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# Pair correlation of zeros of quadratic $L$-functions near the real axis

Keiju Sono (宗野 惠樹)

**Abstract**

In this article, we investigate the non-trivial zeros of quadratic $L$-functions near the real axis. Assuming the Generalized Riemann Hypothesis, we give an asymptotic formula for the weighted pair correlation function of quadratic $L$-functions. From this formula, we prove that there exists a number of " close low lying zeros ".

## 1 Introduction

About forty years ago, H. L. Montgomery [9] published his famous paper titled " The pair correlation of zeros of the zeta function ". Under the assumption of the Riemann Hypothesis (RH), he investigated the function

$$F(\alpha,T)=\left(\frac{T}{2\pi}\log T\right)^{-1}\sum_{0<\gamma,\gamma'\leq T}T^{i\alpha(\gamma-\gamma')}w(\gamma-\gamma'),$$

where $w(u)=4/(4+u^{2})$ and $\gamma$, $\gamma'$ run over the set of the imaginary parts of the non-trivial zeros of the Riemann zeta-function $\zeta(s)$ in $0<\mathrm{Im}(s)\leq T$. (Note that the number of non-trivial zeros of $\zeta(s)$ in this domain is asymptotic to $(1/2\pi)T\log T$.) He obtained some asymptotic formula for $F(\alpha,T)$ $(0<\alpha\leq 1-\epsilon)$, and using this formula, he obtained several results on the distances of the non-trivial zeros. For example, under the assumption of the RH, he proved that at least 67% of the non-trivial zeros are simple , and that

$$\liminf_{n\rightarrow\infty}\frac{(\gamma_{n+1}-\gamma_{n})\log\gamma_{n}}{2\pi}\leq\lambda<1$$

holds for specific $\lambda$, where $\gamma_{n}$ denotes the imaginary part of the $n$-th non-trivial zero of $\zeta(s)$ in the upper half plane.

Later, Montgomery's idea was extended to many types of $L$-functions or other situations. For example, Özlük [10] investigated the non-trivial zeros of the Dirichlet $L$-functions near the real axis. Assuming the Generalized Riemann Hypothesis (GRH), he proved that at least 86% of such zeros are simple in some sense. One of the other interesting generalizations is the work of Hejhal [5].

From his explicit formula of the Riemann zeta-function, he constructed certain asymptotic formula for the function involving the pairs of three distinct zeros of $\zeta(s)$. Further, the result of Hejhal was generalized by Rudnick and Sarnak [14], and the $n$-level correlation of the zeros of principal $L$-functions was obtained. In particular, their results agree with the prediction for the Gaussian unitary ensemble of random matrix theory. Today there are several papers considering such kind of problem ($n$-level density). For example, see [1], [2], [3], [6], [7], [8], [12], [13].

Our aim in this paper is to investigate the pair correlation of the zeros of the quadratic $L$-functions near the real axis. As a prior research, Özlük and Snyder [11] investigated such zeros. Under the assumption of GRH, they studied the asymptotic behavior of the function

$$G_K(\alpha, D) = \left(\frac{1}{2}K\left(\frac{1}{2}\right)D\right)^{-1} \sum_{d\neq 0} e^{-\frac{\pi d^2}{D^2}} \sum_{\rho\in Z_d} K(\rho)D^{i\alpha\gamma}$$

as $D \to \infty$ for $|\alpha| < 2$, where $\rho = 1/2 + i\gamma$ runs over the set of all non-trivial zeros of $L(s, \chi_d)$, the quadratic $L$-function associated to the Kronecker symbol $\chi_d = (d/\cdot)$, and $K(s)$ is some weight function. From their asymptotic formula, they proved that assuming the GRH, not more than 6.25% of all integers $d$ have the property that $L(s, \chi_d)$ vanishes at the central point $s = 1/2$. Subsequently Soundararajan [16] unconditionally proved that $L(1/2, \chi_d) \neq 0$ for at least 87.5% of all fundamental discriminants $d$.

In this paper, assuming the GRH (including RH), we investigate the function $F_K(\alpha, D)$ defined as follows. Let $K(s)$ be analytic in $-1 < \mathrm{Re}(s) < 2$ and satisfy $K(1/2 - it) = K(1/2 + it)$ for any $t \in \mathbf{R}$. Moreover, we assume that its Mellin inverse transform

$$a(x) := \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} K(s)x^{-s}ds \tag{1.1}$$

converges absolutely for any $-1 < c < 2$, $x > 0$, and that $a(x)$ is real, non-negative, belongs to $C^1$ class, and has a support in $[A, B]$ for some $0 < A < B < \infty$. Then, $K(s)$ is given by the Mellin transform of $a(x)$:

$$K(s) = \int_0^\infty a(x)x^s\frac{dx}{x}. \tag{1.2}$$

For $d \in \mathbf{Z}$, let $\chi_d = (d/\cdot)$ be the Kronecker symbol and $L(s, \chi_d)$ be the $L$-function associated to $\chi_d$. We denote the set of non-trivial zeros of $L(s, \chi_d)$ by $Z_d$. For $x > 0$, $D > 0$, we put

$$f_K(x, D) = \sum_d e^{-\frac{\pi d^2}{D^2}} \sum_{\rho_1,\rho_2\in Z_d} K(\rho_1)\overline{K(\rho_2)}x^{\rho_1+\overline{\rho_2}},$$

and for $\alpha \in \mathbf{R}$, we define the correlation function $F_K(\alpha, D)$ by

$$\begin{aligned} F_K(\alpha, D) &= \left[\frac{1}{xD\log D} f_K(x,D)\right]_{x=D^\alpha} \\ &= \frac{1}{D\log D}\sum_d e^{-\frac{\pi d^2}{D^2}} \sum_{\rho_1,\rho_2\in Z_d} K(\rho_1)\overline{K(\rho_2)} D^{i\alpha(\gamma_1-\gamma_2)}, \end{aligned} \tag{1.3}$$

where $\rho_j = 1/2 + i\gamma_j$ for $j = 1, 2$. Then, the main theorem is stated as follows:

**Theorem 1.1.** *Assuming the Generalized Riemann Hypothesis, for any small $\delta > 0$, we have*

$$\begin{aligned} F_K(\alpha, D) = L(1)\alpha &+ a(D^{-\alpha})^2 D^{-\alpha}\log D + a(D^{-\alpha})\cdot O(\alpha D^{-\frac{\alpha}{2}}\log D) \\ &+ a(D^{-\alpha})^2\cdot O(D^{-\alpha}) + O(\min\{1,\ \alpha D^{-\alpha}\log^2 D\}) \\ &+ O(\min\{D^\alpha(\log D)^{-1},\ \alpha^2 D^{-\alpha}\log^3 D\}) + o(1) \end{aligned} \tag{1.4}$$

*uniformly for $0 < \alpha < 1-\delta$ as $D \to \infty$, where*

$$L(1) = \int_0^\infty a(x)^2 dx.$$

*The implied constants depend only on $K(s)$ and $\delta > 0$.*

In the next section, we introduce the outline of the proof. The author recommends the reader to see the preprint [15] to check the detailed computations. Several results on the average gaps of the non-trivial zeros can be obtained. Among others, in Section 3, we prove that there are quite a few pairs of zeros $(1/2 + i\gamma_1, 1/2 + i\gamma_2)$ of $L(s, \chi_d)$ $(d \in \mathbf{Z}\backslash\{0\})$ near the real axis satisfying $0 < |\gamma_1 - \gamma_2| \le (2\pi\lambda)/\log D$, if $\lambda$ is large to a certain extent.

# 2 The proof of Theorem 1.1 (outline)

We start from Özlük's explicit formula

$$\begin{aligned} \sum_{\rho\in Z_d} K(\rho)x^\rho &= K(1)E(\chi_d)x - \sum_{n=1}^\infty a\left(\frac{n}{x}\right)\Lambda(n)\left(\frac{d}{n}\right) \\ &\quad + a\left(\frac{1}{x}\right)\log\left(\frac{|d|}{\pi}\right) + O(\min\{x,\ \log|d|\log x\}) \quad (x \ge 1), \end{aligned}$$

introduced in [11]. Here, $E(\chi) = 1$ if $\chi$ is a principal character, and otherwise $E(\chi) = 0$. The error term is interpreted as $O(1)$ if $x = 1$. Since the main terms of the right hand side are real, we have

$$
\begin{aligned}
&\sum_{\rho_1,\rho_2\in Z_d} K(\rho_1)\overline{K(\rho_2)}x^{\rho_1+\overline{\rho_2}}\\
&=\Bigg\{K(1)E(\chi_d)x-\sum_{n=1}^{\infty}a\left(\frac{n}{x}\right)\Lambda(n)\left(\frac{d}{n}\right)\\
&\qquad\qquad +a\left(\frac{1}{x}\right)\log\left(\frac{|d|}{\pi}\right)+O(\min\{x,\ \log|d|\log x\})\Bigg\}^2\\
&=K(1)^2E(\chi_d)^2x^2+\sum_{n,m=1}^{\infty}a\left(\frac{n}{x}\right)a\left(\frac{m}{x}\right)\Lambda(n)\Lambda(m)\left(\frac{d}{n}\right)\left(\frac{d}{m}\right)\\
&\quad+a\left(\frac{1}{x}\right)^2\log^2\left(\frac{|d|}{\pi}\right)-2K(1)E(\chi_d)x\sum_{n=1}^{\infty}a\left(\frac{n}{x}\right)\Lambda(n)\left(\frac{d}{n}\right)\\
&\quad+2K(1)E(\chi_d)xa\left(\frac{1}{x}\right)\log\left(\frac{|d|}{\pi}\right)-2a\left(\frac{1}{x}\right)\log\left(\frac{|d|}{\pi}\right)\sum_{n=1}^{\infty}a\left(\frac{n}{x}\right)\Lambda(n)\left(\frac{d}{n}\right)\\
&\quad+O\left(\max\left\{K(1)E(\chi_d)x,\sum_{n=1}^{\infty}a\left(\frac{n}{x}\right)\Lambda(n)\left(\frac{d}{n}\right),a\left(\frac{1}{x}\right)\log\left(\frac{|d|}{\pi}\right)\right\}\right)\\
&\qquad\times O\left(\min\{x,\ \log|d|\log x\}\right)\\
&\quad+O\left(\min\{x^2,\ \log^2|d|\log^2 x\}\right).
\end{aligned}
$$

By multiplying both sides by $e^{-\frac{\pi d^2}{D^2}}$ and taking the sum over $d$, we have

$$
f_K(x,D)=\sum_{i=1}^{6}M_i+\sum_{i=1}^{4}O_i, \tag{2.1}
$$

where

$$
\begin{aligned}
M_1&=K(1)^2x^2\sum_d E(\chi_d)^2e^{-\frac{\pi d^2}{D^2}},\\
M_2&=\sum_{n,m=1}^{\infty}a\left(\frac{n}{x}\right)a\left(\frac{m}{x}\right)\Lambda(n)\Lambda(m)\sum_d e^{-\frac{\pi d^2}{D^2}}\left(\frac{d}{n}\right)\left(\frac{d}{m}\right),\\
M_3&=a\left(\frac{1}{x}\right)^2\sum_d e^{-\frac{\pi d^2}{D^2}}\log^2\left(\frac{|d|}{\pi}\right),\\
M_4&=-2K(1)x\sum_{n=1}^{\infty}a\left(\frac{n}{x}\right)\Lambda(n)\sum_d e^{-\frac{\pi d^2}{D^2}}E(\chi_d)\left(\frac{d}{n}\right),\\
M_5&=2K(1)xa\left(\frac{1}{x}\right)\sum_d e^{-\frac{\pi d^2}{D^2}}E(\chi_d)\log\left(\frac{|d|}{\pi}\right),\\
M_6&=-2a\left(\frac{1}{x}\right)\sum_{n=1}^{\infty}a\left(\frac{n}{x}\right)\Lambda(n)\sum_d e^{-\frac{\pi d^2}{D^2}}\left(\frac{d}{n}\right)\log\left(\frac{|d|}{\pi}\right),
\end{aligned}
$$

$$O_1 = O\left(\min\{O_{11},\ O_{12}\}\right), \quad O_2 = O\left(\min\{O_{21},\ O_{22}\}\right),$$
$$O_3 = O\left(\min\{O_{31},\ O_{32}\}\right), \quad O_4 = \left(\min\{O_{41},\ O_{42}\}\right),$$

with

$$O_{11} = K(1)x^2 \sum_d e^{-\frac{\pi d^2}{D^2}} E(\chi_d), \quad O_{12} = K(1)x \log x \sum_d e^{-\frac{\pi d^2}{D^2}} E(\chi_d) \log|d|,$$

$$O_{21} = x \sum_{n=1}^{\infty} a\left(\frac{n}{x}\right) \Lambda(n) \sum_d e^{-\frac{\pi d^2}{D^2}} \left(\frac{d}{n}\right),$$

$$O_{22} = \log x \sum_{n=1}^{\infty} a\left(\frac{n}{x}\right) \Lambda(n) \sum_d e^{-\frac{\pi d^2}{D^2}} \left(\frac{d}{n}\right) \log|d|,$$

$$O_{31} = xa\left(\frac{1}{x}\right) \sum_d e^{-\frac{\pi d^2}{D^2}} \log\left(\frac{|d|}{\pi}\right),$$

$$O_{32} = a\left(\frac{1}{x}\right) \log x \sum_d e^{-\frac{\pi d^2}{D^2}} \log|d| \log\left(\frac{|d|}{\pi}\right),$$

$$O_{41} = x^2 \sum_d e^{-\frac{\pi d^2}{D^2}}, \quad O_{42} = \log^2 x \sum_d e^{-\frac{\pi d^2}{D^2}} \log^2|d|.$$

First, by a standard technique, we find that the error terms $O_1$, $O_2$, $O_3$, $O_4$ are evaluated by

$$O_1 \ll \min\{x^2 D^{1/2}, x D^{1/2} \log x \log D\},$$
$$O_2 \ll \min\{x^2 D, x D \log x \log D, x^{3/2} \log^2 x \log D\},$$
$$O_3 \ll \min\{x D \log D, D \log x \log^2 D\},$$
$$O_4 \ll \min\{x^2 D, D \log^2 x \log^2 D\}.$$

Next, by using partial summation and prime number theorem, the main terms except for $M_2$ are obtained as follows:

$$M_1 = IK(1)^2 x^2 D^{1/2} - \frac{1}{2} K(1)^2 x^2 + O(x^2 D^{-1/2}),$$

$$M_3 = a\left(\frac{1}{x}\right) \{D \log^2 D + O(D \log D)\},$$

$$M_4 = -2IK(1)^2 D^{1/2} x^2 + O(D^{1/2} x^{3/2} \log^2 x) + O(\min\{x^2, x^3 D^{-1/2}\}),$$

$$M_5 \ll xa\left(\frac{1}{x}\right) D^{1/2} \log D,$$

$$M_6 \ll a\left(\frac{1}{x}\right) \{x^{3/2} \log x \log D + D x^{1/2} \log x \log D\},$$

where $I = 4^{-1}\pi^{-1/4}\Gamma(1/4)$. These are obtained by some computations similar to those in the paper of Özlük and Snyder [11], hence we omit the detail. Finally, we compute

$$M_2 = \sum_{k,l=1}^{\infty} \sum_{p,q \in P} a\left(\frac{p^k}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q) \sum_d e^{-\frac{\pi d^2}{D^2}} \left(\frac{d}{p^k}\right)\left(\frac{d}{q^l}\right),$$

where $P$ denotes the set of all prime numbers. It will be convenient to keep in mind that only $k, l$ satisfying $k, l \ll \log x$ contribute to the sum above, since $a(x)$ has a support in such range. First, we evaluate the contribution of the part $p = 2$ to $M_2$. The contribution of the part $p = q = 2$ is

$$\ll \sum_{k,l=1}^{\infty} a\left(\frac{2^k}{x}\right) a\left(\frac{2^l}{x}\right) \sum_d e^{-\frac{\pi d^2}{D^2}} \ll D \log^2 x. \tag{2.2}$$

The contribution of the part $p = 2$, $q \geq 3$, $l \geq 2$ is

$$\begin{aligned} &\ll \sum_k \sum_{l \geq 2} \sum_{q \in P} a\left(\frac{2^k}{x}\right) a\left(\frac{q^l}{x}\right) (\log q) \sum_d e^{-\frac{\pi d^2}{D^2}} \\ &\ll D x^{\frac{1}{2}} \log^2 x. \end{aligned} \tag{2.3}$$

Since $(\cdot/2^k q)$ is a non-principal character whose conductor is at most $2q$, by combining partial summation and Pólya-Vinogradov inequality for the sum involving quadratic characters, we find that

$$\sum_d e^{-\frac{\pi d^2}{D^2}} \left(\frac{d}{2^k}\right)\left(\frac{d}{q}\right) \ll q^{\frac{1}{2}} \log q$$

for primes $q \geq 3$. Therefore, the contribution of the part $p = 2$, $q \geq 3$, $l = 1$ is

$$\begin{aligned} &\sum_k \sum_{q \in P_{\geq 3}} a\left(\frac{2^k}{x}\right) a\left(\frac{q}{x}\right) (\log 2)(\log q) \sum_d e^{-\frac{\pi d^2}{D^2}} \left(\frac{d}{2^k}\right)\left(\frac{d}{q}\right) \\ &\ll (\log x) \sum_{q \in P} a\left(\frac{q}{x}\right) (\log q) \cdot q^{\frac{1}{2}} \log q \\ &\ll x^{\frac{3}{2}} \log^2 x, \end{aligned} \tag{2.4}$$

where $P_{\geq 3}$ denotes the set of all prime numbers greater than 2. By (2.2), (2.3) and (2.4), the contribution of the part $p = 2$ to $M_2$ is at most $O(Dx^{1/2}\log^2 x + x^{3/2}\log^2 x)$. The contribution of the part $q = 2$ is the same. Hence

$$\begin{aligned} M_2 &= \sum_{k,l=1}^{\infty} \sum_{p,q \in P_{\geq 3}} a\left(\frac{p^k}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q) \sum_d e^{-\frac{\pi d^2}{D^2}} \left(\frac{d}{p^k}\right)\left(\frac{d}{q^l}\right) \\ &\quad + O(Dx^{\frac{1}{2}} \log^2 x + x^{\frac{3}{2}} \log^2 x) \\ &=: \sum_{k,l=1}^{\infty} M_2^{(k,l)} + O(Dx^{\frac{1}{2}} \log^2 x + x^{\frac{3}{2}} \log^2 x), \end{aligned} \tag{2.5}$$

say. Moreover, by prime number theorem and partial summation, we have

$$\begin{aligned} M_2^{(k,l)} &\ll \sum_{p,q} a\left(\frac{p^k}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q) \sum_d e^{-\frac{\pi d^2}{D^2}} \\ &\ll klDx^{\frac{1}{k}+\frac{1}{l}} \end{aligned}$$

uniformly for each $k, l$. Hence the contribution of the part $k \geq 3, l \geq 2$ or $k \geq 2, l \geq 3$ is at most $O(Dx^{5/6}\log^4 x)$. Therefore,

$$M_2 = M_2^{(1,1)} + M_2^{(2,2)} + 2\sum_{l \geq 2} M_2^{(1,l)} + O(Dx^{\frac{5}{6}}\log^4 x + x^{\frac{3}{2}}\log^2 x). \tag{2.6}$$

By the computation above, $M_2^{(2,2)}$ is evaluated by

$$M_2^{(2,2)} \ll Dx. \tag{2.7}$$

Next, we compute $M_2^{(1,l)}$ for $l \geq 1$.
A) First, we consider the case that $l$ is odd. We decompose

$$\begin{aligned} M_2^{(1,l)} &= \left( \sum_{\substack{p,q \in P_{\geq 3} \\ p=q}} + \sum_{\substack{p,q \in P_{\geq 3} \\ p \neq q}} \right) a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q) \sum_d e^{-\frac{\pi d^2}{D^2}} \left(\frac{d}{p}\right)\left(\frac{d}{q}\right) \\ &=: M_{2,1}^{(1,l)} + M_{2,2}^{(1,l)}. \end{aligned} \tag{2.8}$$

Easily, we find that

$$M_{2,1}^{(1,1)} = L(1)Dx\log x + O(Dx + x\log x),$$

$$M_{2,1}^{(1,l)} \ll lDx^{1/l}\log x \quad (l \geq 2).$$

Next, we compute $M_{2,2}^{(1,l)}$.
a) If $x = o(D^{1/2})$, by prime number theorem and Pólya-Vinogradov inequality, we obtain

$$M_{2,2}^{(1,l)} \ll lx^{3/2(1+1/l)}\log^2 x.$$

b) If $D^{1/2-\delta} \ll x \ll D^{1-\delta}$ $(\delta > 0)$, by the translation formula of twisted theta function, we find that

$$M_{2,2}^{(1,l)} = D\sum_{p \geq 3} \sum_{\substack{q \geq 3 \\ q \neq p}} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q) \frac{1}{\sqrt{pq}} \sum_m \left(\frac{m}{pq}\right) e^{-\frac{\pi m^2 D^2}{p^2 q^2}}.$$

We decompose this by

$$M_{2,2}^{(1,l)} = M_s^{(1,l)} - M_p^{(1,l)} - M_q^{(1,l)} + M_{pq}^{(1,l)} + E, \tag{2.9}$$

where

$$M_s^{(1,l)} = D\sum_{p\geq 3}\sum_{\substack{q\geq 3\\ q\neq p}} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right)(\log p)(\log q)\frac{1}{\sqrt{pq}}\sum_{m=\square} e^{-\frac{\pi m^2 D^2}{p^2q^2}},$$

$$M_p^{(1,l)} = D\sum_{p\geq 3}\sum_{\substack{q\geq 3\\ q\neq p}} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right)(\log p)(\log q)\frac{1}{\sqrt{pq}}\sum_{\substack{m=\square\\ p|m}} e^{-\frac{\pi m^2 D^2}{p^2q^2}},$$

$$M_q^{(1,l)} = D\sum_{p\geq 3}\sum_{\substack{q\geq 3\\ q\neq p}} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right)(\log p)(\log q)\frac{1}{\sqrt{pq}}\sum_{\substack{m=\square\\ q|m}} e^{-\frac{\pi m^2 D^2}{p^2q^2}},$$

$$M_{pq}^{(1,l)} = D\sum_{p\geq 3}\sum_{\substack{q\geq 3\\ q\neq p}} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right)(\log p)(\log q)\frac{1}{\sqrt{pq}}\sum_{\substack{m=\square\\ pq|m}} e^{-\frac{\pi m^2 D^2}{p^2q^2}},$$

and

$$E = D\sum_{p\geq 3}\sum_{\substack{q\geq 3\\ q\neq p}} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right)(\log p)(\log q)\frac{1}{\sqrt{pq}}\sum_{m\neq\square}\left(\frac{m}{pq}\right) e^{-\frac{\pi m^2 D^2}{p^2q^2}}.$$

Here, $\square$ denotes the square of natural numbers. Using partial summation and prime number theorem, we obtain

$$M_s^{(1,l)} = ID^{1/2}l^{-1}K(1)K(1/l)x^{1+1/l}+O(l^{-1}D^{1/2}x^{1+1/2l}\log^2 x)+O(Dx^{1/2+1/2l}),$$

$$M_p^{(1,l)} \ll lD^{1/2}x^{1/l} + Dx^{1/2+1/2l},$$

$$M_q^{(1,l)} \ll l^{-1}D^{1/2}x\log x + Dx^{1/2+1/2l},$$

$$M_{pq}^{(1,l)} \ll Dx^{1/2+1/2l},$$

and after slightly complicated computations (we use the assumption of the GRH to evaluate the sum involving quadratic symbols), the error term $E$ is evaluated by

$$E \ll Dx^{1/l} + x^{1+1/l}\log^4 x.$$

Hence we obtain

$$\begin{aligned} M_{2,2}^{(1,l)} &= ID^{\frac12}l^{-1}K(1)K\left(\frac{1}{l}\right)x^{1+\frac1l} + O(l^{-1}D^{\frac12}x^{1+\frac{1}{2l}}\log^2 x) + O(Dx^{\frac12+\frac{1}{2l}}) \\ &\quad + O(x^{1+\frac1l}\log^4 x) \end{aligned} \tag{2.10}$$

for odd $l$ and $D^{1/2-\delta} \ll x \ll D^{1-\delta}$. Therefore we have

$$\begin{aligned} M_2^{(1,l)} &= ID^{\frac12}l^{-1}K(1)K\left(\frac{1}{l}\right)x^{1+\frac1l} + O(l^{-1}D^{\frac12}x^{1+\frac{1}{2l}}\log^2 x) + O(lDx^{\frac1l}\log x) \\ &\quad + O(Dx^{\frac12+\frac{1}{2l}}) + O(x^{1+\frac1l}\log^4 x) \end{aligned} \tag{2.11}$$

for odd $l$ and $D^{1/2-\delta} \ll x \ll D^{1-\delta}$. If $l=1$, $D^{1/2-\delta} \ll x \ll D^{1-\delta}$, we have

$$M_2^{(1,1)} = L(1)Dx\log x + IK(1)^2 D^{\frac{1}{2}}x^2 + O(Dx + D^{\frac{1}{2}}x^{\frac{3}{2}}\log^2 x + x^2\log^4 x). \quad (2.12)$$

On the other hand, for odd $l \geq 3$ and $x = o(D^{1/2})$, we have

$$M_2^{(1,l)} \ll lDx^{\frac{1}{l}}\log x + lx^{\frac{3}{2}(1+\frac{1}{l})}\log^2 x, \quad (2.13)$$

and for $l=1$, $x = o(D^{1/2})$, we have

$$M_2^{(1,1)} = L(1)Dx\log x + O(Dx + x^3\log^2 x). \quad (2.14)$$

B) Next, we consider the case that $l$ is even. In this case, we have

$$\begin{aligned} M_2^{(1,l)} &= \sum_{p,q} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right)(\log p)(\log q)\sum_d \left(\frac{d}{p}\right) e^{-\frac{\pi d^2}{D^2}} \\ &- \sum_{p,q} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right)(\log p)(\log q)\sum_{q|d} \left(\frac{d}{p}\right) e^{-\frac{\pi d^2}{D^2}}. \end{aligned} \quad (2.15)$$

Since

$$\sum_d \left(\frac{d}{p}\right) e^{-\frac{\pi d^2}{D^2}} \ll \sqrt{p}\log p,$$

the first term of the right hand side of (2.15) is

$$\begin{aligned} &\sum_{p,q} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right)(\log p)(\log q)\sum_d \left(\frac{d}{p}\right) e^{-\frac{\pi d^2}{D^2}} \\ &\ll \sum_p a\left(\frac{p}{x}\right)\sqrt{p}\log^2 p \sum_q a\left(\frac{q^l}{x}\right)\log q \\ &\ll x^{\frac{3}{2}}\log x \cdot lx^{\frac{1}{l}} \\ &\ll lx^{\frac{3}{2}+\frac{1}{l}}\log x. \end{aligned} \quad (2.16)$$

The second term of the right hand side of (2.15) is

$$\begin{aligned} &\sum_{p,q} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right)(\log p)(\log q)\sum_{q|d} \left(\frac{d}{p}\right) e^{-\frac{\pi d^2}{D^2}} \\ &= \sum_p a\left(\frac{p}{x}\right)(\log p)\sum_q \left(\frac{q}{p}\right) a\left(\frac{q^l}{x}\right)(\log q)\sum_d \left(\frac{d}{p}\right) e^{-\frac{\pi q^2 d^2}{D^2}} \\ &\ll \sum_p a\left(\frac{p}{x}\right)(\log p)\sum_q a\left(\frac{q^l}{x}\right)(\log q)\left|\sum_d \left(\frac{d}{p}\right) e^{-\frac{\pi q^2 d^2}{D^2}}\right|. \end{aligned} \quad (2.17)$$

Now, since $q$ satisfies $q \leq (Bx)^{1/l} \ll D^{1-\delta} \ll D$, we have $D/q \gg 1$. Therefore, by using Pólya-Vinogradov inequality, we have

$$\sum_d \left(\frac{d}{p}\right) e^{-\frac{\pi q^2 d^2}{D^2}} \ll \sqrt{p}\log p. \quad (2.18)$$

Therefore,

$$
\begin{aligned}
&\sum_{p,q} a\left(\frac{p}{x}\right) a\left(\frac{q^l}{x}\right) (\log p)(\log q) \sum_{q|d} \left(\frac{d}{p}\right) e^{-\frac{\pi d^2}{D^2}} \\
&\ll \sum_p a\left(\frac{p}{x}\right) \sqrt{p} \log^2 p \sum_q a\left(\frac{q^l}{x}\right) \log q \qquad (2.19) \\
&\ll l x^{\frac{3}{2}+\frac{1}{l}} \log x.
\end{aligned}
$$

By combining (2.16) and (2.19), we obtain

$$M_2^{(1,l)} \ll l x^{\frac{3}{2}+\frac{1}{l}} \log x \tag{2.20}$$

for even $l$. Now, we have computed or evaluated $M_2^{(1,l)}$ for all $l$. If $x = o(D^{1/2})$, by (2.13) and (2.20), we have

$$\sum_{l \geq 2} M_2^{(1,l)} \ll D x^{\frac{1}{3}} \log x + x^2 \log^2 x. \tag{2.21}$$

By inserting (2.7), (2.14), (2.21) into (2.6), we obtain

$$M_2 = L(1) D x \log x + O(Dx + x^3 \log^2 x).$$

If $D^{1/2-\delta} \ll x \ll D^{1-\delta}$, by (2.11),

$$\sum_{l \geq 3,\ \text{odd}} M_2^{(1,l)} \ll D^{\frac{1}{2}} x^{\frac{4}{3}} + D x^{\frac{2}{3}} + x^{\frac{4}{3}} \log^4 x. \tag{2.22}$$

(Notice that $K(1/l) \ll l$.) On the other hand, by (2.20),

$$\sum_{l \geq 2,\ \text{even}} M_2^{(1,l)} \ll x^2 \log x. \tag{2.23}$$

By inserting (2.7), (2.12), (2.22), (2.23) into (2.6), we obtain

$$
\begin{aligned}
M_2 = L(1) D x \log x &+ I K(1)^2 D^{\frac{1}{2}} x^2 \\
&+ O(Dx + D^{\frac{1}{2}} x^{\frac{3}{2}} \log^2 x + x^2 \log^4 x)
\end{aligned}
$$

for $D^{1/2-\delta} \ll x \ll D^{1-\delta}$.

Now, we have computed or evaluated all the terms appearing in (2.1), hence we obtain the asymptotic formula for $f_K(x, D)$. By dividing this by $xD \log D$ and putting $x = D^\alpha$ ($\alpha > 0$), we obtain the asymptotic formula in Theorem 1.1.

# 3 The pairs of close zeros near the real axis

The asymptotic formula (1.4) of Theorem 1.1 is useful to investigate the distribution of low lying zeros of quadratic $L$-functions. As a corollary, we prove that assuming the GRH and simple zero conjecture for each relevant $L$-functions, there exists a number of "close zeros" near the real axis. First, we mention to the $L$-functions associated to Kronecker symbols. If $d \not\equiv 3 \pmod 4$, $\chi_d = (d/\cdot)$ becomes a Dirichlet character modulo $4|d|$ or $|d|$. In this case, we denote the conductor of $\chi_d$ by $d^*$. If $d \equiv 3 \pmod 4$, the $L$-function associated to $\chi_d$ is expressed by

$$L(s, \chi_d) = \frac{1}{1 - \left(\frac{2}{d}\right) 2^{-s}} \prod_{p \geq 3} \frac{1}{1 - \eta_4(p) \left(\frac{p}{d}\right) p^{-s}},$$

where $\eta_4$ is the non-principal character of modulo 4. In this case, we denote the conductor of $\eta_4(\cdot)(\cdot/d)$ by $d^*$. We define the constants $A_+^*$, $A_-^*$ by

$$A_-^* = \liminf_{D\to\infty} \frac{1}{D\log D} \sum_d e^{-\frac{\pi d^2}{D^2}} \log d^*, \quad A_+^* = \limsup_{D\to\infty} \frac{1}{D\log D} \sum_d e^{-\frac{\pi d^2}{D^2}} \log d^*. \tag{3.1}$$

Notice that since $\log d^* \leq \log d + O(1)$, we have $A_+^* \leq 1$.

**Corollary 3.1.** *Assume the Generalized Riemann Hypothesis and that all non-trivial zeros of* $L(s, \chi_d)$ *are simple. Then, for* $0 < \lambda < 1$, *we have*

$$\begin{aligned} &\frac{1}{D\log D} \sum_d e^{-\frac{\pi d^2}{D^2}} \sum_{\substack{\rho_1, \rho_2 \in Z_d \\ 0 < |\gamma_1 - \gamma_2| \leq \frac{2\pi\lambda}{\log D}}} K(\rho_1) K(\rho_2) \\ &\geq \frac{2}{3}\lambda - \frac{2}{9}\lambda^2 - \frac{\cos 2\pi\lambda}{6\pi^2} + \frac{\sin 2\pi\lambda}{12\pi^3\lambda} - B_+^* + o(1) \end{aligned} \tag{3.2}$$

*as* $D \to \infty$, *where* $B_+^* = A_+^*/3$ *and*

$$K(s) = \left( \frac{e^{s-\frac{1}{2}} - e^{-s+\frac{1}{2}}}{2s-1} \right)^2.$$

*In particular, if* $\lambda > \lambda_0 = 0.6073$, *we have*

$$\sum_d e^{-\frac{\pi d^2}{D^2}} \sum_{\substack{\rho_1, \rho_2 \in Z_d \\ 0 < |\gamma_1 - \gamma_2| \leq \frac{2\pi\lambda}{\log D}}} K(\rho_1) K(\rho_2) \gg D \log D \tag{3.3}$$

*as* $D \to \infty$.

*Proof.* We use Selberg's minorant function

$$h(u) = \left( \frac{\sin \pi u}{\pi u} \right)^2 \frac{1}{1-u^2}.$$

This function is bounded and satisfies $h(u) \leq 1$, $h(u) < 0$ if $|u| > 1$. The Fourier transform of $h(u)$ is given by

$$\hat{h}(\alpha) = \begin{cases} 1 - |\alpha| + \frac{\sin 2\pi|\alpha|}{2\pi} & (|\alpha| \leq 1) \\ 0 & (|\alpha| > 1) \end{cases}$$

(for example, see [4]). For $0 < \lambda < 1$, we give lower and upper bounds for the integral

$$\int_{-\infty}^{\infty} F_K(\alpha, D) \cdot \lambda \hat{h}(\lambda\alpha) d\alpha.$$

First, since the integrant is non-negative and $1/\lambda > 1$, by (1.4), we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} F_K(\alpha, D) \cdot \lambda \hat{h}(\lambda\alpha) d\alpha \\
&\geq \int_{-1}^{1} F_K(\alpha, D) \cdot \lambda \hat{h}(\lambda\alpha) d\alpha \\
&= \lambda L(1) \int_{-1}^{1} |\alpha| \left\{1 - |\lambda\alpha| + \frac{\sin 2\pi\lambda|\alpha|}{2\pi}\right\} d\alpha \qquad (3.4)\\
&+ \lambda \log D \int_{-1}^{1} a(D^{-|\alpha|})^2 D^{-|\alpha|} \left\{1 - |\lambda\alpha| + \frac{\sin 2\pi\lambda|\alpha|}{2\pi}\right\} d\alpha + o(1) \\
&= \frac{2}{3}\lambda - \frac{2}{9}\lambda^2 - \frac{\cos 2\pi\lambda}{6\pi^2} + \frac{\sin 2\pi\lambda}{12\pi^3\lambda} + o(1).
\end{aligned}$$

In the computation above, we used

$$L(1) = \int_0^{\infty} a(v)^2 dv = \frac{1}{3},$$

$$\int_{-\infty}^{\infty} a(D^{-\alpha})^2 D^{-\alpha} d\alpha = \frac{1}{\log D} \int_0^1 a(v)^2 dv = \frac{1}{6 \log D}.$$

On the other hand,

$$\begin{aligned}
&\int_{-\infty}^{\infty} F_K(\alpha, D) \cdot \lambda \hat{h}(\lambda\alpha) d\alpha \\
&= \frac{1}{D \log D} \sum_d e^{-\frac{\pi d^2}{D^2}} \sum_{\rho_1, \rho_2 \in Z_d} K(\rho_1) K(\rho_2) h\left(\frac{(\gamma_1 - \gamma_2)\log D}{2\pi\lambda}\right). \qquad (3.5)
\end{aligned}$$

Now, since $h((\gamma_1 - \gamma_2)\log D/(2\pi\lambda))$ is negative if $|\gamma_1 - \gamma_2| > (2\pi\lambda)/\log D$, by (3.5), we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} F_K(\alpha, D) \cdot \lambda \hat{h}(\lambda\alpha) d\alpha \\
&\leq \frac{1}{D \log D} \sum_d e^{-\frac{\pi d^2}{D^2}} \sum_{\substack{\rho_1, \rho_2 \in Z_d \\ |\gamma_1 - \gamma_2| \leq \frac{2\pi\lambda}{\log D}}} K(\rho_1) K(\rho_2) h\left(\frac{(\gamma_1 - \gamma_2)\log D}{2\pi\lambda}\right) \\
&= \frac{1}{D \log D} \sum_d e^{-\frac{\pi d^2}{D^2}} \sum_{\rho \in Z_d} m_\rho K(\rho)^2 \\
&+ \frac{1}{D \log D} \sum_d e^{-\frac{\pi d^2}{D^2}} \sum_{\substack{\rho_1, \rho_2 \in Z_d \\ 0 < |\gamma_1 - \gamma_2| \leq \frac{2\pi\lambda}{\log D}}} K(\rho_1) K(\rho_2) h\left(\frac{(\gamma_1 - \gamma_2)\log D}{2\pi\lambda}\right)
\end{aligned}$$

$$\leq B_+^* + \frac{1}{D\log D}\sum_d e^{-\frac{\pi d^2}{D^2}} \sum_{\substack{\rho_1,\rho_2\in Z_d \\ 0<|\gamma_1-\gamma_2|\leq\frac{2\pi\lambda}{\log D}}} K(\rho_1)K(\rho_2) + o(1),$$

where $m_\rho$ is the multiplicity of the zero of $L(s,\chi_d)$ at $s = \rho$, and by our assumption, this equals 1. By combining (3.4) and above, we obtain (3.2). Since $B_+^* = A_+^*/3 \leq 1/3$, the right hand side of (3.2) becomes positive if $\lambda > \lambda_0 = 0.6073$. Therefore, we obtain (3.3). $\square$

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Tokyo Denki University
Muzaigakuendai, Inzai,
Chiba, Japan
E-mail address:souno@mail.dendai.ac.jp