y) \rightarrow (0, 1)$ , there is a map  $\phi : \downarrow C(X, Y) \rightarrow \downarrow C(X, Y)$  such that  $\phi(\downarrow C(X, Y)) \cap \mathcal{A} = \emptyset$  and  $\rho_H(\phi(\downarrow f), \downarrow f) < \epsilon(\downarrow f)$  for each  $\downarrow f \in \downarrow C(X, Y)$ . Since  $\mathcal{Z}$  is a  $Z$ -set, there exists a map  $\psi : \downarrow C(X, Y) \rightarrow \downarrow C(X, Y) \setminus \mathcal{Z}$  such that  $\rho_H(\psi(\downarrow f), \downarrow f) < \epsilon(\downarrow f)/2$  for every  $\downarrow f \in \downarrow C(X, Y)$ . Fix a point  $y_0 \in Y \setminus \{\mathbf{0}\}$  with  $d_Y(\mathbf{0}, y_0) \leq 1$  and let

$$t(\downarrow f) = \min\{\epsilon(\downarrow f), \rho_H(\psi(\downarrow f), \mathcal{Z}), \text{diam } Y\}/2 > 0$$

for each  $\downarrow f \in \downarrow C(X, Y)$ , where  $\rho_H(\psi(\downarrow f), \mathcal{Z})$  means the usual distance between the point  $\psi(\downarrow f)$  and the subset  $\mathcal{Z}$  in  $\downarrow C(X, Y)$  and  $\text{diam } Y$  means the diameter of  $Y$ .

We consider the case that  $x \notin X_0$  (the case that  $x \in X_0$  can be proved without using Lemma 2.2). Using the Digging Lemma 2.2, we can find maps  $\xi : \downarrow C(X, Y) \rightarrow \downarrow C(X, Y)$  and  $\delta : \downarrow C(X, Y) \rightarrow (0, 1)$  such that for each  $\downarrow f \in \downarrow C(X, Y)$ ,

- (a)  $\rho_H(\xi(\downarrow f), \psi(\downarrow f)) < t(\downarrow f)/2$ ,
- (b)  $\xi(\downarrow f)(B_{d_X}(x, \delta(\downarrow f))) = \{\{\mathbf{0}\}\}$ .

For each  $\downarrow f \in \downarrow C(X, Y)$ , let

$$\eta(\downarrow f) = \bigcup_{x' \in B_{d_X}(x, \delta(\downarrow f))} \{x'\} \times [\mathbf{0}, \gamma(\mathbf{0}, y_0, t(\downarrow f)(\delta(\downarrow f) - d_X(x, x'))/(2\delta(\downarrow f)))],$$

where  $\gamma : Y^2 \times \mathbf{I} \rightarrow Y$  is as in Lemma 2.1. We define a map  $\phi : \downarrow C(X, Y) \rightarrow \downarrow C(X, Y)$  by  $\phi(\downarrow f) = \xi(\downarrow f) \cup \eta(\downarrow f)$ , which is the desired map.  $\square$

We show the implication (2)  $\rightarrow$  (3) in the main theorem.

**Proposition 3.4.** *If  $X_0$  is not dense in  $X$ , then  $\downarrow C(X, Y)$  is a  $Z_\sigma$ -set in itself.*

*Sketch of proof.* Take a countable dense subset  $\{x_n \mid n \in \mathbb{N}\}$  of  $X \setminus X_0$ . For each  $n, m \in \mathbb{N}$ , the set

$$\mathcal{F}_{n,m} = \{\downarrow f \in \downarrow C(X, Y) \mid d_Y(f(x_n), \mathbf{0}) \geq 1/m\}$$

is closed in  $\downarrow C(X, Y)$ . Applying the Digging Lemma 2.2 to the point  $x_n$ , we can easily show that each  $\mathcal{F}_{n,m}$  is a  $Z$ -set in  $\downarrow C(X, Y)$ .

Let  $\mathcal{F} = \bigcap_{n \in \mathbb{N}} \bigcap_{m \in \mathbb{N}} (\downarrow C(X, Y) \setminus \mathcal{F}_{n,m})$ . We prove that the closure  $\overline{\mathcal{F}}$  of  $\mathcal{F}$  in  $\downarrow C(X, Y)$  is a  $Z$ -set. As is easily observed,

$$\mathcal{F} = \{\downarrow f \in \downarrow C(X, Y) \mid f(x_n) = \mathbf{0} \text{ for each } n \in \mathbb{N}\}.$$

Since  $X_0$  is not dense in  $X$ , we can choose a point  $x \in X \setminus \overline{X_0}$ , where  $\overline{X_0}$  is the closure of  $X_0$ . Then for every  $\downarrow f \in \overline{\mathcal{F}}$ , we have  $\downarrow f(x) = \{\mathbf{0}\}$ . According to Lemma 3.3,  $\overline{\mathcal{F}}$  is a  $Z$ -set in  $\downarrow C(X, Y)$ . Consequently,  $\downarrow C(X, Y) = \overline{\mathcal{F}} \cup \bigcup_{m,n \in \mathbb{N}} \mathcal{F}_{n,m}$  is a  $Z_\sigma$ -set in itself.  $\square$

## 4 Topological type of $\downarrow C(X, Y)$

Historically, the notion of infinite-dimensional manifolds arose in the field of functional analysis to classify linear spaces and convex sets topologically. Techniques from this theory has been used for the study on function spaces, and hence typical infinite-dimensional manifolds, especially their model spaces, have been detected among many function spaces. From the end of 1980s to the beginning of 1990s, many researchers investigated topological types of function spaces of real-valued continuous functions on countable spaces equipped with the topology of pointwise convergence, refer to [4].

We can consider that spaces of hypo-graphs give certain geometric aspect to function spaces with the topology of pointwise convergence. Let  $\mathbf{Q} = \mathbf{I}^{\mathbb{N}}$  be the Hilbert cube and  $\mathbf{c}_0 = \{(x_i)_{i \in \mathbb{N}} \in \mathbf{Q} \mid \lim_{i \rightarrow \infty} x_i = 0\}$ . In the case that  $Y = \mathbf{I}$  and  $\mathbf{0} = 0$ , we can regard

$$\downarrow \text{USC}(X, \mathbf{I}) = \{\downarrow f \mid f : X \rightarrow \mathbf{I} \text{ is upper semi-continuous}\}$$

as a subspace in  $\text{Cld}(X \times \mathbf{I})$ . In [8], the following theorem is shown:

**Theorem 4.1.** *Suppose that  $X$  is infinite and locally connected. Then  $\downarrow \text{USC}(X, \mathbf{I}) = \overline{\downarrow C(X, \mathbf{I})}$  and the pair  $(\downarrow \text{USC}(X, \mathbf{I}), \downarrow C(X, \mathbf{I}))$  is homeomorphic to  $(\mathbf{Q}, \mathbf{c}_0)$ .*

For spaces  $W_1$  and  $W_2$ , the symbol  $(W_1, W_2)$  means that  $W_2 \subset W_1$ . We recall that a pair  $(W_1, W_2)$  of spaces is homeomorphic to  $(Z_1, Z_2)$  if there exists a homeomorphism  $f : W_1 \rightarrow Z_1$  such that  $f(W_2) = Z_2$ . In the paper [7], the above result is generalized as follows:

**Theorem 4.2.** *If  $X$  is infinite and has only a finite number of isolated points, then the pair  $(\overline{\downarrow C(X, Y)}, \downarrow C(X, Y))$  is homeomorphic to  $(\mathbf{Q}, \mathbf{c}_0)$ .*

The space  $\mathbf{c}_0$  is not a Baire space. In fact, it is a  $Z_\sigma$ -set in itself. According to the main theorem, we can establish the following immediately.

**Corollary 4.3.** *If  $\downarrow C(X, Y)$  is homeomorphic to  $\mathbf{c}_0$ , then the set of isolated points is not dense in  $X$ .*

Z. Yang and X. Zhou [9] strengthened Theorem 4.1 as follows:

**Theorem 4.4.** *The pair  $(\downarrow USC(X, \mathbf{I}), \downarrow C(X, \mathbf{I}))$  is homeomorphic to  $(\mathbf{Q}, \mathbf{c}_0)$  if and only if the set of isolated points of  $X$  is not dense.*

It is still unknown whether the same result holds or not in our setting.

*Problem 1.* If the set of isolated points of  $X$  is not dense, then is the pair  $(\overline{\downarrow C(X, Y)}, \downarrow C(X, Y))$  homeomorphic to  $(\mathbf{Q}, \mathbf{c}_0)$ ?

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