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Dynamic Investment Decisions with an American Exchange Option*

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1 Introduction

An exchange option is a simple contingent claim written on two assets, which gives its holder the right to exchange one asset for another. European prototypes of the exchange option were independently developed by Fischer [11] and Margrabe [24], which are special cases of a general European exchange option (EEO) studied by McDonald and Siegel [25]. For the perpetual American exchange option (AEO), McDonald and Siegel [26] obtained explicit formulas for the option value and the early exercise boundary; see also Liu and Liu [21]. There has been, however, insufficient research on the finite-lived AEO, compared with the standard American option written on a single asset.

For a finite-lived AEO, a very limited number of *approximations* have been developed so far, due to the analytical difficulty; see Carr [7, 8], Paxson [28], Armada et al. [2] and Lindset [20] for infinitely compounded options approximations, and Andrikopoulos [1] and Liu [22] for quadratic approximations. From the view point of dynamic (i.e., time-dependent) and long-term investment decisions, these approximations have grave faults: The compound-option approximations, all of which are based on Geske and Johnson [13] are inappropriate for decision making, because it evaluates the early exercise boundary only at a few points of time. On the other hand, the quadratic approximations based on MacMillan [23] and Barone-Adesi and Whaley [3] have been known to be inaccurate for cases with long maturity. The aim of this research is to analyze the finite-lived AEO quantitatively by the Laplace-Carson transform (LCT) approach [16], obtaining some qualitative findings in dynamic investment decisions.

This paper is organized as follows: In Section 2, as a basic framework of AEO analysis, we introduce a change of numeraire to obtain a one-dimensional dynamics for the two target assets. With this dynamics, we provide some preliminary results for the EEO as well as AEO. In particular, we show that the AEO value can be represented by the value of an American vanilla call option with unit strike. In Section 3, following Kimura [16], we apply the LCT approach to this American call to obtain the LCTs of the value and EEB for the AEO. In Section 4, we show computational results for some particular cases with the aid of numerical Laplace transform inversion. Finally, as applications of our analysis, we show a few other contingent claims of which value can be evaluated by the AEO.

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2 Basic Framework

2.1 Change of Numeraire

For an economy with finite time period $[0, T]$, consider a pair of assets with price processes $(S_t^1)_{t \geq 0}$ and $(S_t^2)_{t \geq 0}$. For S_0^i $(i = 1, 2)$ given, assume that $(S_t^i)_{t \geq 0}$ is a risk-neutralized diffusion process described by the SDE

$$\frac{\mathrm{d}S_t^i}{S_t^i} = (r - \delta_i)\mathrm{d}t + \sigma_i \mathrm{d}W_t^i, \quad t \in [0, T] \tag{1}$$

where $r > 0$ is the risk-free rate of interest, $\delta_i \geq 0$ is the continuous dividend rate of asset i, and $\sigma_i > 0$ is the volatility coefficient of $(S_t^i)_{t \geq 0}$. Assume that all of these coefficients (r, δ_i, σ_i) are constant. In (1), $(W_t^i)_{t \geq 0}$ $(i = 1, 2)$ denote one-dimensional standard Brownian motion processes with constant correlation ρ $(|\rho| < 1)$, defined on a filtered probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, \mathbb{P})$, where $(\mathcal{F}_t)_{t \geq 0}$ is the natural filtration corresponding to $(W_t^1, W_t^2)_{t \geq 0}$, and the probability measure $\mathbb{P}$ is chosen so that each of assets has mean rate of return r.

For these two assets, consider an American option to exchange one asset for another with payoff

$$(S_t^2 - S_t^1)^+ = S_t^1(S_t - 1)^+, \qquad S_t \equiv \frac{S_t^2}{S_t^1}$$

upon exercise. With the numeraire $S_t^1 \mathrm{e}^{\delta_1 t}$, define the equivalent measure $\mathbb{Q}$ on $\mathcal{F}_T$ by

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_{\mathcal{F}_T} = \exp\left\{-\tfrac{1}{2}\sigma_1^2 T + \sigma_1 W_T^1\right\} \tag{2}$$

By Itô's lemma, $(S_t)_{t \geq 0}$ under $\mathbb{Q}$ has the dynamics

$$\frac{\mathrm{d}S_t}{S_t} = (\delta_1 - \delta_2 - \rho\sigma_1\sigma_2 + \sigma_1^2)\mathrm{d}t + \sigma_2 \mathrm{d}W_t^2 - \sigma_1 \mathrm{d}W_t^1 \tag{3}$$

By the Girsanov theorem, $(\widehat{W}_t^i)_{t \geq 0}$ $(i = 1, 2)$ defined by

$$\mathrm{d}\widehat{W}_t^1 = \mathrm{d}W_t^1 - \sigma_1 \mathrm{d}t, \quad \text{and} \quad \mathrm{d}\widehat{W}_t^2 = \mathrm{d}W_t^2 - \rho\sigma_1 \mathrm{d}t$$

are Brownian motion processes under $\mathbb{Q}$, and hence $(W_t)_{t \geq 0}$ defined by

$$\mathrm{d}W_t = \frac{1}{\sigma}\left(\sigma_2 \mathrm{d}\widehat{W}_t^2 - \sigma_1 \mathrm{d}\widehat{W}_t^1\right)$$

is also a Brownian motion under $\mathbb{Q}$, where $\sigma = \sqrt{\sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2}$. Hence, under the measure $\mathbb{Q}$, we obtain the SDE

$$\frac{\mathrm{d}S_t}{S_t} = (\delta_1 - \delta_2)\mathrm{d}t + \sigma \mathrm{d}W_t \tag{4}$$

which means that $(S_t)_{t \geq 0}$ is a geometric Brownian motion with drift $\delta_1 - \delta_2$ and volatility σ. From the relation $(S_t^2 - S_t^1)^+ = S_t^1(S_t - 1)^+$, we see that the exchange option is equivalent to S^1 vanilla call options with unit strike, written on a single underlying asset with continuous dividend rate δ_2 and volatility σ, in a financial market with interest rate δ_1.

2.2 European Exchange Option

Let

$$e(S_t^1, S_t^2, t) = \mathbb{E}_t\left[\mathrm{e}^{-r(T-t)}(S_T^2 - S_T^1)^+\right]$$

denote the value of the EEO at time $t \in [0, T]$ with maturity date T, where $\mathbb{E}_t[\,\cdot\,] \equiv \mathbb{E}^{\mathbb{P}}[\,\cdot\,|\,\mathcal{F}_t]$ denotes the conditional expectation under the measure $\mathbb{P}$. Then, by the change of numeraire (2), we have the McDonald and Siegel formula [25]:

$$\begin{aligned}
e(S_t^1, S_t^2, t) &= S_t^1 \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\delta_1\tau}(S_T - 1)^+\right] \\
&= S_t^1 c(S_t, t) \\
&= S_t^1\left\{S_t\mathrm{e}^{-\delta_2\tau}\Phi\big(d_+(S_t, 1, \tau)\big) - \mathrm{e}^{-\delta_1\tau}\Phi\big(d_-(S_t, 1, \tau)\big)\right\} \\
&= S_t^2\mathrm{e}^{-\delta_2\tau}\Phi\big(d_+(S_t^2, S_t^1, \tau)\big) - S_t^1\mathrm{e}^{-\delta_1\tau}\Phi\big(d_-(S_t^2, S_t^1, \tau)\big),
\end{aligned} \tag{5}$$

where $\mathbb{E}_t^{\mathbb{Q}}[\,\cdot\,] \equiv \mathbb{E}^{\mathbb{Q}}[\,\cdot\,|\,\mathcal{F}_t]$, $\tau = T - t$, $c(S_t, t)$ denotes the vanilla call value, $\Phi(\,\cdot\,)$ is the standard normal cumulative distribution function given by

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \mathrm{e}^{-\frac{1}{2}y^2}\mathrm{d}y, \quad x \in \mathbb{R},$$

and

$$d_\pm(x, y, \tau) = \frac{\log(x/y) + (\delta_1 - \delta_2 \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}.$$

Remark 1 The McDonald and Siegel formula (5) yields various known results as its special cases: It coincides with the results of Fischer [11] if $\delta_2 = 0$; Margrabe [24] if $\delta_i = 0$ $(i = 1, 2)$; Merton [27] for call if $\delta_1 = r$, $\delta_2 = \delta$ and $\sigma_1^2 = 0$; Merton [27] for put if $\delta_1 = \delta$, $\delta_2 = r$ and $\sigma_2^2 = 0$; Black and Scholes [4] for call if $\delta_1 = r$, $\delta_2 = 0$ and $\sigma_1^2 = 0$; and Black and Scholes [4] for put if $\delta_1 = 0$, $\delta_2 = r$ and $\sigma_2^2 = 0$.

2.3 American Exchange Option

Let $E(S_t^1, S_t^2, t)$ denote the value of the AEO at time $t \in [0, T]$ with maturity date T. In the absence of arbitrage opportunities, the value $E(S_t^1, S_t^2, t)$ is a solution of an *optimal stopping problem*

$$E(S_t^1, S_t^2, t) = \operatorname*{ess\,sup}_{\tau_e \in [t,T]} \mathbb{E}_t\left[\mathrm{e}^{-r(\tau_e - t)}(S_{\tau_e}^2 - S_{\tau_e}^1)^+\right], \tag{6}$$

where τ_e is a stopping time of the filtration $(\mathcal{F}_t)_{t\geq 0}$ and the random variable $\tau_e^* \in [t, T]$ is called an *optimal stopping time* if it gives the supremum value of $E(S^1, S^2, t)$. Under the measure $\mathbb{Q}$, we have

$$E(S_t^1, S_t^2, t) = S_t^1 \operatorname*{ess\,sup}_{\tau_e \in [t,T]} \mathbb{E}_t^{\mathbb{Q}}\left[\mathrm{e}^{-\delta_1(\tau_e - t)}(S_{\tau_e} - 1)^+\right] = S_t^1 C(S_t, t), \tag{7}$$

where $C(S_t, t)$ is the value of an American vanilla call option with unit strike, written on an underlying asset with dividend rate δ_2 and volatility σ, in a market with interest rate δ_1.

Due to Kim [15], Jacka [14] and Carr et al. [9], the value $C(S_t, t)$ can be represented as the sum of the European vanilla call value and the early exercise premium, i.e.,

$$C(S_t, t) = c(S_t, t) + \pi(S_t, t), \tag{8}$$

where

$$c(S_t, t) = S_t e^{-\delta_2(T-t)}\Phi\big(d_+(S_t, 1, T-t)\big) - e^{-\delta_1(T-t)}\Phi\big(d_-(S_t, 1, T-t)\big),$$

$$\pi(S_t, t) = \int_t^T \Big\{\delta_2 S_t e^{-\delta_2(u-t)}\Phi\big(d_+(S_t, B(u), u-t)\big) - \delta_1 e^{-\delta_1(u-t)}\Phi\big(d_-(S_t, B(u), u-t)\big)\Big\}\,\mathrm{d}u,$$

and $B(t)$ ($t \in [0, T]$) is the *early exercise boundary* (EEB) of the American vanilla call with unit strike, which is given by solving the integral equation

$$B(t) - 1 = c\big(B(t), t\big) + \pi\big(B(t), t\big). \tag{9}$$

The integral representation (8) shows that the American exchange option can be valued by using an approximation for EEB; see Kimura [18] for a survey.

Let

$$\mathcal{E} \equiv \mathcal{E}(S_t^1, S_t^2, t) = \{(S_t^1, S_t^2, t) : E(S_t^1, S_t^2, t) = (S_t^2 - S_t^1)^+\}$$

denote the immediate exercise region for an American exchange option. Clearly,

$$\mathcal{E} = \{(S_t^1, S_t^2, t) : S_t^2 \geq S_t^1 B(t)\}. \tag{10}$$

For the region $\mathcal{E}$, Broadie and Detemple [6, Proposition 3.3] proved

Proposition 1 The optimal exercise region $\mathcal{E}$ satisfies

(i) $(S_t^1, S_t^2, t) \in \mathcal{E}$ implies $S_t \geq 1$;

(ii) $(S_t^1, S_t^2, t) \in \mathcal{E}$ implies $(S_t^1, \alpha S_t^2, t) \in \mathcal{E}$ for $\alpha \geq 1$;

(iii) $(S_t^1, S_t^2, t) \in \mathcal{E}$ implies $(\alpha S_t^1, \alpha S_t^2, t) \in \mathcal{E}$ for $\alpha > 0$;

(iv) $S^1 = 0$ implies immediate exercise is optimal for all $S^2 > 0$.

The American call value $C(S, t)$ ($S \equiv S_t$ in brief) and the EEB $B(t)$ also can be obtained by jointly solving a *free boundary problem*, which is specified by the Black-Scholes-Merton partial differential equation (PDE)

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial C^2}{\partial S^2} + (\delta_1 - \delta_2) S \frac{\partial C}{\partial S} - \delta_1 C = 0 \tag{11}$$

for $S < B(t)$, together with the boundary conditions

$$\left|\begin{array}{l} \lim\limits_{S \downarrow 0} C(S, t) = 0, \\ \lim\limits_{S \uparrow B(t)} C(S, t) = B(t) - 1, \\ \lim\limits_{S \uparrow B(t)} \dfrac{\partial C}{\partial S} = 1. \end{array}\right. \tag{12}$$

and the terminal condition

$$C(S, T) = (S - 1)^+. \tag{13}$$

In Section 3, we will apply the LCT approach to the PDE (11) with the conditions (12) and (13).

3 A Laplace Transform Approach

With the change of variables $\tau = T - t$, let $\widetilde{S}_\tau = S_{T-\tau} = S_t$ for $\tau \geq 0$. We refer to $(\widetilde{S}_\tau)_{\tau\geq 0}$ as the *backward running process* of $(S_t)_{t\geq 0}$. For the backward running process $(\widetilde{S}_\tau)_{\tau\geq 0}$, let

$$\widetilde{C}(\widetilde{S}_\tau, \tau) = C(S_{T-\tau}, T-\tau) = C(S_t, t),$$
$$\widetilde{B}(\tau) = B(T-\tau) = B(t).$$

For $\lambda > 0$, define the Laplace-Carson transform (LCT) of these time-reversed quantities as

$$C^*(S,\lambda) = \mathcal{LC}[\widetilde{C}(S,\tau)] \equiv \int_0^\infty \lambda e^{-\lambda\tau}\widetilde{C}(S,\tau)d\tau$$
$$B^*(\lambda) = \mathcal{LC}[\widetilde{B}(\tau)] \equiv \int_0^\infty \lambda e^{-\lambda\tau}\widetilde{B}(\tau)d\tau$$

Similarly, define the time-reversed quantities $\tilde{c}(\widetilde{S}_\tau, \tau)$, $\tilde{\pi}(\widetilde{S}_\tau, \tau)$, the LCTs $c^*(S,\lambda)$, $\pi^*(S,\lambda)$ and so on.

For the LCT $c^*(S,\lambda)$ of the European vanilla call option value, Kimura [16, Theorem 3.2] obtained

Proposition 2

$$c^*(S,\lambda) = \begin{cases} \xi_1(S), & S < 1 \\ \xi_2(S) + \dfrac{\lambda S}{\lambda+\delta_2} - \dfrac{\lambda}{\lambda+\delta_1}, & S \geq 1, \end{cases}$$

where for $i = 1, 2$

$$\xi_i(S) = \frac{1}{\theta_1-\theta_2}\frac{\lambda}{\lambda+\delta_2}\left(1 - \frac{\delta_1-\delta_2}{\lambda+\delta_1}\theta_{3-i}\right)S^{\theta_i},$$

and the parameters $\theta_1 \equiv \theta_1(\lambda) > 1$ and $\theta_2 \equiv \theta_2(\lambda) < 0$ are two real roots of the quadratic equation

$$\tfrac{1}{2}\sigma^2\theta^2 + (\delta_1 - \delta_2 - \tfrac{1}{2}\sigma^2)\theta - (\lambda + \delta_1) = 0.$$

Using a symmetric relation between put and call options, Kimura [17, Theorem 2] derived the LCTs for $C^*(S,\lambda)$ and $B^*(\lambda)$:

Proposition 3

$$C^*(S,\lambda) = \begin{cases} S-1, & S \geq B^* \\ c^*(S,\lambda) + \pi^*(S,\lambda), & S < B^*, \end{cases}$$

where $c^*(S,\lambda)$ is the LCT for the European vanilla call value given in Proposition 2,

$$\pi^*(S,\lambda) = \frac{1}{\theta_1}\left\{\frac{\delta_2 B^*}{\lambda+\delta_2} - \theta_2\xi_2(B^*)\right\}\left(\frac{S}{B^*}\right)^{\theta_1}, \quad S < B^*,$$

and $B^* \equiv B^*(\lambda)$ (≥ 1) is a unique positive solution of the functional equation

$$\lambda B^{*\theta_2} + \delta_2\theta_2 B^* + \delta_1(1-\theta_2) = 0. \tag{14}$$

From the relation (7), we have

Theorem 1 Let $E^*(S^1, S^2, \lambda) = \mathcal{LC}[\widetilde{E}(S^1, S^2, \tau)]$ be the LCT of the American exchange option value. Then, we have

$$E^*(S^1,S^2,\lambda) = \begin{cases} S^2 - S^1, & S^2 \geq S^1 B^* \\ S^1\left\{\xi_2\left(\dfrac{S^2}{S^1}\right) + \pi^*\left(\dfrac{S^2}{S^1},\lambda\right)\right\} + \dfrac{\lambda S^2}{\lambda+\delta_2} - \dfrac{\lambda S^1}{\lambda+\delta_1}, & S^1 \leq S^2 < S^1 B^* \\ S^1\left\{\xi_1\left(\dfrac{S^2}{S^1}\right) + \pi^*\left(\dfrac{S^2}{S^1},\lambda\right)\right\}, & S^2 < S^1. \end{cases}$$

Using the Abelian theorems of Laplace transforms, we can characterize asymptotic behaviors of the EEB at a time close to expiration and at infinite time to expiration, which is

Theorem 2 For the time-reversed EEB $\widetilde{B}(\tau)$ $(\tau \geq 0)$, we have

$$\widetilde{B}(0) = B(T) = \max\left(\frac{\delta_1}{\delta_2}, 1\right),$$

and

$$\lim_{\tau\to\infty} \widetilde{B}(\tau) \equiv B_\infty = \frac{\delta_1}{\delta_2}\frac{\theta_2^\circ - 1}{\theta_2^\circ} = \frac{\theta_1^\circ}{\theta_1^\circ - 1},$$

where $\theta_i^\circ = \lim_{\lambda\to 0}\theta_i(\lambda)$ $(i = 1, 2)$.

From Theorems 1 and 2, we can obtain the perpetual AEO value.

Theorem 3 Let $E_\infty(S^1, S^2) = \lim_{T\to\infty} \widetilde{E}(S^1, S^2, \tau)$ be the perpetual value of the American exchange option. Then, for $S^2 < B_\infty S^1$, we have

$$E_\infty(S^1,S^2) = \frac{B_\infty S^1}{\theta_1^\circ}\left(\frac{S^2}{B_\infty S^1}\right)^{\theta_1}$$

Remark 2 Note that the perpetual AEO value given in Theorem 3 can be rewritten as

$$E_\infty(S^1,S^2) = (B_\infty - 1)S^1\left(\frac{S^2/S^1}{B_\infty}\right)^{\theta_1}, \tag{15}$$

which coincides with the McDonald and Siegel result [26, Equation (4)].

4 Computational Results

As shown in Theorems 2 and 3, Laplace transforms are useful to do asymptotic analysis via the Abelian theorem. However, the primary value of the transforms is in time-dependent analysis of the original functions via analytical or numerical transform inversion. In particular, numerical inversion is most important when a transform cannot be analytically inverted by manipulating tabled formulas, which is the normal case in option pricing problems. Numerical inversion is also important when a Laplace transform is implicitly defined, e.g., as a solution of a certain functional equation. Actually, this is the case of our problem: To invert the LCT $E^*(S^1, S^2, \lambda)$, we first have to solve the nonlinear equation in (14) for B^*. Among many numerical methods

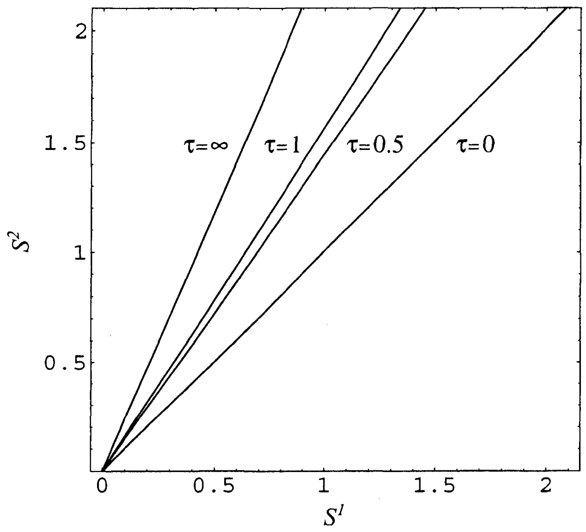


Figure 1: Exercise region $\mathcal{E}$ with different option lives ($T = 1$, $\tau = 0, 0.5, 1, \infty$, $\delta_1 = 0.03$, $\delta_2 = 0.01$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\rho = 0.5$)

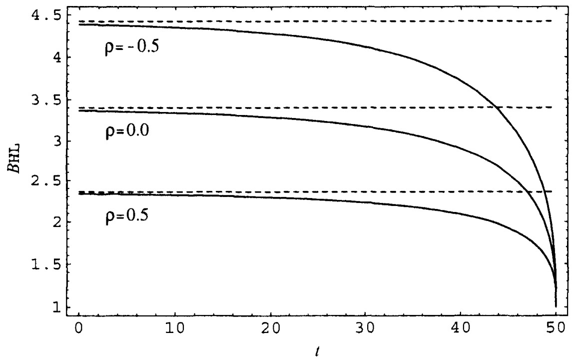


Figure 2: Early exercise boundaries for long maturity ($T = 50$, $\delta_1 = 0.03$, $\delta_2 = 0.01$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\rho = -0.5, 0, 0.5$)

for Laplace transform inversion, the Gaver-Stehfest method [12, 30] is especially convenient for such implicitly defined Laplace transforms, since it works with the transform evaluated only at real arguments. The option value $E^*(S^1, S^2, \lambda)$ in Theorem 1 has different representations according to the values S^1, S^2 and B^*, which raises an issue of selecting a valid expression to be evaluated when λ is a complex number. With the Gaver-Stehfest method, this case-switching issue is resolved. Hence, we use the Gaver-Stehfest method in this paper.

Figure 1 illustrates the dynamic behavior of the optimal exercise region $\mathcal{E}$ as a function of the time to mature τ. In this figure, the upper domain of each line indicates the exercise region, whereas the lower domain indicates a continuation region. Note that the region $\mathcal{E}$ satisfies the properties shown in Proposition 1. The exercise region gets larger as time $t \in [0, T]$ grows, and it becomes the upper half of the S^1-S^2 plane at maturity $t = T$ (i.e., $\tau = 0$). We see from the figure that the *speed* of the boarder line of $\mathcal{E}$ is not constant but it becomes faster as t grows.

We also see that there is a definite gap between the cases $\tau = \infty$ and $\tau = T = 1$, which means that the perpetual result is too permissible to miss an optimal timing of investment.

To see a time lag between the investment decisions based on the perpetual and finite-lived AEOs more definitely, Figure 2 plots the EEBs for AEOs with long maturity ($T = 50$, $\rho = -0.5, 0.0, 0.5$). The perpetual exercise levels are depicted in the dashed lines. We immediately see that the McDonald and Siegel formula (15) for $E_\infty(S^1, S^2)$ gives a much higher estimate for the finite-lived AEO value, so that the perpetual level B_∞ tends to lag behind the optimal timing. This tendency becomes more clear when the two assets are highly and negatively correlated. Although it depends on economic conditions, we need $T > 30$ (years) to justify the usage of the perpetual results in finite-lived cases.

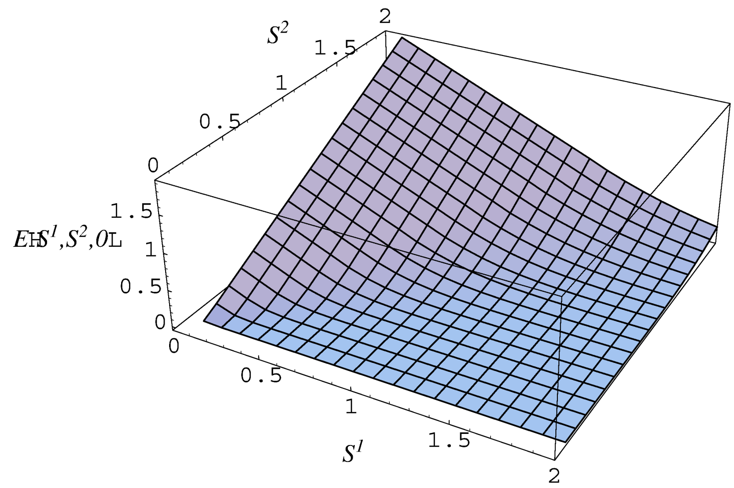


Figure 3: 3D-surface of the AEO value on the S^1-S^2 plane ($T = 1$, $t = 0$, $\delta_1 = 0.03$, $\delta_2 = 0.01$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\rho = 0.5$)

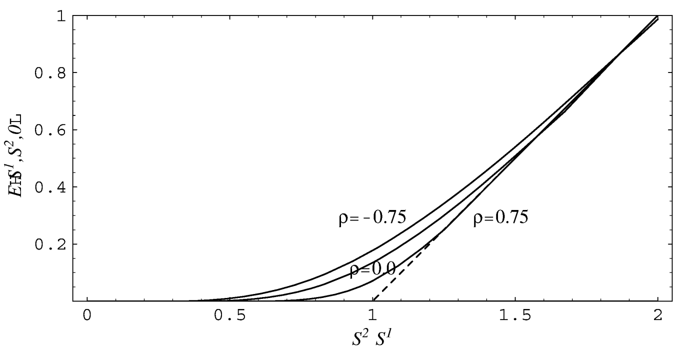
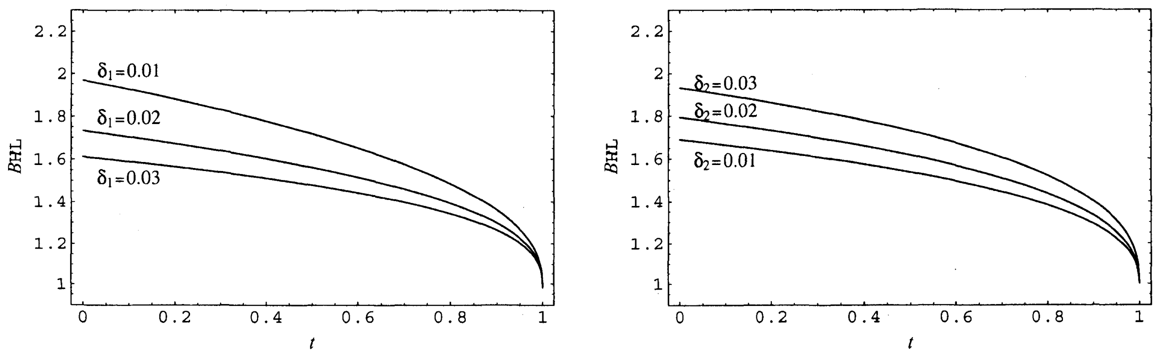


Figure 4: AEO values ($T = 1$, $t = 0$, $\delta_1 = 0.03$, $\delta_2 = 0.01$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\rho = -0.75, 0.0, 0.75$)

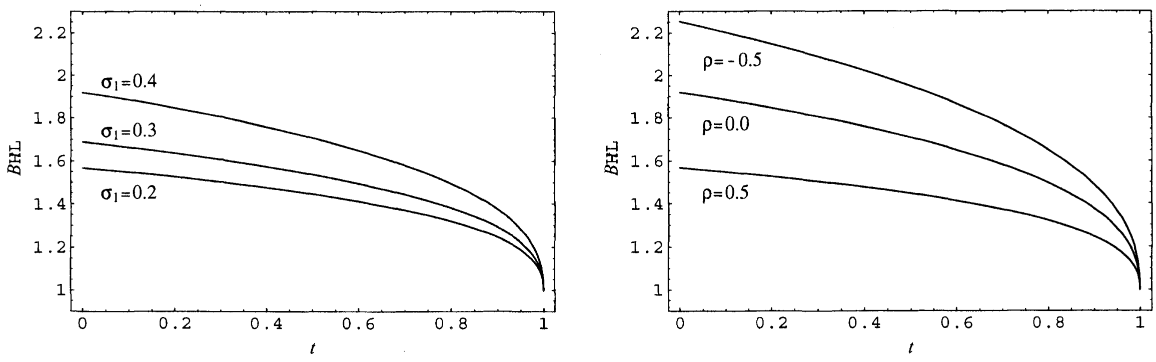
Figure 3 illustrates a 3-dimensional surface of the AEO value $E(S^1, S^2, 0)$ on the S^1-S^2 plane ($(S^1, S^2) \in [0,2]^2$). We see that the value is almost negligible in the lower-half plane, and that the surface is almost rectilinear in the upper-half plane. This observation also can be ensured in Figure 4 that plots the AEO values as a function of $S = S^2/S^1 \in [0,2]$ as well as the intrinsic value $(S-1)^+$ in the dashed line. From Figure 4, we see that the AEO value approaches the intrinsic value as ρ grows, i.e., when the two assets are highly and positively correlated.

Figures 5 and 6 plot EEBs $\left(B(t)\right)_{t\in[0,T]}$ for some AEOs with $T = 1$. The plots are restricted to the cases $\delta_1 \geq \delta_2$, because the Gaver-Stehfest method performs very poorly if $\delta_1 < \delta_2$. We see that the boundary value is an increasing (decreasing) function of δ_2 and σ (δ_1 and ρ), and that the EEBs for the American exchange and vanilla call options are quite alike in their shape.



(a) $\delta_1 = 0.01, 0.02, 0.03$, $\delta_2 = 0.00$ (b) $\delta_1 = 0.03$, $\delta_2 = 0.01, 0.02, 0.03$

Figure 5: Early exercise boundaries ($T = 1$, $\sigma_1 = \sigma_2 = 0.3$, $\rho = 0.5$).



(a) $\sigma_1 = 0.2, 0.3, 0.4$, $\sigma_2 = 0.3$, $\rho = 0.5$ (b) $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\rho = -0.5, 0, 0.5$

Figure 6: Early exercise boundaries ($T = 1$, $\delta_1 = 0.03$, $\delta_2 = 0.01$).

5 Applications to Other Options

The results of this paper can be directly applied some other contingent claims as follows:

Exchange options with proportional caps

The value of an American capped exchange option with proportional cap, i.e., with payoff

$$(S_t^2 - S_t^1)^+ \wedge LS_t^1, \quad \text{where} \quad L > 0$$

is given by $S^1 C^{\widehat{L}}(S_t, t)$, where $C^{\widehat{L}}(S_t, t)$ is the value of an American capped call option with unit strike and cap $\widehat{L} \equiv L + 1$, written on a single asset with price $S_t \equiv S_t^2 / S_t^1$. The immediate exercise region $\mathcal{E}_c$ for the American capped exchange option is given by

$$\mathcal{E}_c = \left\{ (S_t^1, S_t^2, t) : S_t^2 \geq S_t^1 \left(B(t) \wedge \widehat{L} \right) \right\}$$

See Broadie and Detemple [5].

Options on the product with random exercise price

A contract with payoff

$$(S_t^1 S_t^2 - K S_t^1)^+ = S_t^1 (S_t^2 - K)^+$$

is an option to exchange one asset for another where value of the asset to be received is a product of two prices, which is often referred to as a *product option.* An example is an option on the Nikkei index with an exercise price quoted in Japanese yen, where S_t^2 represents the yen-value of the Nikkei, S_t^1 the $/¥ exchange rate, and K the yen-exercise price; see Dravid et al. [10].

The maximum/minimum of two prices

The worse or better performing of two underlying assets can be valued in terms of an exchange option, since

$$\min(S_t^1, S_t^2) = S_t^2 - (S_t^2 - S_t^1)^+$$
$$\max(S_t^1, S_t^2) = S_t^1 + (S_t^2 - S_t^1)^+$$

That is, the minimum (maximum) of two prices is equivalent to the value of a portfolio consisting of a long holding of an asset and an exchange option in the short (long) position; see Rubinstein [29].

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