数理解析研究所講究録
第 1932 巻 2015 年 21-24

Dynamical decomposition theorems

Masatoshi Hiraki[1] and Hisao Kato[2]
[1]Graduate School of Pure and Applied Sciences, University of Tsukuba
[2]Institute of Mathematics, University of Tsukuba

Abstract. In this article, we study some dynamical decomposition theorems of spaces related to given homeomorphisms. First, we introduce new notions of 'bright spaces' and 'dark spaces' of homeomorphisms except n times, and by use of the notions we show some dynamical decomposition theorems of spaces related to given homeomorphisms. Next, we show that if $f : X \to X$ is a homeomorphism of an n-dimensional separable metric space X with zero-dimensional set of periodic points, then X can be decomposed into a zero-dimensional bright space of f except n times and an $(n-1)$-dimensional dark space of f except n times, and also by use of dark spaces, we can show some decomposition theorems of X related to dimension theory and dynamical systems. Finally, we study dynamical decompositions of continuum-wise expansive homeomorphisms.

1 Introduction

In this article, we assume that all spaces are separable metric spaces and dimension means the topological dimension dim. Also, let $\mathbb{N}$ and $\mathbb{Z}$ denote the set of natural numbers and the set of integers, respectively. If A is a subset of a space X, then $\mathrm{cl}(A)$, $\mathrm{bd}(A)$ and $\mathrm{int}(A)$ denote the closure, the boundary and the interior of A in X, respectively. For a collection $\mathcal{G}$ of subsets of X,

$$\mathrm{ord}(\mathcal{G}) = \sup\{\mathrm{ord}_x(\mathcal{G}) |\ x \in X\},$$

where $\mathrm{ord}_x(\mathcal{G})$ is the number of members of $\mathcal{G}$ which contains x.

We introduce new notions of 'bright spaces' and 'dark spaces' of homeomorphisms except n times, and by use of the notions we prove some dynamical decomposition theorems of spaces related to given homeomorphisms. For a homeomorphism $f : X \to X$ of a space X and $k \in \mathbb{N}$, let $P_k(f)$ denote the set of points of period $\leq k$. Also, $P(f)$ denotes the set of all periodic poins of f. A subset Z of X is a *bright space* of f except n times ($n \in \{0\} \cup \mathbb{N}$) if for any $x \in X$,

$$|\{p \in \mathbb{Z}|\ f^p(x) \notin Z\}| \leq n,$$

where $|A|$ denotes the cardinality of a set A. Also we say that $L = X - Z$ is a *dark space* of f except n times. Note that for any $x \in X$, $|O_f(x) \cap L| \leq n$, where $O_f(x) = \{f^p(x)|\ p \in \mathbb{Z}\}$ denotes the orbit of x, and also note that $L \cap P(f) = \phi$. For a dark space L of f except n times and $0 \leq j \leq n$, we put

$$A_f(L,j) = \{x \in X|\ |\{p \in \mathbb{Z}|\ f^p(x) \in L\}| = j\}\ (= \{x \in X|\ |O_f(x) \cap L| = j\}).$$

$A_f(L,j)$ denotes the set of all point $x \in X$ whose orbit $O_f(x)$ appears in L just j times. Note that $P(f) \subset A_f(L,0)$ and $A_f(L,j)$ is f-invariant, i.e. $f(A_f(L,j)) = A_f(L,j)$ and $A_f(L,i) \cap A_f(L,j) = \phi$ if $i \neq j$. Hence we have the f-invariant decomposition related to the dark space L as follows;

$$X = A_f(L,0) \cup A_f(L,1) \cup \cdots \cup A_f(L,n).$$

2 Dynamical decomposition theorems of homeomorphisms with zero-dimensional sets of periodic points

It is well-known that a space X has at most dimension n ($n \in \{0\} \cup \mathbb{N}$) (i.e. $\dim X \leq n$) if and only if X can be represented as a union of $(n+1)$ zero-dimensional subspaces of X (see [2, 12]). The following proposition may be known.

Proposition 2.1. *Suppose that X is a space with $\dim X = n$ $(< \infty)$ and $f : X \to X$ is a homeomorphism. Then there exist f-invariant zero-dimensional dense G_δ-sets $A_f(j)$ $(j = 0, 1, 2, ..., n)$ of X such that*

$$X = A_f(0) \cup A_f(1) \cup \cdots \cup A_f(n).$$

In [1], Arts, Fokkink and Vermeer proved the following interesting theorem of dynamical systems of homeomorphisms under some dimensional conditions of periodic points.

Theorem 2.2. ([1, Theorem 8]) *Suppose that $f : X \to X$ is a homeomorphism of a (metric) space X with $\dim X \leq n$ $(< \infty)$. Then there exists a dense G_δ-set Z of X such that $\dim Z = 0$ and*

$$X = Z \cup f(Z) \cup f^2(Z) \cup \cdots \cup f^n(Z)$$

if and only if $\dim P_k(f) < k$ for each $1 \leq k \leq n$.

In this article, under the condition of $\dim P(f) \leq 0$, we prove more chaotic decomposition theorems of dynamical systems of homeomorphisms. In [3, 4, 5, 8, 9], we studied some dynamical properties of homeomorphisms with zero-dimensional set of periodic points. Now, we need the following lemma.

Lemma 2.3. (cf. [4, Lemma 3.5] and [3, Lemma 2.2]) *Suppose that X is a space with $\dim X = n$ $(< \infty)$ and $f : X \to X$ is a homeomorphism with $\dim P(f) \leq 0$. Let F be an F_σ-set of X with $\dim F \leq 0$. Then for each $j \in \mathbb{N}$, there is a locally finite countable open cover $\mathcal{C}(j) = \{C(j)_\alpha \mid \alpha \in \mathbb{N}\}$ of X such that*
(1) $\mathrm{mesh}(\mathcal{C}(j)) < 1/j$,
(2) $\mathrm{ord}(\mathcal{G}) \leq n$, where $\mathcal{G} = \{f^p(\mathrm{bd}(C(j)_\alpha)) \mid \alpha \in \mathbb{N},\ j \in \mathbb{N}$ and $p \in \mathbb{Z}\}$ and
(3) $F \cap L = \phi$, where $L = \cup\{(\mathrm{bd}(C(j)_\alpha)) \mid \alpha \in \mathbb{N},\ j \in \mathbb{N}\}$.

The following theorem is a key result.

Theorem 2.4. *Suppose that X is a space with $\dim X = n$ $(< \infty)$ and $f : X \to X$ is a homeomorphism. Then there exists a bright space Z of f except n times such that Z is a zero-dimensional dense G_δ-set of X and the dark space $L = X - Z$ of f is a $(n-1)$-dimensional F_σ-set of X if and only if $\dim P(f) \leq 0$.*

Corollary 2.5. *Suppose that X is a space with $\dim X = n$ $(< \infty)$ and $f : X \to X$ is a homeomorphism. Then there exists a zero-dimensional G_δ-dense set Z of X such that for any $(n+1)$ integers $k_0 < k_1 < \cdots < k_n$,*

$$X = f^{k_0}(Z) \cup f^{k_1}(Z) \cup \cdots \cup f^{k_n}(Z)$$

if and only if $\dim P(f) \leq 0$.

Theorem 2.6. *Suppose that X is a space with $\dim X = n$ $(< \infty)$ and $f : X \to X$ is a homeomorphism with $\dim P(f) \leq 0$. If L is a dark space of f except n times such that L is an F_σ-set of X and $\dim\ (X - L) \leq 0$, then $\dim A_f(L, j) = 0$ for each $j = 0, 1, 2, ..., n$. In particular, there is the f-invariant zero-dimensional decomposition of X related to the dark space L:*

$$X = A_f(L, 0) \cup A_f(L, 1) \cup \cdots \cup A_f(L, n).$$

Finally, as a special case we consider the case that $f : X \to X$ is a continuum-wise expansive homeomorphism of a compact metric space X. A homeomorphism $f : X \to X$ of a compact metric space (X, d) is *expansive* (see [11]) if there is $c > 0$ such that for any $x, y \in X$ with $x \neq y$, there is an integer $k \in \mathbb{Z}$ such that $d(f^k(x), f^k(y)) \geq c$. Similarly, a homeomorphism $f : X \to X$ of a compact metric space (X, d) is *continuum-wise expansive* (see [6, 7]) if there is $c > 0$ such that for any nondegenerate subcontinuum A of X, there is an integer $k \in \mathbb{Z}$ such that diam $f^k(A) \geq c$. Note that every expansive homeomorphism is continuum-wise expansive. Such $c > 0$ is called an *expansive constant* for f. It is known that if a compact metric space X admits a continuum-wise expansive homeomorphism f on X, then $\dim X < \infty$ and every minimal set of f is zero-dimensional (see [11] and [6]). Moreover, $\dim I_0(f) \leq 0$, where

$$I_0(f) = \bigcup\{M \mid M \text{ is a zero-dimensional } f\text{-invariant closed set of } X\}$$

(see [7, Proposition 2.5]). In particular, $\dim P(f) \leq 0$. We need the following proposition.

Proposition 2.7. ([6, Proposition 5.1]) *Suppose that $f : X \to X$ is a homeomorphism of a compact metric space X. Then the following are equivalent.*
(1) *f is continuum-wise expansive.*
(2) *There is $\delta > 0$ such that if $\mathcal{C}$ is any finite open cover of X with* mesh$(\mathcal{C}) < \delta$ *and any $\gamma > 0$, there is a sufficiently large natural number N such that if $A, B \in \mathcal{C}$, each component of $f^{-n}(\mathrm{cl}(A)) \cap f^n(\mathrm{cl}(B))$ has diameter less than γ for each $n \geq N$.*

In the case of continuum-wise expansive homeomorphisms, by use of compact dark spaces we obtain the following decomposition theorem.

Theorem 2.8. *Suppose that X is a compact metric space with* $\dim X = n$ $(< \infty)$ *and $f : X \to X$ is a continuum-wise expansive homeomorphism. Then there exists a compact $(n-1)$-dimensional dark space L of f except n times such that* $\dim A_f(L, j) = 0$ *for each $j = 0, 1, 2, ..., n$. In particular, there is the f-invariant zero-dimensional decomposition of X related to the compact dark space L:*

$$X = A_f(L, 0) \cup A_f(L, 1) \cup \cdots \cup A_f(L, n).$$

Remark. (1) In Theorem 2.8, the bright space $Z = X - L$ of f is open in X and n-dimensional. (2) In Theorem 2.8, suppose that $\dim X = 1$. Then L is a compact zero-dimensional dark space of f except 1 time such that $\dim A_f(L, j) = 0$ for each $j = 0, 1$ if and only if L is a zero-dimensional compactum such that $f^i(L) \cap L = \emptyset$ for any $i \in \mathbb{N}$ and dim $(X - \cup_{i \in \mathbb{Z}} f^i(L)) = 0$.

Example. Let $f : I = [0, 1] \to I$ be the 'tent' map of the unit interval I defined by $f(x) = 2x$ for $0 \leq x \leq 1/2$ and $f(x) = 2 - 2x$ for $1/2 \leq x \leq 1$. Consider the inverse limit

$$X = \{(x_i)_{i=1}^{\infty} \in I^{\infty} \mid f(x_{i+1}) = x_i \text{ for } i \in \mathbb{N}\} \subset I^{\infty}$$

of f and the shift map $\tilde{f} : X \to X$ defined by $\tilde{f}((x_i)_{i=1}^{\infty}) = (f(x_i))_{i=1}^{\infty}$. Then $\tilde{f}$ is a continuum-wise exapnsive homeomorphism of the Knaster continuum X. Consider the subset

$$L = \{(x_i)_{i=1}^{\infty} \in X \mid x_1 = 1\}.$$

Then we can easily see that L is a zero-dimensional compactum (in fact, a Cantor set) such that $\tilde{f}^i(L) \cap L = \phi$ for any $i \in \mathbb{N}$ and dim $(X - \cup_{i \in \mathbb{Z}} \tilde{f}^i(L)) = 0$ and hence L is a compact zero-dimensional dark space L of $\tilde{f}$ except 1 time such that $\dim A_{\tilde{f}}(L, 0) = 0$. In fact, $X = A_{\tilde{f}}(L, 0) \cup A_{\tilde{f}}(L, 1)$ is a zero-dimensional decomposition of the Knaster continuum X.

References

[1] J. M. Arts, R. J. Fokkink and J. Vermeer, A dynamical decomposition theorem, Acta Math. Hung., 94(3), 2002, 191-196.

[2] R. Engelking, Theory of Dimensions Finite and Infinite , Heldermann Verlag, Lemgo, 1995.

[3] Y. Ikegami, H. Kato and A. Ueda, Eventual colorings of homeomorphisms, J. Math. Soc. Japan, 65, No 2 (2013), 375-387.

[4] Y. Ikegami, H. Kato and A. Ueda, Dynamical systems of finite-dimensional metric spaces and zero-dimensional covers, Topology Appl. 160 (2013), 564-574.

[5] Y. Ikegami, H. Kato and A. Ueda, On eventual coloring numbers, Topology Proceedings, to appear.

[6] H. Kato, Continuum-wise expansive homeomorphisms, Canadian J. of Mathematics, 45 (1993), 576-598.

[7] H. Kato, Minimal sets and chaos in the sense of Devaney on continuum-wise expansive homeomorphisms, Lecture Notes in Pure and Appl. Math. 170, Dekker, New York, 1995.

[8] H. Kato, A note on metric compactifications and periodic points of maps, Topology Appl. 160 (2013), 1406-1409.

[9] H. Kato, Periodic points, compactifications and eventual colorings of maps, Topology Appl. 160 (2013), 685-691.

[10] J. Kulesza, Zero-dimensional covers of finite dimensional dynamical systems, Ergod. Th. Dynam. Sys. 15 (1995), 939-950.

[11] R. Mañé, Expansive homeomorphisms and topological dimension, Trans. Amer. Math. Soc. 252 (1979), 313-319.

[12] J. van Mill, The Infinite-Dimensional Topology of Function Spaces, North-Holland publishing Co., Amsterdam, 2001.