数理解析研究所講究録
第 1934 巻 2015 年 14-20

Test vectors and central L-values for GL(2)

Daniel File, Kimball Martin, Ameya Pitale

Abstract

This note is a write-up of our results on test vectors for GL(2) and their applications to central L-values, presented by the third author in the RIMS conference on automorphic forms (2014, Jan 20-24). This note also serves as a summary of our recent paper [2].

1 Local results

Let F be a local non-archimedean field of characteristic 0. Let $\mathfrak{o}, \mathfrak{p}, \varpi$ be the ring of integers, prime ideal and uniformizer for F. Let $G=\mathrm{GL}_2(F)$. Let L be a degree 2 extension of F (could be $F\oplus F$). Let π be an irreducible, admissible representation of G with conductor $c(\pi)$. Let Λ be a character of $L^{\times}$ with conductor $c(\Lambda)$. Embed $L^{\times}$ in $\mathrm{GL}_2(F)$ as a torus $T(F)$. Assume that $\Lambda|_{F^{\times}}=\omega_{\pi}$, the central character of π.

A natural question to ask is whether $\mathrm{Hom}_{T(F)}(\pi, \Lambda)$ is zero or not. In particular, we want to know if there exists a linear functional $\ell : V_{\pi}\rightarrow \mathbb{C}$ such that $\ell\neq 0$ and $\ell(\pi(t)v)=\Lambda(t)\ell(v)$ for all $t\in T(F)$ and $v\in V_{\pi}$. It is a theorem of Waldspurger [10] that

$$\dim \mathrm{Hom}_{T(F)}(\pi, \Lambda)\leq 1.$$

More precise information, in terms of epsilon factors, is given by the work of Saito [8] and Tunnell [9].

$$\dim \mathrm{Hom}_{T(F)}(\pi, \Lambda)=\frac{1+\epsilon(1/2, \pi_{L}\times\Lambda)\omega_{\pi}(-1)}{2},$$

where π_{L} is the base change of π to $\mathrm{GL}_2(L)$. Let us assume that the above condition is satisfied for π and Λ and let $0\neq\ell\in \mathrm{Hom}_{T(F)}(\pi, \Lambda)$. A vector $v\in V_{\pi}$ is called a *test vector* for ℓ if $\ell(v)\neq 0$. For applications, we need test vectors satisfying further conditions. For this purpose, let us define a *good test vector* for ℓ to be a test vector v for ℓ satisfying the following two conditions.

i) We have $v\in V_{\pi}^{K}$, where K is a compact subgroup of G such that $\dim V_{\pi}^{K}=1$.

ii) The compact subgroup K in i) depends only on the conductors $c(\pi)$ and $c(\Lambda)$ of π and Λ.

Let us remark on the above two conditions. The first is a matter of computational convenience. In a trace formula application to central L-values, we need to compute some local integrals which simplify greatly if the test vector satisfies condition i) above. The second condition is crucial for average L-value applications. We wish to take an average over all new forms of a fixed level, which would mean that the local non-archimedean representation will have a fixed conductor but otherwise completely arbitrary. In [3], Gross and Prasad obtained the first results on good test vectors for $c(\pi)=0$ or $c(\Lambda)=0$ (under some further conditions). Let us define

$$K_{n}:=\begin{bmatrix}\mathfrak{o}^{\times} & \mathfrak{o}\\ \mathfrak{p}^{n} & 1+\mathfrak{p}^{n}\end{bmatrix}\cap \mathrm{GL}_2(\mathfrak{o}), \text{ for } n\geq 0.$$

The third author is supported by NSF grant DMS-1100541.

Let us now state our first local result for the split case.

1.1 Theorem. *Let $L = F \oplus F$. Assume that π and Λ are unitary. Then* $\mathrm{Hom}_{T(F)}(\pi, \Lambda) \neq 0$, *and for* $0 \neq \ell \in \mathrm{Hom}_{T(F)}(\pi, \Lambda)$ *there exists a good test vector with*

$$K = \begin{bmatrix} 1 & \varpi^{-c(\Lambda)} \\ & 1 \end{bmatrix} K_{c(\pi)} \begin{bmatrix} 1 & -\varpi^{-c(\Lambda)} \\ & 1 \end{bmatrix}.$$

Proof. In the split case, the torus $T(F)$ is given by the diagonal matrices in G. Let the character Λ be given by $\Lambda(\mathrm{diag}(x,y)) = \Lambda_1(x)\Lambda_2(y)$ for characters Λ_1, Λ_2 of $F^\times$. Assume, without loss of generality, that $c(\Lambda_1) \geq c(\Lambda_2)$. Write $\Lambda_1 = |\cdot|^{1/2-s_0}\mu_0$ for $s_0 \in \mathbb{C}$ and unitary character μ_0 of $F^\times$ such that $\mu_0(\varpi) = 1$. The character Λ_2 is determined by the relation $\Lambda_1\Lambda_2 = \omega_\pi$. Let ψ be an additive character of F with conductor $\mathfrak{o}$ and let π be given by its Whittaker model $\mathcal{W}(\pi, \psi)$. For $W \in \mathcal{W}(\pi, \psi)$, $s \in C$, μ a unitary character of $F^\times$, define the zeta integral

$$Z(s, W, \mu^{-1}) := \int\limits_{F^\times} W(\begin{bmatrix} x & \\ & 1 \end{bmatrix})|x|^{s-1/2}\mu^{-1}(x)d^\times x,$$

and define $\ell : \mathcal{W}(\pi, \psi) \to \mathbb{C}$ by

$$\ell(W) := \frac{Z(s_0, W, \mu_0^{-1})}{L(s_0, \mu_0^{-1} \otimes \pi)}.$$

By the theory of zeta integrals, the above function is well-defined and belongs to $\mathrm{Hom}_{T(F)}(\pi, \Lambda)$. Let $W_0 \in V_\pi^{K_{c(\pi)}}$ such that $W_0(1) = 1$. Then, one can check that

$$\ell(\pi(\begin{bmatrix} 1 & \varpi^{-c(\Lambda)} \\ & 1 \end{bmatrix})W_0) \neq 0.$$

This gives the theorem. See [2] for details. ∎

Let us remark here that there is no condition on the conductors of π and Λ. One can relax the unitarity condition (see [2]) but these are satisfied for global applications. When L is a field extension, we have the following result.

1.2 Theorem. *Let L be a field. Assume that $c(\Lambda) \geq c(\pi) > 0$. Then* $\mathrm{Hom}_{T(F)}(\pi, \Lambda) \neq 0$, *and for* $0 \neq \ell \in \mathrm{Hom}_{T(F)}(\pi, \Lambda)$ *there exists a good test vector with*

$$K = \begin{bmatrix} \mathfrak{o}^\times & \mathfrak{p}^{c(\Lambda)} \\ \mathfrak{p}^{c(\pi)-c(\Lambda)} & 1+\mathfrak{p}^{c(\pi)} \end{bmatrix} \cap \mathrm{GL}_2(F) = hK_{c(\pi)}h^{-1},$$

with

$$h = \begin{bmatrix} \varpi^{c(\Lambda)-c(\pi)} & \\ & 1 \end{bmatrix} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix}.$$

Proof. If π is an irreducible principal series or a twist of the Steinberg representation by a ramified character, then we use the induced model for π. We define $\ell : V_\pi \to \mathbb{C}$ by the integral

$$\ell(f) := \int\limits_{Z(F)\backslash T(F)} f(t)\Lambda^{-1}(t)dt.$$

Here, $Z(F)$ is the center of G. Since $Z(F)\backslash T(F)$ is compact, this integral always converges. It can be shown that $\ell \neq 0$ for any $c(\pi)$ and $c(\Lambda)$ (see Section 4.1 of [2]). Let f_0 be the new form in the induced model. An explicit calculation shows that, if $c(\Lambda) \geq c(\pi) > 0$, then $\ell(\pi(h)f_0) \neq 0$, for h as in the statement of the theorem.

If π is the twist of the Steinberg representation by an unramified character, we can in fact do more than in the statement of the theorem. Let $\mathcal{B}(\pi, \Lambda)$ be the Λ-Waldspurger model for π consisting of functions on G transforming by Λ under left translation by $T(F)$. Let B_0 be the new form for π in $\mathcal{B}(\pi, \Lambda)$. Then B_0 is completely determined by its values on the representatives of the double cosets $T(F)\backslash G/K_1$. Using Hecke operators and Atkin-Lehner operators, we can find the explicit value of B_0 for all these representatives. The theorem follows.

If π is a supercuspidal representation, we use Mackey theory. We realize $\pi = c - \mathrm{Ind}_J^G \rho$, where J is a compact (modulo center) subgroup of G and ρ a representation of J. Finding an appropriate ℓ amounts to understanding the intertwining between ρ and Λ. This boils down to looking at the double cosets for $T(F)\backslash G/J$ and showing that only one of them can support the intertwining : $T(F)hJ$, with h as in the statement of the theorem. Then, we show that the translate of the new form by h is indeed a test vector for ℓ. See [2] for details. ∎

Let us remark here that the above calculations do not work for $c(\Lambda) < c(\pi)$. The crucial point is that, under the hypothesis $c(\Lambda) \geq c(\pi)$, certain multiplicative characters give rise to additive characters, which end up giving central values of epsilon factors which are non-zero.

2 Global applications

Let F be a number field and L a quadratic extension. Let π be a cuspidal, automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ with trivial central character. Let Λ be an idele class character of $\mathbb{A}_L^\times$ such that $\Lambda|_{\mathbb{A}_F^\times} = 1$. We are interested in the central value of the L-function $L(1/2, \pi_L \times \Lambda) = L(1/2, \pi, \theta_\Lambda)$. Here, π_L is the base change of π to $\mathrm{GL}_2(\mathbb{A}_L)$ and θ_Λ is the theta function corresponding to Λ. There are several properties of these central values that are of interest – non-vanishing, positivity, sub-convexity, arithmetic properties. The main tool to study this central L-value is the relation it has to period integrals. Let us explain this.

Let D be a quaternion algebra over F containing L such that π has a Jacquet-Langlands transfer π' to $D^\times(\mathbb{A}_F)$. Note that we allow the possibility that D is the matrix algebra. As before, one can embed $L^\times$ as a torus T in $D^\times$. For $\phi \in \pi'$, define the integral

$$P_D(\phi) = \int_{Z(\mathbb{A}_F)T(F)\backslash T(\mathbb{A}_F)} \phi(t)\Lambda^{-1}(t)dt.$$

In [10], using the theory of theta lifts, Waldspurger proved the beautiful formula

$$\frac{|P_D(\phi)|^2}{(\phi, \phi)} = \zeta(2) \prod_v \alpha_v(L, \Lambda, \phi) \frac{L(1/2, \pi_L \times \Lambda)}{L(1, \pi, Ad)}.$$

Here, $(\cdot, \cdot)$ is an inner product on π'. The terms $\alpha_v(L, \Lambda, \phi)$ are certain local integrals which are 1 for almost all places v and $L(s, \pi, Ad)$ is the adjoint L-function of π. It turns out that, excepting for one unique choice of D, the period integrals $P_D(\phi)$ are trivially zero for local reasons. If one fixes this unique D, one gets the criteria for non-vanishing: $L(1/2,\pi_L\times\Lambda)\neq 0$ if and only if $P_D(\phi)\neq 0$.

In [5], Jacquet and Chen have studied the same L-value using a relative trace formula. They get a similar formula with the additional information that the local integrals are squares. This immediately leads to the positivity result $L(1/2,\pi_L\times\Lambda)\geq 0$. For any further information regarding the central L-values, one needs a more explicit formula, i.e., one needs to compute the local integrals for suitable choices of ϕ. For this, one needs to choose ϕ such that the local component of ϕ are precisely the good test vectors from the previous section. Using the Jacquet-Chen formula and the test vectors from the work of Gross and Prasad, explicit central value formulas were obtained by Martin and Whitehouse [6] assuming π and Λ have disjoint ramification.

In case of joint ramification, we obtain local test vectors from Theorems 1.1 and 1.2, which yields the desired global test vector ϕ. Let us now describe the L-value formula obtained by us more precisely.

Denote the absolute value of the discriminants of F and L by Δ and Δ_L. Let $e(L_v/F_v)$ be the ramification degree of L_v/F_v. Let S_{inert} be the set of places of F inert in L. Let $S(\pi)$ (resp. $S(\Lambda)$) be the set of finite places of F where π (resp. Λ) is ramified, $S_1(\pi)$ (resp. $S_2(\pi)$) the set of places where $c(\pi_v)=1$ (resp. $c(\pi_v)\geq 2$), and $S_0(\pi)=S_2(\pi)\cup\{v\in S_1(\pi):L_v/F_v$ is ramified and Λ_v is unramified$\}$. Denote by $c(\Lambda)$ the absolute norm of the conductor of Λ.

2.1 Theorem. *Let π be a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_F)$ with trivial central character and Λ a character of $\mathbb{A}_L^\times/L^\times\mathbb{A}_F^\times$. Assume $\epsilon(1/2,\pi_L\otimes\Lambda)=1$. If $v<\infty$ is inert in L and $c(\pi_v)$, $c(\Omega_v)>0$, then assume that $c(\Omega_v)\geq c(\pi_v)$. Then, one can choose a test vector $\phi\in\pi'$ such that*

$$\frac{|P_D(\phi)|^2}{(\phi,\phi)}=\frac{1}{2}\sqrt{\frac{\Delta}{c(\Lambda)\Delta_L}}L_{S(\Lambda)}(1,\eta)L_{S(\pi)\cup S(\Lambda)}(1,\eta)L_{S(\pi)\cap S(\Lambda)}(1,1_F)L^{S(\pi)}(2,1_F)$$
$$\times\prod_{v\in S(\pi)\cap S(\Lambda)^c}e(L_v/F_v)\prod_{v|\infty}C_v(L,\pi,\Lambda)\cdot\frac{L^{S_0(\pi)}(1/2,\pi_L\otimes\Lambda)}{L^{S_0(\pi)}(1,\pi,Ad)}.$$

Here $(\cdot,\cdot)$ is the standard inner product on π' with respect to the measure on $D^\times(\mathbb{A}_F)$ which is the product of local Tamagawa measures. Also, C_v, for $v|\infty$ are explicit non-zero numbers obtained from the archimedean computation.

Proof. Let S be the set of all places of F including the archimedean ones and where any of L,π or Λ are ramified. For $v\in S_{\text{inert}}$, define

$$\tilde{J}_{\pi_v'}(f_v)=\int_{G'(F_v)}f_v(g)(\pi_v'(g)e_v',e_v')\,dg_v,$$

where e_v' is a norm 1 vector such that $\pi_v'(t)e_v'=\Lambda_v(t)e_v'$ for all $t\in T(F_v)$. For $v\in S-S_{\text{inert}}$, set

$$\tilde{J}_{\pi_v'}(f_v)=\sum_W\int_{F_v^\times}\pi_v'(f_v)W(\begin{bmatrix}a&\\&1\end{bmatrix})\Lambda(\begin{bmatrix}a&\\&1\end{bmatrix})^{-1}\,d^\times a\overline{\int_{F_v^\times}W(\begin{bmatrix}a&\\&1\end{bmatrix})\Lambda(\begin{bmatrix}a&\\&1\end{bmatrix})^{-1}\,d^\times a},$$

where $d^\times a$ is the local Tamagawa measure and W runs over an orthonormal basis for the local Whittaker model $\mathcal{W}(\pi_v, \psi_v)$. Let $f = \prod f_v \in C_c^\infty(G'(\mathbb{A}_F))$ with f_v the unit element of the Hecke algebra for $v \notin S$. Let $\pi'(f)$ be an orthogonal projection onto a 1-dimensional subspace $\langle\phi\rangle$. Then Jacquet-Chen prove the following identity in [5].

$$\frac{|P_D(\phi)|^2}{(\phi,\phi)} = \frac{1}{2}\prod_S \tilde{J}_{\pi'_v}(f_v) \prod_{v\in S_{\text{inert}}} 2\epsilon(1,\eta_v,\psi_v)L(0,\eta_v) \times \frac{L_S(1,\eta)L^S(1/2,\pi_L\otimes\Lambda)}{L^S(1,\pi,Ad)}.$$

One can choose f so that it picks out the global test vector ϕ having local components as the good test vectors. If π_v or Λ_v is unramified or $v|\infty$, the integral $\tilde{J}_{\pi'_v}(f_v)$ has been computed in [6]. In the joint ramification case, we compute using the test vectors from the previous local section. If $c(\pi_v) \geq 2$, then

$$\tilde{J}_{\pi_v}(f_v) = q_v^{-c(\Lambda_v)}\frac{L(1,1_{F_v})L(1,\eta_v)}{e(L_v/F_v)}.$$

If $c(\pi_v) = 1$, then

$$\tilde{J}_{\pi_v}(f_v) = q_v^{-c(\Lambda_v)}\frac{L(1,1_{F_v})L(1,\eta_v)}{e(L_v/F_v)L(2,1_{F_v})}.$$

Note that, in the above case, we can take $D_v^\times = \mathrm{GL}_2(F_v)$ and hence, $\pi_v = \pi'_v$. Putting all the terms together, one gets the result of the theorem. ∎

If $F = \mathbb{Q}$ and π corresponds to a holomorphic new form of square free level N with $N|c(\Lambda)$, then the above formula simplifies considerably:

2.2 Corollary. *Let f be a normalized holomorphic modular eigenform of weight k and square-free level N. Let S be the set of primes $p|N$ which split in L. Let Λ be any ideal class character of L such that $N|c(\Lambda)$ and $\epsilon(1/2, f\times\Lambda) = 1$. Then*

$$\frac{|P_D(\phi)|^2}{(\phi,\phi)} = \frac{C_\infty(L,f,\Lambda)}{2^{k+1}\sqrt{c(\Lambda)\Delta_L}}L_{S(\Lambda)}(1,\eta)^2\prod_{p|N}(1+p^{-1})^{\epsilon_p} \times \frac{L^S(1/2,f\times\Lambda)}{\langle f,f\rangle},$$

where ϵ_p is $+1$ if p splits in L and -1 otherwise, and $\langle\cdot,\cdot\rangle$ is the Petersson inner product.

Note that, in the above theorem and corollary, we still need to have additional information about the period to obtain properties of the central L-value. In certain special cases, one can have a better understanding of the periods. For example, in [4], the author considers a totally real field F and π corresponding to a new form f with parallel weight $(2,\cdots,2)$. Using the techniques of Cornut and Vatsal, and Theorem 2.1 above, the author obtains properties of $\mathrm{ord}_\lambda(L^{\mathrm{alg}}(1/2,f,\Lambda))$.

Another application of the explicit central value formula is to obtain an explicit average value formula for central L-values. As is clear from the theorem below, the formula for averages is much simpler, in particular, it does not involve the mysterious period anymore. Hence, non-vanishing results are immediate. Let us state the average value result next.

2.3 Theorem. *Let F be a totally real number field with $d = [F:\mathbb{Q}]$. Let $\mathcal{F}(\mathfrak{N},2\mathbf{k})$ be the set of cuspidal automorphic representations of $\mathrm{GL}_2(\mathbb{A}_F)$ associated to the holomorphic Hilbert* *modular eigen newforms of weight* $2\mathbf{k}$ *and level* $\mathfrak{N}$, *with* $\mathbf{k} = (k_1, \dots, k_d) \neq (1, \dots, 1)$ *and* $\mathfrak{N}$ *squarefree. Let* L *be a totally imaginary quadratic extension of* F, *which is inert and unramified above each place* $\mathfrak{p}|\mathfrak{N}$. *Fix a unitary character* Λ *of* $\mathbb{A}_L^\times / L^\times \mathbb{A}_F^\times$, *and let* $\mathfrak{C}$ *be the norm of its conductor in* F. *Suppose* $\mathfrak{N} = \mathfrak{N}_0\mathfrak{N}_1$ *and* $\mathfrak{C} = \mathfrak{C}_0\mathfrak{N}_1$ *with* $\mathfrak{N}_0$, $\mathfrak{N}_1$ *and* $\mathfrak{C}_0$ *all coprime. Assume* $\mathfrak{N}_1$ *is odd, and that the number of primes dividing* $\mathfrak{N}_0$ *has the same parity as* d. *Further assume that for each infinite place* v *of* F, $k_v > |m_v|$ *where* $\Lambda_v(z) = (z/\bar{z})^{m_v}$.

Then, if $|\mathfrak{N}_0| > d_{L/F}(|\mathfrak{C}_0|/|\mathfrak{N}_1|)^{h_F}$, *where* h_F *is the class number of* F, *we have*

$$\prod_{v|\infty} \binom{2k_v - 2}{k_v - m_v - 1} \sum_{\mathfrak{N}'} \sum_{\pi \in \mathcal{F}(\mathfrak{N}', 2\mathbf{k})} \frac{L(1/2, \pi_L \otimes \Lambda)}{L^{S(\mathfrak{N})}(1, \pi, Ad)}$$
$$= 2^{2-d} \Delta^{3/2} |\mathfrak{N}| L_{S(\mathfrak{N}_0)}(2, 1_F) L_{S(\mathfrak{N}_1)}(1, 1_F) L^{S(\mathfrak{C}_0)}(1, \eta),$$

where $\mathfrak{N}'$ *runs over ideals dividing* $\mathfrak{N}$ *which are divisible by* $\mathfrak{N}_0$, *and* $S(\mathfrak{J})$ *denotes the set of all primes dividing* $\mathfrak{J}$.

The theorem is proven by computing the geometric sides of a trace formula. The Theorem specializes to [1, Thm 1.1] in the case that $\mathfrak{N}$ and $\mathfrak{C}$ are coprime, i.e., $\mathfrak{N} = \mathfrak{N}_0$. One can use the above formula together with formulas for smaller levels to get both explicit bounds and asymptotics for average values over just the forms of exact level $\mathfrak{N}$. We do this in the case $\mathfrak{N}_1$ is prime. This immediately implies $L(1/2, \pi_L \otimes \Lambda) \neq 0$ for some $\pi_L \in \mathcal{F}(\mathfrak{N}, 2\mathbf{k})$.

Lastly, we include another application of Theorem 2.3 when $\mathfrak{N} = \mathfrak{N}_0$. Here, having an exact formula for the average value over newforms allows us to deduce the nonvanishing mod p of the algebraic part $L^{\mathrm{alg}}(1/2, \pi_L \otimes \Lambda)$ of the central value for p suitably large.

2.4 Theorem. *With notation and assumptions as in Theorem 2.3, suppose* $|\mathfrak{N}| > d_{L/F}|\mathfrak{C}|^{h_F}$, $\mathfrak{N}$ *is coprime to* $\mathfrak{C}$, *and, for each* $v|\infty$, m_v *is even. Let* p *be an odd rational prime satisfying* $p > q + 1$ *for all primes* $q \in S(\Omega)$, *and* $\mathcal{P}$ *a prime of* $\bar{\mathbb{Q}}$ *above* p. *Then there exists* $\pi \in \mathcal{F}(\mathfrak{N}, 2\mathbf{k})$ *such that*

$$L^{\mathrm{alg}}(1/2, \pi_L \otimes \Lambda) \not\equiv 0 \mod \mathcal{P}.$$

This generalizes a theorem of Michel and Ramakrishnan [7] on the case $F = \mathbb{Q}$ and $\mathfrak{N} = N$ is prime. The parity condition on m_v ensures that Λ is algebraic and that the above central value is critical.

References

[1] Feigon, B., Whitehouse, D.: *Averages of central L-values of Hilbert modular forms with an application to subconvexity*, Duke Math. J., **149**, no. 2, 347-410 (2009)

[2] File, D., Martin, K., Pitale, A.: *Test vectors and central L-values for* GL_2. arXiv: 1310.1765

[3] Gross, B. H., Prasad, D.: *Test vectors for linear forms.* Math. Ann., **291**, no. 2, 343-355 (1991)

[4] Hamieh, A.: *Special values of anticyclotomic L-functions modulo* λ. arXiv: 1302.3249

REFERENCES

[5] JACQUET, H., CHEN, N.: *Positivity of quadratic base change L-functions.* Bull. Soc. Math. France, **129**, no. 1, 33-90 (2001)

[6] MARTIN, K., WHITEHOUSE, D.: *Central L-values and toric periods for* GL(2). Int. Math. Res. Not. IMRN, 141-191 (2009)

[7] MICHEL, P., RAMAKRISHNAN, D.: *Consequences of the Gross-Zagier formulae: stability of average L-values, subconvexity, and non-vanishing mod p.* Number theory, analysis and geometry, Springer, New York, 437-459 (2012)

[8] SAITO, H.: *On Tunnell's formula for characters of* GL(2). Compositio Math., **85**, no. 1, 99-108 (1993)

[9] TUNNELL, J.B.: *Local ϵ-factors and characters of* GL(2). Amer. J. Math., **105**, no. 6, 1277-1307 (1983)

[10] WALDSPURGER, J.-L.: *Sur les valeurs de certaines fonctions L automorphes en leur centre de symétrie.* Compositio Math., **54**, no. 2, 173-242 (1985)